Prize offered for the solution of a dynamic blocking problem

Posted by A. Bressan on January 19, 2011

Statement of the problem

Fire is initially burning on the unit disc in the plane $\mathbb{R}^2$, and propagates with unit speed in all directions. To block its spreading, a barrier is constructed in real time, at a certain speed $\sigma > 0$. That means: the total length of the wall constructed up to time t is equal to σt.

Question: how large should the construction speed σ be, in order to be able block the fire?

Mathematical formulation

At each time $t \geq 0$, we denote by $R(t) \subset \mathbb{R}^2$ the region burned by the fire. At the initial time we assume that this region is the open unit disc:

$$R(0) \;=\; R_0 \;=\; \{x \in \mathbb{R}^2\,;\;\; |x| < 1\}. \tag{1}$$

If no barriers are constructed, then the fire propagates with unit speed in all directions. Hence the burned set at time $t > 0$ is explicitly described by

$$R(t) \;=\; \{x \in \mathbb{R}^2\,;\;\; d(x, R_0) \leq t\} \;=\; \{x \in \mathbb{R}^2\,;\;\; |x| < 1 + t\}.$$

We now assume that the spreading of the fire can be controlled by erecting walls, or barriers. These are curves in the plane that cannot be crossed by the advancing fire.

Definition 1. *A set valued map $t \mapsto \gamma(t) \subset \mathbb{R}^2$ is an* **admissible strategy** *if the following conditions hold.*

(H1) For every $t_1 \leq t_2$ one has $\gamma(t_1) \subseteq \gamma(t_2)$.

(H2) Each $\gamma(t)$ is a rectifiable curve with length $m_1(\gamma(t)) \;=\; \sigma t$.

Here m_1 denotes the standard one-dimensional Hausdorff measure. One should think of $\gamma(t)$ as the portion of the wall constructed up to time t, while $\sigma > 0$ is the construction speed. Notice that we do not require that the set $\gamma(t)$ be connected. For example, $\gamma(t)$ may be the union of two arcs, one with length $\sigma t/3$ and the other with length $2\sigma t/3$.

By constructing walls, the size of the burned set is reduced.

Definition 2. *Let $t \mapsto \gamma(t)$ be an admissible strategy. The corresponding set reached by the fire at time t is then defined as*

$$R^\gamma(\tau) \doteq \Big\{ x(\tau)\,;\;\; x(\cdot) \text{ absolutely continuous}, \;\; x(0) \in R_0\,, \;\; |\dot x(t)| \leq 1 \;\text{ for a.e. } t \in [0,\tau]\,, \\ x(t) \notin \gamma(t) \;\text{ for all } t \in [0,\tau] \Big\}. \tag{2}$$

Here the upper dot denotes a time derivative: $\dot x = \frac{dx}{dt}$. In other words, $R^\gamma(t)$ is the set reached by trajectories of the fire which start inside R_0, move at speed ≤ 1, and do not cross the constructed wall.

Definition 3. *We say that the admissible strategy $\gamma(\cdot)$* **blocks the fire** *if the total region burned by the fire*

$$R^\gamma_\infty \doteq \bigcup_{\tau \geq 0} R^\gamma(\tau) \tag{3}$$

is a bounded set.

I am offering a prize of US $ 500 to the first person who solves the following

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Problem: Determine the critical value $\sigma^* \in I\!R$, such that

- if the construction speed is $\sigma > \sigma^*$, then an admissible blocking strategy exists.

- if the construction speed is $\sigma < \sigma^*$, then an admissible blocking strategy does not exist.

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REMARKS

1. This problem was first proposed in [1], where it is proved that a blocking strategy exists if $\sigma > 2$ and cannot exist if $\sigma < 1$. Hence we already know that the critical value must satisfy $1 \leq \sigma^* \leq 2$. The problem could be solved by proving the following

Conjecture: *If $\sigma \leq 2$, then no admissible blocking strategy exists.*

2. As shown in [4], one can reformulate the above problem in terms of one single set $\Gamma \subset I\!R^2$, rather than a family of sets $\gamma(t)$, growing in time. We recall here the main idea.

Given a rectifiable set $\Gamma \subset I\!R^2$, the corresponding set burned by the fire at time t is defined as

$$R^\Gamma(\tau) \doteq \Big\{ x(\tau)\,;\;\; x(\cdot) \text{ absolutely continuous}, \;\; x(0) \in R_0\,, \;\; |\dot x(t)| \leq 1 \;\text{ for a.e. } t \in [0,\tau]\,, \\ x(t) \notin \Gamma \;\text{ for all } t \in [0,\tau] \Big\}. \tag{4}$$

Given a construction speed σ, we say that the barrier Γ is **admissible** if

$$m_1\Big(\Gamma \cap \overline{R^\Gamma(t)}\Big) \;\leq\; \sigma t \qquad\qquad \text{for all } t \geq 0\,. \tag{5}$$

Here the upper bar denotes the closure of a set. An equivalent problem is now the following:

Determine for which values of σ there exists an admissible barrier Γ such that the connected component of $I\!R^2 \setminus \Gamma$ containing R_0 is bounded.

3. If $\sigma \leq 2$, then the fire cannot be blocked by a barrier which is a simple closed curve. Indeed, let P be the position of the last brick of the wall (fig. 1, left). Then there exists a trajectory ω for the fire, starting inside R_0 and reaching P without crossing Γ, having total length

$$m_1(\omega) \;<\; \frac{1}{2}\, m_1(\Gamma)\,. \tag{6}$$

In other words, the fire reaches P and gets out *before* the wall is completed.

Any blocking barrier constructed with speed ≤ 2 (if it exists at all) must have a more complicated topology. For example, one could first construct a simple arc with the only purpose of slowing down the advancement of the fire, then construct a closed curve, entirely surrounding the fire.

Notice that, if Γ is not a simple closed curve (fig. 1, right), then the estimate (6) can fail.

Figure 1.

4. If a blocking strategy exists, then there exist a second blocking strategy that minimizes the sum of the total area burned by the fire plus the total length of the barrier. One can try to use necessary conditions for optimality to derive some properties of this optimal barrier (or to reach a contradiction, ruling out its existence).

5. In this problem, the initial set R_0 is irrelevant. If a blocking strategy exists for some (nonempty) bounded open set R_0, then some blocking strategy exists for every other bounded open set.

6. As shown in fig. 2, in the case $\sigma > 2$, an admissible blocking strategy consists of the union of two arcs of logarithmic spirals:

$$\Gamma \doteq \Big\{(r\cos\theta\,,\ r\sin\theta)\,;\quad -\pi \leq \theta \leq \pi\,,\quad r = e^{\lambda|\theta|}\Big\}.$$

Here $\lambda = \dfrac{2}{\sqrt{\sigma^2-4}}$, so that $\sigma = \dfrac{2\sqrt{1+\lambda^2}}{\lambda}$. The closed curve Γ is admissible because the portion of Γ touched by the fire at time $t \geq 0$ has length

$$\begin{aligned}
m_1\Big(\Gamma \cap \overline{R^\Gamma(t)}\Big) &= m_1\Big(\{x\in\Gamma\,;\ \ |x|\leq 1+t\}\Big)\\
&= 2\int_0^{\lambda^{-1}\cdot\ln(1+t)} \sqrt{r^2(\theta)+\dot r^2(\theta)}\,d\theta \;=\; 2\int_0^{\lambda^{-1}\cdot\ln(1+t)} e^{\lambda\theta}\cdot\sqrt{1+\lambda^2}\,d\theta\\
&= 2\sqrt{1+\lambda^2}\cdot\frac{1}{\lambda}\Big[(1+t)-1\Big] \;=\; \sigma\,t\,.
\end{aligned}$$

Clearly, this construction breaks down when $\sigma \leq 2$.

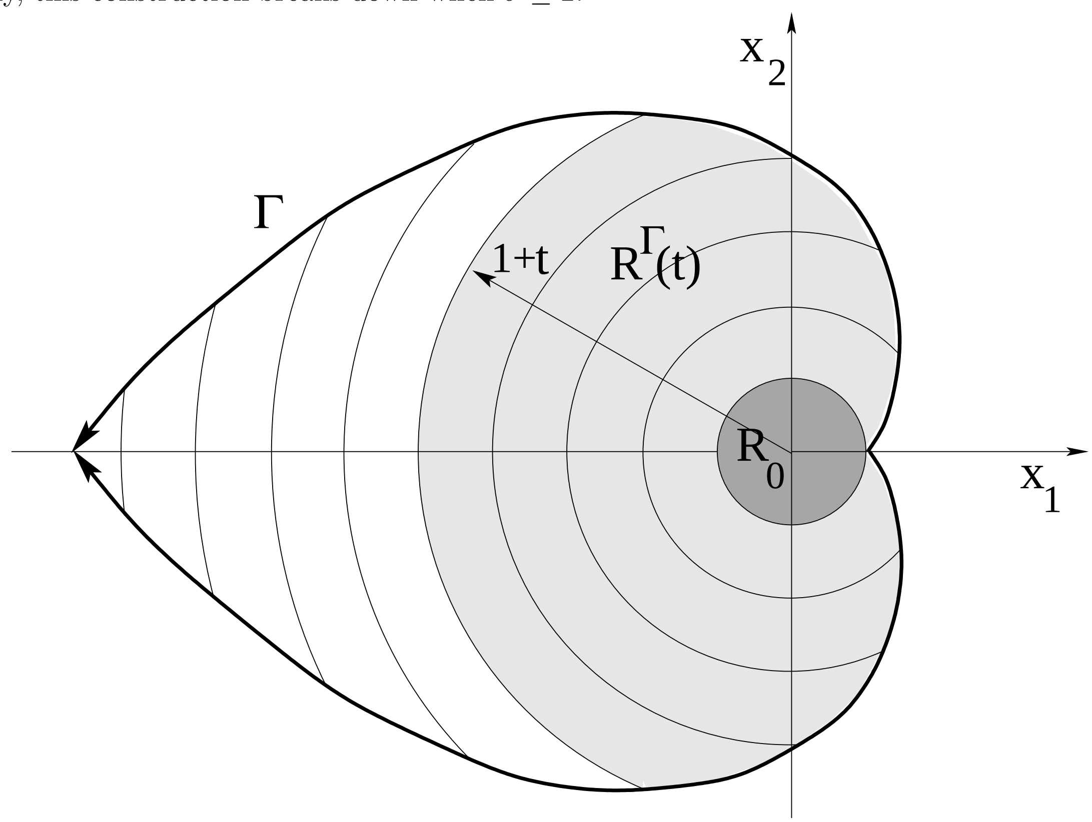

Figure 2. A blocking strategy consisting of two arcs of logarithmic spirals.

7. For $\sigma > 1$, an admissible strategy consists in constructing a single spiral-like curve along the edge of the fire (fig. 3). However, the analysis in [2] shows that this curve eventually closes into itself, encircling the fire, only if $\sigma > \sigma^\dagger = 2.614430844\ldots$
Therefore this strategy cannot be successful when $\sigma \leq 2$.

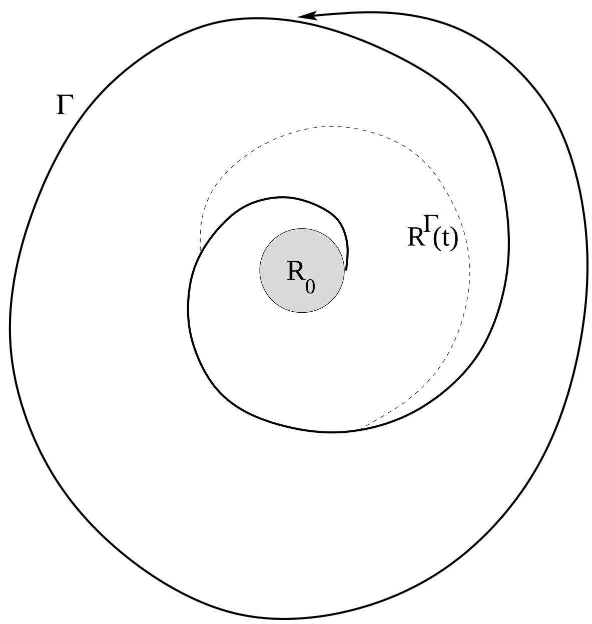

Figure 3. A blocking strategy consisting of one single spiral-like curve.

References

[1] A. Bressan, Differential inclusions and the control of forest fires, *J. Differential Equations* (special volume in honor of A. Cellina and J. Yorke), **243** (2007), 179-207.

[2] A. Bressan, M. Burago, A. Friend, and J. Jou, Blocking strategies for a fire control problem, *Analysis and Applications* **6** (2008), 229–246.

[3] A. Bressan and T. Wang, The minimum speed for a blocking problem on the half plane, *J. Math. Anal. Appl.*, **356** (2009), 133-144.

[4] A. Bressan and T. Wang, Equivalent formulation and numerical analysis of a fire confinement problem, *ESAIM; Control, Optimization and Calculus of Variations* **16** (2010), 974-1001.

Some related pictures and animations can be found on the web page:

http://www.math.psu.edu/wang_t/research.htm