Weighted Irrigation Plans

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August 16, 2021

Abstract

We model an irrigation network where lower branches must be thicker in order to support the weight of the higher ones. This leads to a countable family of ODEs, one for each branch, that must be solved by backward induction. Having introduced conditions that guarantee the existence and uniqueness of solutions, our main result establishes the lower semicontinuity of the corresponding cost functional, w.r.t. pointwise convergence of the irrigation plans. In turn, this yields the existence of an optimal irrigation plan, in the presence of these additional weights.

1 Introduction

In the classical irrigation problem with Gilbert cost [7], water is pumped out from a well and transported to finitely many locations $P_1, \ldots, P_n$ by a network of pipes. The total cost is computed by

$$\sum [\text{flux of water through the pipe}]^\alpha \times [\text{length of the pipe}]. \tag{1.1}$$

Here the sum ranges over all pipes in the network, while $\alpha \in [0,1]$ is a fixed exponent.

This model is appropriate for an irrigation network built at ground level. On the other hand, sometimes one would like to model a network as a free standing structure. For example, in [3] the authors considered tree branches transporting water and nutrients from the root to all the leaves. In this case, one should take into account that the lower portion of each branch bears the weight of the upper part. As a result, the thickness (and hence cost per unit length) of the lower portion should be greater, even if the flux remains the same. This is indeed observed in nature, where the thickness of tree branches decreases in a continuous fashion, as one moves toward the tip.

Aim of this paper is to develop a general framework to describe this situation. As a first step, consider a single branch with length ℓ, parameterized by arc-length $s \in [0, \ell]$, oriented from the root toward the tip. To account for the variable thickness of this branch we introduce a weight function $W = W(s)$. Assuming that the flux is constant along the entire branch, this will satisfy an ODE of the form

$$W'(s) \;=\; -\,f(W(s)), \tag{1.2}$$

where f is a non-negative, continuous function. A natural set of assumptions on f is

(A1) *The function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous on $[0, +\infty[$, twice continuously differentiable for $s > 0$, and satisfies*

$$f(0) = 0, \qquad f'(s) > 0, \qquad f''(s) \leq 0 \qquad \textit{for all} \;\; s > 0. \tag{1.3}$$

A typical example is $f(s) = cs^\beta$, for some $c \geq 0$ and $0 < \beta \leq 1$.

To illustrate the main idea, we describe how to construct a family of weights $W_i(\cdot)$ in a network consisting of finitely many branches γ_i, $i = 1, \ldots, N$. This will be achieved by induction, starting from the tip of each branch and proceeding backward toward the root.

Let each branch $\gamma_i : [0, \ell_i] \mapsto \mathbb{R}^d$ be parameterized by arc-length, oriented from the root toward the tip. As shown in Fig. 1, call $P_i = \gamma_i(\ell_i)$ the endpoint of the arc γ_i and consider a measure μ consisting of finitely many point masses $m_i \geq 0$ located at points P_i. It is assumed that, for each node P_i, there is a unique path (i.e., a concatenation of arcs γ_j) connecting P_i to the origin.

Call

$$\mathcal{O}(i) \;=\; \Big\{ j \in \{1, \ldots, N\}\,;\;\; \gamma_j(0) = P_i \Big\} \tag{1.4}$$

the set of branches originating from the node $P_i = \gamma_i(\ell_i)$, i.e., from the tip of the i-th branch. Moreover, consider the sets of indices inductively defined by

$$\begin{aligned} \mathcal{I}_1 \;&\doteq\; \Big\{ i \in \{1, \ldots, N\}\,;\;\; \mathcal{O}(i) = \emptyset \Big\}, \\ \mathcal{I}_{p+1} \;&\doteq\; \Big\{ i \in \{1, \ldots, N\}\,;\;\; \mathcal{O}(i) \subseteq \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p \Big\} \setminus (\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_p)\,. \end{aligned} \tag{1.5}$$

Roughly speaking, $\mathcal{I}_1$ is the set of outer-most branches. Branches in $\mathcal{I}_p$ originate from the tips of branches in $\mathcal{I}_{p+1}$, etc. Since the graph contains no loops, the set $\{1, \ldots, N\}$ is the disjoint union of the sets $\mathcal{I}_p$, $p \geq 1$.

For each branch $i \in \{1, \ldots, N\}$, a weight function $W_i(\cdot)$ can now be defined in terms of the following rules:

(i) The weight at the tip of the i-th branch is

$$W_i(\ell_i) \;=\; \overline{W}_i \;\doteq\; m_i + \sum_{j \in \mathcal{O}(i)} W_j(0+). \tag{1.6}$$

(ii) Along each branch γ_i, the weight $W_i(\cdot)$ is absolutely continuous and satisfies the ODE

$$W_i'(s) \;=\; -\,f(W_i(s)) \qquad\qquad s \in \,]0, \ell_i]\,. \tag{1.7}$$

According to (i)-(ii), the solution can be computed by induction on the entire tree, first on all branches $i \in \mathcal{I}_1$, then on all branches $i \in \mathcal{I}_2$, etc. For sake of definiteness, we assume

$$m_i > 0 \qquad \text{for all } i \in \mathcal{I}_1 . \tag{1.8}$$

This guarantees that the flux along every branch is strictly positive. In turn, by (1.3), it implies that the backward Cauchy problem (1.6)-(1.7) on $[0, \ell_i]$ has a unique solution.

Example 1.1 When $f(s) = cs^\beta$, the Cauchy problem (1.6)-(1.7) can be solved explicitly. Namely:

$$\begin{aligned} \overline{W}_i^{1-\beta} - W_i^{1-\beta}(s) &= c(1-\beta)(s-\ell_i), \\ W_i(s) &= \Big(\overline{W}_i^{1-\beta} + c(1-\beta)(\ell_i - s)\Big)^{\frac{1}{1-\beta}}. \end{aligned} \tag{1.9}$$

In particular, from (1.6) we deduce the inductive rule

$$\overline{W}_i = m_i + \sum_{j\in\mathcal{O}(i)} \Big(\overline{W}_j^{1-\beta} + c(1-\beta)\ell_j\Big)^{\frac{1}{1-\beta}}. \tag{1.10}$$

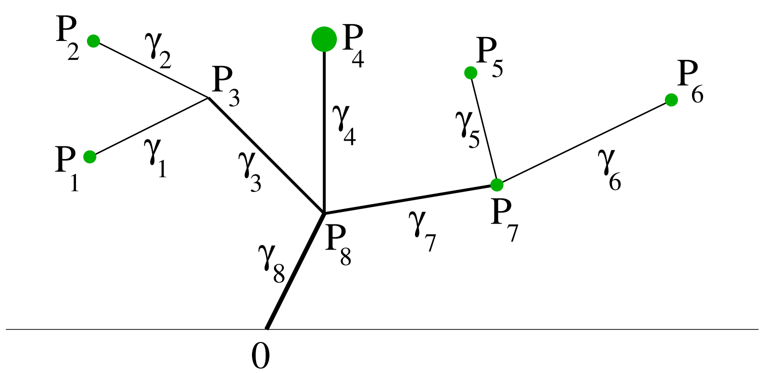

Figure 1: According to (1.5), the branches of this tree are partitioned according to $\mathcal{I}_1 = \{1,2,4,5,6\}$, $\mathcal{I}_2 = \{3,7\}$, $\mathcal{I}_3 = \{8\}$. Weights can be constructed by induction, solving the backward Cauchy problems (1.6)-(1.7) first along the arcs γ_i, $i \in \mathcal{I}_1$, then for $i \in \mathcal{I}_2$, etc.

In the presence of a weight function W, for a given $\alpha \in]0,1]$ the **total weighted cost of the irrigation network** is then defined as

$$E^{W,\alpha} \doteq \sum_i \int_0^{\ell_i} [W_i(s)]^\alpha \, ds. \tag{1.11}$$

Remark 1.2 In the case where $f \equiv 0$, the weight functions are constant along every branch. Moreover, the boundary conditions (1.6) imply

$$W_i(s) = \text{flux through } \gamma_i , \qquad\qquad s \in [0, \ell_i] .$$

Hence the total weighted cost (1.11) coincides with the Gilbert cost (1.1).

Example 1.3 In the special case where $\beta = \alpha$, so that $f(s) = cs^\alpha$, in view of (1.9) this cost is computed by

$$E^{W,\alpha} \doteq \sum_i \int_0^{\ell_i} \Big(\overline{W}_i^{1-\alpha} + c(1-\alpha)(\ell_i - s)\Big)^{\frac{\alpha}{1-\alpha}} ds. \tag{1.12}$$

When $\alpha = 1 - \alpha = 1/2$, the formulas (1.10) and (1.12) further simplify to

$$\overline{W}_i = m_i + \sum_{j\in\mathcal{O}(i)} \Big(\overline{W}_j^{1/2} + \frac{c\ell_j}{2}\Big)^2, \qquad E^{W,\alpha} \doteq \sum_i \Big(\ell_i \overline{W}_i^{1/2} + \frac{c\ell_i^2}{4}\Big).$$

Aim of the present paper is to extend the theory of optimal irrigation networks [1, 2, 3, 10, 19, 20], accounting for the presence of weights in the cost function. In essence, this requires the solution of a countable family of ODEs with measure-valued right hand sides, one for each branch of the network.

In the case of a finite network, where μ consists of finitely many atoms, our definition reduces to (1.11). For a general network, irrigating a positive Radon measure μ, the weighted cost will be defined as a limit of an increasing sequence of approximations. For any $\varepsilon > 0$, these approximations are obtained by restricting the transport plan to a finite family of paths where the flux is $\geq \varepsilon$.

Besides proving that this family of weights can be uniquely determined, our main results include the lower semicontinuity of the corresponding weighted irrigation cost. In turn, this yields the existence of optimal weighted irrigation plans.

The remainder of the paper is organized as follows. In Section 2 we review the basic definitions of irrigation plans, and give a statement of the main results. To simplify the discussion, we here assume that all paths are parameterized by arc-length, and that the irrigation plan has the single path property. Both of these assumptions will be removed in subsequent sections.

The more technical part of the paper begins with Section 3, which collects various results on optimal irrigation plans. For later use, we also include some lemmas on ODEs with measure-valued right hand side, formulated as integral equations. In Section 4 we work out a detailed construction of the weight functions, and define the total weighted cost of an irrigation plan. The lower semicontinuity of the weighted cost, w.r.t. pointwise convergence of the particle paths, is stated in Theorem 5.1 and proved in the remainder of Section 5.

In Section 6 we consider a more general model where the increase in the thickness of each branch, as one moves from the tip toward the root, depends also on the inclination of the branch itself. The ODE (1.2) is thus replaced by

$$W'(s) = -f(\dot\gamma(s), W(s)), \tag{1.13}$$

where $s \mapsto \gamma(s)$ is a parameterization of the branch. We here assume that f is continuous in both variables, and that the map $v \mapsto f(v, W)$ is positively homogeneous and convex w.r.t. the variable $v \in \mathbb{R}^d$. We show that all previous results, including the lower semicontinuity of the weighted irrigation cost, remain valid in this more general case.

Finally, in Section 7 we prove the existence of an optimal weighted irrigation plan for a given measure μ, and the lower semicontinuity of the weighted irrigation cost w.r.t. weak convergence of measures: $\mu_n \rightharpoonup \mu$. We remark that, as a further consequence of these results, the optimization problems for tree branches considered in [4, 5] still have solutions when the cost functional includes the presence of weights.

The problem of determining which measures have a finite or infinite weighted irrigation cost, depending on the dimension of their support, is discussed in the companion paper [18]. An interesting open question is whether, in the presence of weights, an optimal irrigation plan can be computed using a suitable Modica-Mortola approximation based on Γ-convergence, as in [12, 13, 14, 16]. For a general introduction to the theory of ramified transport we refer to [1].

2 Statement of the main results

In this section we review the definition of irrigation plans, we outline the construction of the weight functions W, and provide a statement of our main results. Somewhat more general theorems will be proved in the remaining sections.

Definition 2.1 *Let μ be a positive Radon measure on $\mathbb{R}^d$, with total mass $M \doteq \mu(\mathbb{R}^d)$. An* **irrigation plan** *for μ is a function*

$$\chi : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}^d,$$

measurable w.r.t. ξ and continuous w.r.t. t, with the following properties.

(i) **(regularity)** *For a.e. $\xi \in [0, M]$ the map $t \mapsto \chi(\xi, t)$ is 1-Lipschitz and eventually constant. Namely, there exists $\tau(\xi) \geq 0$ such that*

$$\begin{cases} |\chi(\xi,t) - \chi(\xi,t')| \;\leq\; |t - t'| & \text{for all } t, t' \geq 0, \\ \chi(\xi,t) \;=\; \chi(\xi,\tau(\xi)) & \text{for every } t \geq \tau(\xi). \end{cases} \tag{2.1}$$

Throughout the paper, we denote by $\tau(\xi)$ is the smallest time τ such that $\chi(\xi, \cdot)$ is constant for $t \geq \tau$.

(ii) **(χ irrigates the measure μ)** *For all $\xi \in [0, M]$ one has $\chi(\xi, 0) = \mathbf{0} \in \mathbb{R}^d$. Moreover, the push-forward of the Lebesgue measure on $[0, M]$ by the map $\xi \mapsto \chi(\xi, \tau(\xi))$ coincides with μ. In other words, for every Borel set $V \subseteq \mathbb{R}^d$ on has*

$$\mu(V) \;=\; meas\Big\{\xi \in [0, M]\,;\quad \chi(\xi, \tau(\xi)) \in V\Big\}. \tag{2.2}$$

One may think of the Lagrangian variable $\xi \in [0, M]$ as a label for a water particle. At time $t = 0$, all particles are at the origin. Each particle ξ moves along the path $t \mapsto \chi(\xi, t)$, until it reaches its eventual destination $\chi(\xi, \tau(\xi))$. To simplify some of the subsequent formulas, it is convenient to make the further assumption that each path is parameterized by arc-length. In this case, calling

$$\dot\chi(\xi,t) \;=\; \frac{\partial}{\partial t}\,\chi(\xi,t)$$

the partial derivative w.r.t. time, one has

$$|\dot\chi(\xi,t)| \;=\; \begin{cases} 1 & \text{for a.e. } \; t\in[0,\tau(\xi)],\\ 0 & \text{for } \; t>\tau(\xi). \end{cases} \tag{2.3}$$

We recall here another useful property of irrigation plans.

(SPP) *An irrigation plan χ satisfying (2.1)–(2.3) has the* **single path property** *if the following holds. If $\chi(\xi,\tau)=\chi(\xi',\tau')$ for some $\xi,\xi'\in[0,M]$ and $0<\tau\le\tau'$, then*

$$\chi(\xi,t) \;=\; \chi(\xi',t) \qquad \text{for all } t\in[0,\tau]. \tag{2.4}$$

In other words, if several particles go through the same point $x\in\mathbb{R}^d$, then they travel from the origin to x through the same path.

Given an irrigation plan χ, the amount of particles which go through a point $x\in\mathbb{R}^d$ is defined as

$$|x|_\chi \;\doteq\; \text{meas}\Big(\{\xi\in[0,M]\,;\;\; \chi(\xi,t)=x \quad \text{for some } \; t\ge 0\}\Big). \tag{2.5}$$

One may think of $|x|_\chi$ as the **total flux through the point** x. For a given particle $\xi\in[0,M]$, the **multiplicity** is then defined as

$$m(\xi,t) \;\doteq\; |\chi(\xi,t)|_\chi\,. \tag{2.6}$$

Assuming that χ has the single path property, it is clear that for each ξ the function $t\mapsto m(\xi,t)$ is non-increasing.

In the following, to ensure that the irrigation cost is finite, a further assumption will be needed.

(A2) *For a.e. $\xi\in[0,M]$, one has $m(\xi,t)>0$ for every $0\le t<\tau(\xi)$.*

In other words, for any particle ξ and any $t<\tau(\xi)$, there is a positive amount of other particles that travel along the same path $\chi(\xi,\cdot)$, up to time t.

Thinking that water particles are moving through a network of pipes, the flux $|x|_\chi$ measures how big should be the pipe going through the point x. Our present goal is to introduce a weight $W(x)$, which depends not only on the flux through x but also on all further branches of the network originating from x. If these branches are long, the weight $W(x)$ will be large.

For this purpose, we shall use a truncation procedure. Given $\varepsilon>0$, for each ξ we define the stopping time

$$\tau_\varepsilon(\xi) \;\doteq\; \sup\{t\in[0,\tau(\xi)]\,;\;\; m(\xi,t)\ge\varepsilon\}.$$

The truncated irrigation plan is then defined as

$$\chi_\varepsilon(\xi,t) \;=\; \begin{cases} \chi(\xi,t) & \text{if } t\le\tau_\varepsilon(\xi),\\ \chi(\xi,\tau_\varepsilon(\xi)) & \text{if } t>\tau_\varepsilon(\xi). \end{cases} \tag{2.7}$$

Since the total mass of all particles is bounded, the set of all truncated paths can now be written as the union of finitely many Lipschitz curves, say

$$\Big\{\chi(\xi,t)\,;\ \xi\in[0,M],\ t\in[0,\tau_\varepsilon(\xi)]\Big\} \;=\; \bigcup_{i=1}^{N}\gamma_i\,. \tag{2.8}$$

We parameterize each branch $\gamma_i : [0,\ell_i] \mapsto \mathbb{R}^d$ by arc-length, oriented from the root toward the tip. This finite set of branches is naturally endowed with a partial ordering. We write

$$\gamma_i \;\prec\; \gamma_j$$

if there exists a particle ξ and $t < t'$ such that

$$\chi(\xi,t)\in\gamma_i, \qquad \chi(\xi,t')\in\gamma_j\,.$$

As in (1.4), for each $i \in \{1,\ldots,N\}$ the set of branches originating from the tip of the branch γ_i is defined as

$$\mathcal{O}(i) \;=\; \{j\,;\ \gamma_j(0) \;=\; \gamma_i(\ell_i)\}.$$

We can now construct an approximate weight W^ε on every branch γ_i, starting from the uppermost branches (i.e., those for which $\mathcal{O}(i) = \emptyset$) and moving backwards toward the root, by an inductive procedure. More precisely, calling

$$m_i(s) \;=\; |\gamma_i(s)|_\chi\,, \qquad\qquad s\in[0,\ell_i],$$

the multiplicity along the branch γ_i, we define $W_i^\varepsilon(s) = W^\varepsilon(\gamma_i(s))$ to be the solution of the ODE

$$\frac{d}{ds}W_i^\varepsilon(s) \;=\; -f(W_i^\varepsilon(s)) + \frac{d}{ds}m_i(s) \qquad\qquad s\in[0,\ell_i], \tag{2.9}$$

with terminal data

$$W_i^\varepsilon(\ell_i) \;=\; \overline{W}_i \;\doteq\; \sum_{j\in\mathcal{O}(i)} W_j^\varepsilon(0+) + \text{meas}\Big\{\xi\in[0,M]\,;\ \ \chi(\xi,\tau_\varepsilon(\xi)) = \gamma_i(\ell_i)\Big\}. \tag{2.10}$$

To appreciate this construction, a few remarks are in order.

(i) The multiplicity function $s \mapsto m_i(s)$ along γ_i is non-increasing, possibly discontinuous. Hence the ODE (2.9) must be interpreted in distributional sense.

(ii) In the case where this multiplicity m_i is constant, the ODE (2.9) reduces to (1.7).

(iii) In the case where $f \equiv 0$, we wish to recover the identity $W_i^\varepsilon(s) = m_i(s)$. This motivates the boundary condition (2.10). Indeed, in this case, the total amount of particles that reach the terminal point $P_i = \gamma(\ell_i)$ is equal to the amount of particles that move into one of the outgoing branches γ_j with $j \in \mathcal{O}(i)$, plus the amount of particles that stop exactly at P_i.

(iv) In the general case where $f > 0$, the terminal condition (2.10) is an equivalent formulation of (1.6).

Our first main result can be stated as follows.

Theorem A. *Let f satisfy* **(A1)**. *Let μ be a positive, bounded measure on $\mathbb{R}^d$ and let χ be an irrigation plan for μ, satisfying the single path property as well as* **(A2)**.

(i) For any given $\varepsilon > 0$, the relations (2.9)-(2.10) uniquely define a set of weight functions W_i^ε along finitely many branches $\gamma_1, \ldots, \gamma_N$.

(ii) Letting $\varepsilon \downarrow 0$, for a.e. ξ one obtains a unique (possibly unbounded) limit

$$W(\xi,t) \;=\; \lim_{\varepsilon\downarrow 0} W^\varepsilon(\chi_\varepsilon(\xi,t)) \;=\; \sup_{\varepsilon>0} W^\varepsilon(\chi_\varepsilon(\xi,t)), \qquad\qquad 0<t<\tau(\xi). \tag{2.11}$$

When the particle paths are parameterized by arc-length as in (2.3), for a given $\alpha \in [0,1]$, the total cost of the irrigation plan χ is defined as

$$\mathcal{E}^\alpha(\chi) \;\doteq\; \int_{[0,M]} \left(\int_0^{\tau(\xi)} |\chi(\xi,t)|_\chi^{\alpha-1}\, dt \right) d\xi. \tag{2.12}$$

In connection with the weight function W in (2.11), we now introduce the **weighted cost**

$$\mathcal{E}^{W,\alpha}(\chi) \;\doteq\; \int_{[0,M]} \left(\int_0^{\tau(\xi)} \frac{[W(\xi,t)]^\alpha}{m(\xi,t)}\, dt \right) d\xi. \tag{2.13}$$

Remark 2.2 In the case where $f = 0$, and hence $W(\xi,t) = m(\xi,t) \doteq |\chi(\xi,t)|$, the definition (2.13) reduces to (2.12).

More generally, given a continuous function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying the same assumptions (1.3) as f in **(A1)**, we define the **weighted cost of the irrigation plan** χ by setting

$$\mathcal{E}^{W,\psi}(\chi) \;\doteq\; \int_{[0,M]} \left(\int_0^{\tau(\xi)} \frac{\psi(W(\xi,t))}{m(\xi,t)}\, dt \right) d\xi. \tag{2.14}$$

Finally, for a given measure μ, the **weighted irrigation cost** of μ is defined as

$$\mathcal{I}^{W,\psi}(\mu) \;\doteq\; \inf_\chi \mathcal{E}^{W,\psi}(\chi), \tag{2.15}$$

where the infimum is taken over all irrigation plans for the measure μ.

A major part of our analysis will be devoted to establishing the lower semicontinuity of the weighted cost, w.r.t. pointwise convergence of the particle paths. This yields our second main result, on the existence of optimal irrigation plans.

Theorem B. *Let μ be a positive, bounded Radon measure on $\mathbb{R}^d$. Let f and ψ be two continuous functions that both satisfy the assumptions in* **(A1)**. *If μ admits an irrigation plan whose weighted cost is finite, then there exists an irrigation plan with minimum weighted cost.*

In addition, we obtain the lower semicontinuity of the weighted irrigation cost w.r.t. weak convergence of the irrigated measures:

Theorem C. *Let f and ψ both satisfy* **(A1)**. *Let $(\mu_n)_{n\geq 1}$ be a sequence of bounded positive measures on $\mathbb{R}^d$, with uniformly bounded supports, weakly converging to μ. Then*

$$\mathcal{I}^{W,\psi}(\mu) \ \leq \ \liminf_{n\to\infty} \ \mathcal{I}^{W,\psi}(\mu_n). \tag{2.16}$$

It is well known [1] that optimal irrigation plans have the single path property. Moreover, every path can be re-parameterized by arc-length, as in (2.3). At first sight, it would thus seem natural to restrict all the analysis to irrigation plans that satisfy these two properties. Unfortunately, this approach runs into trouble as soon as we wish to establish limit theorems. Indeed, as shown in Fig. 2, both of these properties can fail when we take the limit of a sequence of irrigation plans.

Example 2.3 Referring to Fig. 2, left, assume that every path γ_n is parameterized by arc-length, so that $|\dot\gamma_n(s)| = 1$ for a.e. s. Moreover, assume the uniform convergence $\gamma_n(s) \to \gamma(s)$. Then the map $s \mapsto \gamma(s)$ is 1-Lipschitz. However, the identity $|\dot\gamma(s)| = 1$ fails.

Next, consider the sequence of irrigation plans χ_n shown in Fig. 2, right, where water particles move along two paths $\gamma_{n,1}, \gamma_{n,2}$. These two paths bifurcate after reaching the point P and never touch each other again. Therefore χ_n satisfies the single path property. However, taking the limit as $n \to \infty$ we obtain an irrigation plan χ containing two paths γ_1, γ_2 which bifurcate at P but join together again at Q. Particles can reach the same point Q traveling along two distinct paths. Hence χ does not satisfy the single path property.

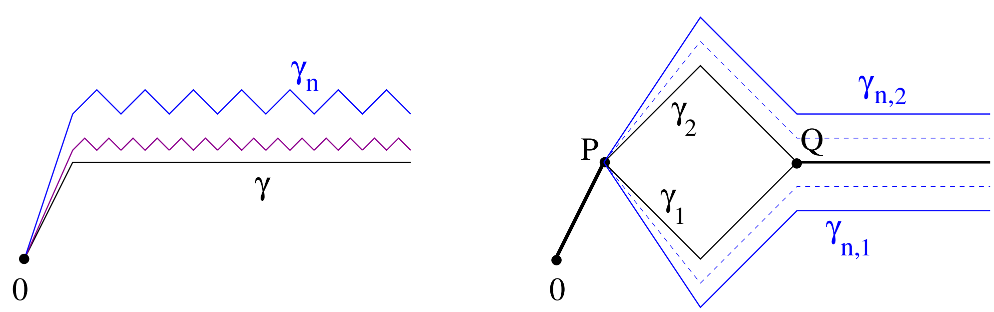

Figure 2: Left: a sequence of paths γ_n which converge uniformly to a path γ. Every γ_n is parameterized by arc-lenth, but the same is not true of the limit. Right: a sequence of irrigation plans χ_n converging to a limit plan χ. Here every χ_n has the single path property, but χ does not.

For the above reasons, in the next sections we shall introduce a more general construction of the weight functions $W = W(\xi, t)$, which applies to all irrigation plans considered in Definition 2.1, without assuming any further property (see Definition 4.3). Toward this goal, the formula (2.6) must be replaced by a somewhat different definition of multiplicity, introduced at (3.7). The difference between these two definitions is explained in Remark 3.7. In this more general setting, Theorem B and C are then restated as Theorem 7.1 and 7.2, and proved within Section 7.

3 Preliminaries

In this section collects various lemmas on irrigation plans and on ODEs with measure-valued right hand side, which will be used later on.

Throughout the following, we say that a map $\gamma(\cdot)$ is 1-Lipschitz if it is Lipschitz continuous with Lipschitz constant 1. We denote by Γ the set of all 1-Lipschitz maps $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}^d$. By Ascoli-Arzela's theorem (see Lemma 3.4 in [1]), this is a compact metric space with the distance

$$d(\gamma_1, \gamma_2) \doteq \sup_{k>1} \frac{1}{k} \Big\{ \max_{s\in[0,k]} |\gamma_1(s) - \gamma_2(s)| \Big\}. \tag{3.1}$$

Notice that (3.1) corresponds to the topology of uniform convergence on compact sets.

Let now $\chi : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}^d$ be an irrigation plan for the measure μ, as in Definition 2.1. Relying on a theorem of Scorza-Dragoni [9, 17], one can construct a partition of the interval $[0, M]$ into countably many disjoint subsets

$$[0, M] = \left(\bigcup_{j=1}^{\infty} K_j \right) \cup \mathcal{N}, \tag{3.2}$$

such that

- each K_j is compact,
- the set $\mathcal{N}$ has measure zero,
- the restriction of χ to each product set $K_j \times \mathbb{R}_+$ is continuous.

Thanks to the above construction, measurability issues can be more easily resolved. For example, we have

Lemma 3.1 *Let $K \subseteq [0, M]$ be a compact subset such that χ is continuous restricted to $K \times \mathbb{R}_+$. Then the map $\xi \mapsto \tau(\xi)$ is lower semicontinuous restricted to K.*

Proof. Indeed, consider a sequence $\xi_n \to \xi$ of points in K. If $\liminf \tau(\xi_n) = +\infty$ there is nothing to prove. Otherwise, by taking a subsequence, we can assume

$$\lim_{n\to\infty} \tau(\xi_n) = \bar{\tau}.$$

By assumption, the continuous functions $\chi(\xi_n, \cdot)$ converge to $\chi(\xi, \cdot)$ uniformly on compact sets. For any $\varepsilon > 0$, all but finitely many of these functions are constant on $[\bar{\tau} + \varepsilon, +\infty[$. Hence also $\chi(\xi, \cdot)$ is constant on this same domain. Since $\varepsilon > 0$ is arbitrary, we conclude that $\chi(\xi, \cdot)$ is constant on $[\bar{\tau}, +\infty[$. Hence $\tau(\xi) \leq \bar{\tau}$, as claimed. □

Corollary 3.2 *Given any $\varepsilon > 0$ there exists a compact set $K \subseteq [0, M]$, with*

$$meas\Big([0, M] \setminus K\Big) \;<\; \varepsilon, \tag{3.3}$$

and such that

(i) the map $\xi \mapsto \chi(\xi, \cdot)$ is continuous restricted to K,

(ii) the map $\xi \mapsto \tau(\xi)$ is continuous restricted to K.

Proof. In connection with the decomposition (3.2), we can choose $K = \cup_{j=1}^{\nu} K_j$ with ν large enough so that (3.3) holds. Since χ is continuous on each K_j, the statement (i) follows immediately. By Lemma 3.1 the map $\xi \mapsto \tau(\xi)$ is measurable on K. By Lusin's theorem, there exists a smaller compact set $K_0 \subseteq K$, still with $\text{meas}([0, M] \setminus K_0) \;<\; \varepsilon$, such that the restriction of $\tau(\cdot)$ to K_0 is continuous. By replacing K with K_0, the conclusion (ii) of the Corollary is satisfied. □

The usual definition of *irrigation cost* involves the multiplicity of a point $x = \chi(\xi, t)$, defined as

$$|x|_\chi \;\doteq\; \text{meas}\Big(\{\xi \in [0, M]\,;\;\; \chi(\xi, t) = x \;\;\text{for some } t \geq 0\}\Big). \tag{3.4}$$

In the present case, this must be replaced by a different concept, related to the single-path property.

Definition 3.3 *We say that two 1-Lipschitz maps $\gamma : [0, t] \mapsto \mathbb{R}^d$ and $\gamma' : [0, t'] \mapsto \mathbb{R}^d$ are* **equivalent** *if they are parameterizations of the same curve. That is, if there exists an interval $[0, T]$ and nondecreasing, Lipschitz continuous surjective maps $s \mapsto \eta(s)$ and $s \mapsto \eta'(s)$ from $[0, T]$ onto $[0, t]$ and $[0, t']$ respectively, such that*

$$\gamma(\eta(s)) \;=\; \gamma'(\eta'(s)) \qquad\qquad \textit{for all } s \in [0, T]. \tag{3.5}$$

If this is the case, we write $\gamma \simeq \gamma'$.

Remark 3.4 Given a 1-Lipschitz map $\gamma : [0, t] \mapsto \mathbb{R}^d$, its arc-length re-parameterization is the map

$$\sigma \;\mapsto\; \gamma(s(\sigma))$$

where, for every σ, one has

$$\int_0^{s(\sigma)} |\dot{\gamma}(\zeta)|\, d\zeta \;=\; \sigma.$$

According to the above definition, two maps $\gamma : [0, t] \mapsto \mathbb{R}^d$ and $\gamma' : [0, t'] \mapsto \mathbb{R}^d$ are equivalent if and only if their arc-length re-parameterizations coincide.

Remark 3.5 In Definition 3.3, one can always take $T = t + t'$ and assume that both functions η, η' are 1-Lipschitz. Indeed, let η, η' be maps from $[0, T]$ onto $[0, t]$ and $[0, t']$ respectively, such that (3.5) holds.

For $s \in [0,T]$, define the nondecreasing, surjective map $\sigma : [0,T] \mapsto [0,\widetilde{T}] \doteq [0, t+t']$ by setting

$$\sigma(s) \doteq \eta(s) + \eta'(s).$$

We then define the maps $\tilde{\eta}, \tilde{\eta}'$ from $[0,\widetilde{T}]$ into $[0,t]$ and $[0,t']$ implicitly, by setting

$$\tilde{\eta}(\sigma(s)) \doteq \eta(s), \qquad \tilde{\eta}'(\sigma(s)) \doteq \eta'(s) \qquad s \in [0,T]. \tag{3.6}$$

We claim that $\tilde{\eta}$ and $\tilde{\eta}'$ are 1-Lipschitz. Indeed, let $\sigma_1 = \sigma(s_1) < \sigma(s_2) = \sigma_2$. Then

$$[\tilde{\eta}(\sigma_2) - \tilde{\eta}(\sigma_1)] + [\tilde{\eta}'(\sigma_2) - \tilde{\eta}'(\sigma_1)] = [\eta(s_2) + \eta'(s_2)] - [\eta(s_1) + \eta'(s_1)] = \sigma_2 - \sigma_1 .$$

The identity $\gamma(\tilde{\eta}(s)) = \gamma'(\tilde{\eta}'(s))$ now follows from (3.5) and (3.6).

Throughout the following, we denote by $\gamma\Big|_{[0,t]}$ the restriction of a map γ to the interval $[0,t]$.

Definition 3.6 *Let $\chi : [0,M] \times \mathbb{R}_+ \mapsto \mathbb{R}^d$ be an irrigation plan for the measure μ. We define an equivalence relation on the set $[0,M] \times \mathbb{R}_+$ by setting $(\xi,t) \sim (\xi',t')$ whenever $\chi(\xi,\cdot)\Big|_{[0,t]} \simeq \chi(\xi',\cdot)\Big|_{[0,t']}$. This means that the maps*

$$s \mapsto \chi(\xi,s), \quad s \in [0,t] \qquad \textit{and} \qquad s \mapsto \chi(\xi',s), \quad s \in [0,t']$$

are equivalent in the sense of Definition 3.3.

The **multiplicity** *of (ξ,t) is then defined as*

$$m(\xi,t) \doteq meas\Big(\{\xi' \in [0,M]\,;\ \ (\xi',t') \sim (\xi,t) \ \ \textit{for some } t' > 0\}\Big). \tag{3.7}$$

Remark 3.7 The multiplicity $m(\xi,t)$ measures the total amount of particles that pass through the point $x = \chi(\xi,t)$ *traveling along exactly the same path as the particle* ξ. If χ has the single path property (see Chapter 7 in [1]), then $m(\xi,t) = |\chi(\xi,t)|_\chi$. However, for a general irrigation plan we only have the inequality

$$m(\xi,t) \le |\chi(\xi,t)|_\chi . \tag{3.8}$$

Notice that one may well have

$$\chi(\xi,t) = \chi(\xi',t') \qquad \text{but} \qquad m(\xi,t) \neq m(\xi',t').$$

The next two lemmas establish various properties of the multiplicity function introduced in Definition 3.6

Lemma 3.8 *For any $\varepsilon > 0$ there exists a compact set $K \subseteq [0,M]$ satisfying (3.3), such that the set-valued function*

$$F(\xi,t) \doteq \{\xi' \in K\,;\ \ (\xi',t') \sim (\xi,t) \ \ \textit{for some} \ \ t' \ge 0\} \tag{3.9}$$

is upper semicontinuous on $K \times \mathbb{R}_+$.

Proof. 1. Given $\varepsilon > 0$, let $K \subseteq [0, M]$ be the compact set constructed in Corollary 3.2. We claim that the graph of F, restricted to $K \times \mathbb{R}_+$, is closed. In other words, assume that

$$\xi_n \to \xi \qquad t_n \to t, \qquad \xi'_n \to \xi' \qquad \text{as} \quad n \to \infty,$$

and moreover

$$(\xi'_n, t'_n) \sim (\xi_n, t_n) \qquad \text{for all } n \geq 1.$$

We need to show that there exists $t' \geq 0$ such that $(\xi, t) \sim (\xi', t')$.

2. By the assumptions, according to Remark 3.5 for every $n \geq 1$ there exists an interval $[0, T_n] = [0, t_n + t'_n]$ and two nondecreasing, 1-Lipschitz, surjective maps

$$\eta_n : [0, T_n] \mapsto [0, t_n], \qquad \eta'_n : [0, T_n] \mapsto [0, t'_n],$$

such that

$$\chi(\xi_n, \eta_n(s)) = \chi(\xi'_n, \eta'_n(s)) \qquad \text{for all } s \in [0, T_n]. \tag{3.10}$$

3. We now observe that, since the map $\xi \mapsto \tau(\xi)$ is continuous on the compact set K, it is uniformly bounded. We can thus assume that the sequence $(t'_n)_{n\geq 1}$ is bounded. Since $t_n \to t < +\infty$ and $T_n = t_n + t'_n$, we have the uniform boundedness of $(T_n)_{n\geq 1}$. By extracting a subsequence, one can assume

$$\lim_{n\to+\infty} t'_n = t', \qquad \lim_{n\to+\infty} T_n = T. \tag{3.11}$$

If $T_n < T$, we extend the maps η_n and η'_n to the interval $[0, T]$ by setting $\eta_n(s) \doteq \eta_n(T_n)$, $\eta'_n(s) \doteq \eta'_n(T_n)$, for all $s \in (T_n, T]$. Using the Ascoli-Arzelà theorem, by possibly extracting a subsequence we achieve the uniform convergence

$$\eta_n(\cdot) \to \eta(\cdot), \qquad \eta'_n(\cdot) \to \eta'(\cdot), \qquad \text{uniformly on } [0, T]. \tag{3.12}$$

Here η and η' are two nondecreasing, 1-Lipschitz, surjective maps from $[0, T]$ onto $[0, t]$ and $[0, t']$ respectively. By the continuity of $\chi(\cdot, \cdot)$ on $K \times \mathbb{R}_+$, from (3.10) we obtain

$$\chi(\xi, \eta(s)) = \lim_{n\to\infty} \chi(\xi_n, \eta_n(s)) = \lim_{n\to\infty} \chi(\xi'_n, \eta'_n(s)) = \chi(\xi', \eta'(s)) \qquad \text{for all } s \in [0, T]. \tag{3.13}$$

Therefore, $(\xi', t') \sim (\xi, t)$. □

Lemma 3.9 *Let $\chi : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}^d$ be an irrigation plan for the measure μ. Then the following holds.*

(i) The map $(\xi, t) \mapsto m(\xi, t)$ is measurable.

(ii) For each $\xi \in [0, M]$, the map $t \mapsto m(\xi, t)$ is non-increasing and left continuous.

(iii) For any fixed $\varepsilon > 0$, the stopping time

$$\tau_\varepsilon(\xi) \doteq \max \Big\{ t \in [0, \tau(\xi)] \,; \quad m(\xi, t) \geq \varepsilon \Big\} \tag{3.14}$$

is a measurable function of $\xi \in [0, M]$.

Proof. **1.** Given $\varepsilon > 0$, let $K \subseteq [0, M]$ be a compact set satisfying the conditions in Lemma 3.8. In terms of the multifunction $(\xi, t) \mapsto F(\xi, t) \subseteq K$ defined at (3.9), this implies the scalar function

$$(\xi, t) \;\mapsto\; \text{meas}(F(\xi, t)) \tag{3.15}$$

is upper semicontinuous restricted to $K \times \mathbb{R}_+$. For $(\xi, t) \in K \times \mathbb{R}_+$ this implies

$$m(\xi, t) - \varepsilon \;\leq\; \text{meas}(F(\xi, t)) \;\leq\; m(\xi, t). \tag{3.16}$$

2. Repeating the above construction for decreasing values of ε, we can find an increasing sequence of compact sets $(K_n)_{n\geq 1}$, with $\text{meas}([0, M] \setminus K_n) < 1/n$, such that

$$m(\xi, t) - \frac{1}{n} \;\leq\; \text{meas}(F_n(\xi, t)) \;\leq\; m(\xi, t)\,. \tag{3.17}$$

Here F_n is the multifunction defined at (3.9), with K replaced by K_n. Notice that the function $(\xi, t) \;\mapsto\; \text{meas}(F_n(\xi, t))$ is upper semicontinuous restricted to $K_n \times \mathbb{R}_+$. Setting

$$m_n(\xi, t) \;\doteq\; \begin{cases} \text{meas}(F_n(\xi, t)) & \text{if } \xi \in K_n\,, \\ \quad 0 & \text{if } \xi \notin K_n\,, \end{cases}$$

by (3.17) we have the pointwise convergence $m_n(\xi, t) \to m(\xi, t)$ for a.e. $\xi \in [0, M]$. Since each m_n is measurable, the same holds for m. This proves (i).

3. By the definition of the multiplicity function in (3.7), it immediately follows that the map $t \mapsto m(\xi, t)$ is non-increasing. To prove its left continuity, fix $(\xi, t) \in [0, M] \times \mathbb{R}_+$ and consider an increasing sequence $t_n \uparrow t$. By monotonicity, it follows

$$\lim_{n\to\infty} m(\xi, t_n) \;=\; \inf_n \, m(\xi, t_n) \;\geq\; m(\xi, t)\,. \tag{3.18}$$

To prove that equality actually holds in (3.18), given any $\varepsilon > 0$, let $K \subseteq [0, M]$ be a compact set satisfying the conditions in Lemma 3.8. By the upper semicontinuity of the multifunction $t \mapsto F(\xi, t)$ one has

$$m(\xi, t) \;\geq\; \text{meas}(F(\xi, t)) \;\geq\; \limsup_{n\to\infty} \; \text{meas}(F(\xi, t_n)) \;\geq\; \limsup_{n\to\infty} \; m(\xi, t_n) - \varepsilon\,. \tag{3.19}$$

Since $\varepsilon > 0$ was arbitrary, this proves statement (ii) of the lemma.

4. To prove (iii), we first observe that, by the arguments in the previous steps **1 - 2**, for each fixed $t > 0$ the map $\xi \mapsto m(\xi, t)$ is measurable. Moreover, by Corollary 3.2 it follows that $\xi \mapsto \tau(\xi)$ is measurable. For every $t > 0$ we have the identity

$$\Big\{\xi \in [0, M]\,;\; \tau_\varepsilon(\xi) \geq t\Big\} \;=\; \Big\{\xi \in [0, M]\,;\; m(\xi, t) \geq \varepsilon\Big\} \bigcap \Big\{\xi \in [0, M]\,;\; \tau(\xi) \geq t\Big\}. \tag{3.20}$$

This implies that the map $\tau_\varepsilon(\cdot)$ is measurable. □

3.1 ODE's with measure-valued right hand side.

For future use, we now prove some results on existence and continuous dependence, for Carathéodory solutions to an ODE backward in time. Since in our equations the right hand side can possibly be a measure, it will be convenient to study directly the corresponding integral equations.

Lemma 3.10 *Let $f : [\varepsilon, +\infty[\mapsto \mathbb{R}_+$ be Lipschitz continuous. For $t \in [0,T]$, let $t \mapsto m(t)$ be a non-increasing function with $m(T) \geq \varepsilon$.*

(i) There exists a unique function $w : [0,T] \mapsto [\varepsilon, +\infty[$ which satisfies the integral equation

$$w(t) \;=\; \int_t^T f(w(s))\,ds + m(t) \qquad \text{for all } t \in [0,T]. \tag{3.21}$$

(ii) If $m_1(t) \leq m_2(t)$ for all $t \in [0,T]$, then the corresponding solutions of (3.21) satisfy

$$w_1(t) \;\leq\; w_2(t) \qquad \text{for all } t \in [0,T]. \tag{3.22}$$

(iii) Consider a sequence of measurable sets $J_n \subseteq [0,T]$ such that $\lim_{n\to\infty} meas(J_n) = 0$, and define the functions

$$f_n(t,\omega) \;\doteq\; \begin{cases} f(\omega) & \text{if } t \notin J_n\,, \\ 0 & \text{if } t \in J_n\,. \end{cases}$$

Let $t \mapsto m_n(t) \in [\varepsilon, +\infty[$ be a sequence of non-increasing functions such that, as $n \to \infty$,

$$\|m_n - m\|_{\mathbf{L}^1([0,T])} \to 0. \qquad m_n(0) \to m(0+). \tag{3.23}$$

Then the solutions to

$$w_n(t) \;=\; \int_t^T f_n(s, \omega_n(s))\,ds + m_n(t) \qquad \text{for all } t \in [0,T]. \tag{3.24}$$

satisfy

$$\|w_n - w\|_{\mathbf{L}^1([0,T])} \to 0, \qquad w_n(0) \to w(0+). \tag{3.25}$$

Proof. 1. Consider the function

$$F(t,z) \;\doteq\; f(m(t) + z). \tag{3.26}$$

We observe that a map $t \mapsto w(t)$ satisfies the integral equation (3.21) if and only if $z(t) = w(t) - m(t)$ provides a Carathéodory solution to the backward Cauchy problem

$$\dot z(t) \;=\; -\,F(t, z(t)), \qquad z(T) = 0. \tag{3.27}$$

Observing that F is measurable in t and uniformly Lipschitz continuous in z, by the standard theory of ODE [8] we conclude that (3.27) has a unique solution $t \mapsto z(t)$. In turn, $w(t) = z(t) + m(t)$ provides the unique solution to (3.21).

2. To prove (ii), for $i = 1, 2$ let z_i be a solution to

$$-\dot{z}_i(t) \;=\; F_i(t, z_i(t)) \;\doteq\; f(m_i(t) + z_i(t)), \qquad\qquad z_i(T) = 0.$$

Since $F_1(t, z) \leq F_2(t, z)$ for all t, z, and both F_1, F_2 are Lipschitz continuous w.r.t. z, a standard comparison argument yields $z_1(t) \leq z_2(t)$ for all $t \in [0, T]$. In turn this implies

$$w_1(t) \;=\; m_1(t) + z_1(t) \;\leq\; m_2(t) + z_2(t) \;=\; w_2(t).$$

3. To prove (iii), set $F_n(t, z) \doteq f_n(t, m(t) + z)$ and let z_n be the solution to

$$\dot{z}_n(t) \;=\; -\,F_n(t, z_n(t)), \qquad\qquad z_n(T) = 0. \tag{3.28}$$

Then the difference $\eta_n(t) \doteq |z_n(t) - z(t)|$ satisfies

$$\begin{aligned}
\eta_n(t) \;&\leq\; \int_t^T \Big| f_n(s, m_n(s) + z_n(s)) - f(m(s) + z(s)) \Big|\, ds \\
&\leq\; \int_t^T \Big| f(m_n(s) + z_n(s)) - f(m(s) + z(s)) \Big| + \chi_{J_n}(s) \cdot \Big| f(m(s) + z(s)) \Big|\, ds \\
&\leq\; \int_t^T L\eta_n(s) + L|m_n(s) - m(s)| + \chi_{J_n}(s) \cdot |f(m(s) + z(s)|\, ds\,.
\end{aligned}$$

Here L is a Lipschitz constant for the function f on $[\varepsilon, +\infty[$, while χ_{J_n} denotes the characteristic function of the set J_n. By Gronwall's inequality one obtains

$$\eta_n(t) \;\leq\; \int_t^T e^{L(s-t)} \Big[L|m_n(s) - m(s)| + \chi_{J_n}(s) \cdot |f(m(s) + z(s))| \Big]\, ds\,. \tag{3.29}$$

Since multiplicity functions are non-increasing, there exists some finite constant $K > 0$ such that

$$|f(m(s) + z(s))| \;\leq\; K\,, \qquad \text{for all } n \geq 1,\;\; s \in [0,\, T]\,. \tag{3.30}$$

Letting $n \to \infty$, by (3.23) and (3.29)-(3.30), since $\lim_{n\to\infty} \mathrm{meas}(J_n) = 0$, we thus have the convergence $\eta_n(t) = |z_n(t) - z(t)| \to 0$ uniformly for $t \in [0, T]$. Recalling that $w_n = z_n + m_n$ and $w = z + m$, from (3.23) it now follows (3.25). □

Lemma 3.11 *Let $m, m_1, \ldots, m_q : [0, \ell] \mapsto [\varepsilon_0, +\infty[$ be non-increasing functions such that*

$$\sum_{i=1}^{q} m_i(s) \;\geq\; m(s), \qquad\qquad \textit{for all } s \in [0, \ell]. \tag{3.31}$$

Assume that f satisfies **(A1)** *and let $w, w_i : [0, \ell] \mapsto [\varepsilon_0, +\infty[$, be solutions to*

$$w(s) \;=\; \int_s^\ell f(w(t))\, dt + m(s), \qquad w_i(s) \;=\; \int_s^\ell f(w_i(t))\, dt + m_i(s), \tag{3.32}$$

respectively. Then, for all $s \in [0, \ell]$, one has

$$\sum_{i=1}^{q} w_i(s) \;\geq\; w(s). \tag{3.33}$$

Proof. Consider the functions

$$\widetilde{w}(s) \doteq \sum_{i=1}^{q} w_i(s), \qquad \widetilde{m}(s) \doteq \sum_{i=1}^{q} m_i(s).$$

Using the properties (1.3) of the function f and the inequality (3.31), for all $s \in [0, \ell]$,

$$\widetilde{w}(s) = \int_s^\ell \sum_i f(w_i(t))\,dt + \sum_i m_i(s) \geq \int_s^\ell f(\widetilde{w}(t))\,dt + \widetilde{m}(s).$$

Since $\widetilde{m}(s) \geq m(s)$, the comparison property stated in (iii) of Lemma 3.10 now implies $\widetilde{w}(s) \geq w(s)$ for all $s \in [0, \ell]$. □

4 Construction of the weight functions

Given an irrigation plan $\chi : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}^d$ and a function f satisfying **(A1)**, in this section we construct the weight function $W = W(\xi, t)$, by taking the supremum of a family of approximations W^ε.

Recalling the equivalence relation introduced in Definition 3.3, we introduce

Definition 4.1 *Given an irrigation plan χ, we say that a path $\gamma : [0, \ell] \mapsto \mathbb{R}^d$, parameterized by arc-length, is ε-**good** if*

$$meas\left(\left\{\xi \in [0, M]\,;\ \ \chi(\xi, \cdot)\Big|_{[0,t]} \simeq \gamma \ \ \textit{for some } t = t(\xi) > 0\right\}\right) \geq \varepsilon. \tag{4.1}$$

The family of all ε-good paths will be denoted by $\mathcal{G}_\varepsilon$.

In other words, γ is ε-good if there is an amount $\geq \varepsilon$ of particles whose trajectory contains γ as its initial portion. A somewhat similar definition can be found in [15].

The family of all curves parameterized by arc-length comes with a natural partial order. Namely, given two maps $\gamma : [0, \ell] \mapsto \mathbb{R}^d$, $\gamma' : [0, \ell'] \mapsto \mathbb{R}^d$, we write $\gamma \preceq \gamma'$ if $\ell \leq \ell'$ and $\gamma'(s) = \gamma(s)$ for all $s \in [0, \ell]$. The next lemma yields a bound on the number of maximal curves, within the family of ε-good paths.

Lemma 4.2 *Given an irrigation plan $\chi : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}^d$ and $\varepsilon > 0$, there can be at most M/ε distinct maximal ε-good paths.*

Proof. Let $\gamma_1, \ldots, \gamma_\nu$ be distinct maximal ε-good paths. For each $i \in \{1, \ldots, \nu\}$, consider the set

$$A_i \doteq \left\{\xi \in [0, M]\,;\ \ \chi(\xi, \cdot)\Big|_{[0,t]} \simeq \gamma_i \ \ \text{for some } t > 0\right\}. \tag{4.2}$$

We claim that all sets A_i are disjoint. Indeed, if $\xi \in A_i \cap A_j$, then

$$\chi(\xi, \cdot)\Big|_{[0,t]} \simeq \gamma_i\,, \qquad \chi(\xi, \cdot)\Big|_{[0,t']} \simeq \gamma_j\,.$$

To fix the ideas, assume $t \leq t'$. Then $\gamma_i \prec \gamma_j$, against the maximality of γ_i. This contradiction proves our claim. In turn this implies $\nu \leq M/\varepsilon$, proving the lemma. □

We now fix $\varepsilon > 0$, and let $\{\widehat\gamma_1, \ldots, \widehat\gamma_\nu\}$ be the set of all maximal ε-good paths for the irrigation plan χ. Along each path $\widehat\gamma_i : [0, \hat\ell_i] \mapsto \mathbb{R}^d$ we define the **multiplicity** $\widehat m_i : [0, \hat\ell_i] \mapsto \mathbb{R}_+$ by setting

$$\widehat m_i(t) \doteq \text{meas}\left(\left\{ \xi \in [0, M] \,;\quad \text{there exists } t' \geq 0 \text{ such that } \chi(\xi, \cdot)\Big|_{[0,t']} \simeq \widehat\gamma_i\Big|_{[0,t]} \right\} \right). \tag{4.3}$$

Otherwise stated, $\widehat m_i(t)$ is the amount of particles that travel along the path $\widehat\gamma_i$, at least up to the point $\widehat\gamma_i(t)$.

To construct the weight functions, we first need to split the maximal paths $\widehat\gamma_i$ into elementary paths γ_k, to which an inductive procedure as in (1.6)-(1.7) can then be applied. With this goal in mind, we define the bifurcation times

$$\tau_{ij} = \tau_{ji} \doteq \max\Big\{ t \geq 0 \,;\quad \widehat\gamma_i(s) = \widehat\gamma_j(s) \text{ for all } s \in [0, t] \Big\}. \tag{4.4}$$

The elementary paths $\gamma_k : [a_k, b_k] \mapsto \mathbb{R}^d$ and the corresponding multiplicity functions m_k are constructed by the following Path Splitting Algorithm.

(PSA) For each $i \in \{1, \ldots, \nu\}$, consider the set

$$\{\tau_{i1}, \ldots, \tau_{i\nu}\} = \{t_{i,1}, \ldots, t_{i,N(i)}\},$$

where the times

$$0 < t_{i,1} < t_{i,2} < \cdots < t_{i,N(i)} = \hat\ell_j \tag{4.5}$$

provide an increasing arrangement of the set of times τ_{ij} where the path $\widehat\gamma_i$ splits apart from other maximal paths. For each $k = 1, \ldots, N(i)$, let $\gamma_{i,k}$ be the restriction of the maximal path $\widehat\gamma_i$ to the subinterval $[t_{i,k-1}, t_{i,k}]$. The multiplicity function $m_{i,k}$ along this path is defined simply as

$$m_{i,k}(t) = \widehat m_i(t) \qquad\qquad t \in [t_{i,k-1}, t_{i,k}]. \tag{4.6}$$

If $\tau_{ij} > 0$, i.e. if the two maximal paths $\widehat\gamma_i$ and $\widehat\gamma_j$ partially overlap, it is clear that some of the elementary paths $\gamma_{i,k}$ will coincide with some $\gamma_{j,l}$. To avoid listing multiple times the same path, we thus remove from our list all paths $\gamma_{j,l} : [t_{j,l-1}, t_{j,l}] \mapsto \mathbb{R}^d$ such that $t_{j,l} \leq \tau_{ij}$ for some $i < j$. After relabeling all the remaining paths, the algorithm yields a family of elementary paths and corresponding multiplicities

$$\gamma_i : [a_i, b_i] \mapsto \mathbb{R}^d, \qquad m_i : [a_i, b_i] \mapsto \mathbb{R}_+\,, \qquad i = 1, \ldots, N. \tag{4.7}$$

For example, the tree shown in Fig. 1 contains 5 maximal paths $\widehat\gamma_1, \ldots, \widehat\gamma_5$. These can be decomposed it into 8 elementary paths $\gamma_1, \ldots, \gamma_8$. Each maximal path is a concatenation of elementary paths, namely

$$\widehat\gamma_1 = \gamma_8 \circ \gamma_3 \circ \gamma_1\,, \qquad \widehat\gamma_2 = \gamma_8 \circ \gamma_3 \circ \gamma_2\,, \qquad \widehat\gamma_3 = \gamma_8 \circ \gamma_4\,, \quad \ldots$$

A set of weight functions W_i on the elementary branches γ_i can now be constructed by a backward inductive procedure, similar to (1.6)-(1.7). As in (1.4), call $\mathcal{O}(i)$ the set of branches originating from the node $P_i = \gamma_i(b_i)$. Moreover, consider the sets of indices $\mathcal{I}_p$ inductively defined at (1.5).

(i) For $p = 1$, on each elementary path $\gamma_i : [a_i, b_i] \mapsto \mathbb{R}^d$ with $i \in \mathcal{I}_1$, the weight $W_i^\varepsilon(t)$ is defined to be the solution of

$$w(t) = \int_t^{b_i} f(w(s))\, ds + m_i(t), \qquad t \in]a_i, b_i]. \tag{4.8}$$

(ii) Next, assume that the weight functions $W_k^\varepsilon(t)$ have already been constructed along all paths $\gamma_k : [a_k, b_k] \mapsto \mathbb{R}^d$ with $k \in \mathcal{I}_{p-1}$.

For $i \in \mathcal{I}_p$, the weight $W_i^\varepsilon(t)$ along the i-th branch is then defined to be the solution of

$$w(t) = \int_t^{b_i} f(w(s))\, ds + m_i(t) + \overline{w}_i\,, \qquad t \in]a_i, b_i]. \tag{4.9}$$

where

$$\overline{w}_i \doteq \sum_{k \in \mathcal{O}(i)} W_k^\varepsilon(a_k+) - \sum_{k \in \mathcal{O}(i)} m_k(a_k+). \tag{4.10}$$

Notice that **(PSA)** implies $b_i = a_k$ for all $k \in \mathcal{O}(i)$. At the end-point $\gamma_i(b_i)$, the weight is

$$W_i^\varepsilon(b_i) = \sum_{k \in \mathcal{O}(i)} W_k^\varepsilon(a_k+) + \left[m_i(b_i) - \sum_{k \in \mathcal{O}(i)} m_k(a_k+) \right].$$

Here the term between brackets can be strictly positive. For example, this will happen if the irrigated measure μ contains a point mass at $\gamma_i(b_i)$.

By induction on p, after finitely many steps we obtain a weight function $W_i^\varepsilon : [a_i, b_i] \mapsto [\varepsilon, +\infty[$ defined on each elementary path γ_i.

Going back to the maximal paths $\widehat{\gamma}_j$ considered in **(PSA)**, the above construction yields a weight $\widehat{W}_{j,k}$ on the restriction of $\widehat{\gamma}_j$ to each subinterval $[t_{j,k-1}, t_{j,k}]$. Along the maximal path $\widehat{\gamma}_j$, the weight $\widehat{W}_j : [0, \hat{\ell}_j] \mapsto \mathbb{R}_+$ is then defined simply by setting

$$\widehat{W}_j(t) = \widehat{W}_{j,k}(t) \qquad \text{if} \quad t \in [t_{j,k-1}, t_{j,k}]. \tag{4.11}$$

Next, in order to construct an approximate weight function $W^\varepsilon : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ on the family of all paths $\chi(\xi, \cdot)$ of the irrigation plan, we consider the **stopping time**

$$\tau_\varepsilon(\xi) = \sup\{t \geq 0\,;\ \ m(\xi, t) \geq \varepsilon\}. \tag{4.12}$$

We then define the corresponding weight function

$$W^\varepsilon(\xi, t) \doteq \begin{cases} \widehat{W}_i(s) & \text{if} \quad t \leq \tau_\varepsilon(\xi), \qquad \chi(\xi, \cdot)\Big|_{[0,t]} \simeq \widehat{\gamma}_i\Big|_{[0,s]}, \\ 0 & \text{if} \quad t > \tau_\varepsilon(\xi). \end{cases} \tag{4.13}$$

Having constructed these approximate weights W^ε, the weight function W is then obtained by letting $\varepsilon \to 0$.

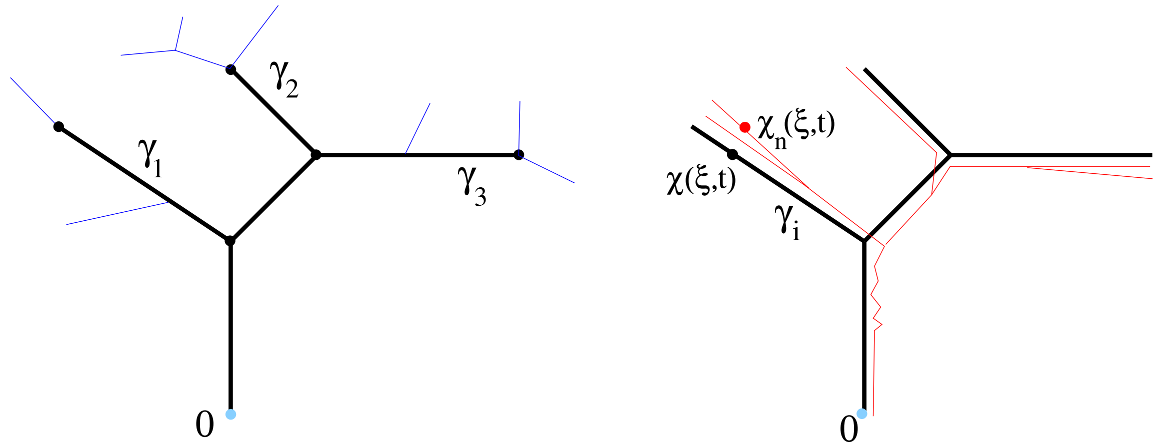

Figure 3: Left: Two finite trees, showing three maximal ε-good paths (thick lines) and 8 maximal ε'-good paths (thin lines), for $0 < \varepsilon' < \varepsilon$. Right: proving the lower semicontinuity of the weighted irrigation cost. Given a sequence of irrigation plans $\chi_n \to \chi$, one can compare the cost of χ restricted to each branch b_i with multiplicity $m(\xi, t) \geq \varepsilon$ with the corresponding costs for the approximating irrigation plans χ_n.

Definition 4.3 *Let* $\chi : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}^d$ *be an irrigation plan satisfying* **(A2)**. *The* **weight function** $W = W(\xi, t)$ *for* χ *is defined as*

$$W(\xi, t) \doteq \sup_{\varepsilon > 0} W^{\varepsilon}(\xi, t). \tag{4.14}$$

Remark 4.4 In the next section we will prove that

$$\varepsilon' < \varepsilon \quad \Longrightarrow \quad W^{\varepsilon}(\xi, t) \leq W^{\varepsilon'}(\xi, t). \tag{4.15}$$

Hence the approximations W^{ε} depend monotonically on ε. As a consequence, we can equivalently write

$$W(\xi, t) = \lim_{\varepsilon \to 0+} W^{\varepsilon}(\xi, t). \tag{4.16}$$

One should be aware that this limit may well be $+\infty$.

Remark 4.5 The assumption **(A2)**, introduced below (3.8), guarantees that the approximation is meaningful. To see what goes wrong when **(A2)** fails, consider the irrigation plan $\chi : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}^2$ defined as

$$\chi(\xi, t) = \begin{cases} (t\xi, t) & \text{if } t \in [0, 1], \\ (\xi, 1) & \text{if } t \geq 1. \end{cases}$$

In this case the multiplicity is $m(\xi, t) = 0$ for all $\xi \in [0, M]$ and $t > 0$. Hence $W^{\varepsilon}(\xi, t) \equiv 0$ for all $\varepsilon > 0$.

Having constructed a family of weights $W(\xi, t)$, we can now define the corresponding irrigation cost. Instead of the function $\psi(s) = s^{\alpha}$ with $0 < \alpha \leq 1$, one can here consider more general cost functions $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, satisfying the same assumptions imposed on f at (1.3). As usual, an upper dot will denote a derivative w.r.t. time.

Definition 4.6 *Let $f, \psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be continuous functions, both satisfying all the assumptions in* **(A1)***. Let χ be an irrigation plan satisfying* **(A2)** *and let $W = W(\xi, t)$ be the corresponding weight function, as in (4.14). If each path is parameterized by arc-length, the* **weighted cost** *is then defined as*

$$\mathcal{E}^{W,\psi}(\chi) \doteq \int_0^M \int_0^{\tau(\xi)} \frac{\psi(W(\xi,t))}{m(\xi,t)}\, dt\, d\xi\,. \tag{4.17}$$

More generally, for an arbitrary parameterization of the paths $\chi(\xi, \cdot)$, the weighted cost is

$$\mathcal{E}^{W,\psi}(\chi) \doteq \int_0^M \int_0^{\tau(\xi)} \frac{\psi(W(\xi,t))}{m(\xi,t)}\, |\dot\chi(\xi,t)|\, dt\, d\xi\,. \tag{4.18}$$

Remark 4.7 In the special case where $f \equiv 0$, the weight function coincides with the multiplicity: $W(\xi, t) = m(\xi, t)$. Taking $\psi(s) = s^\alpha$ for some $0 < \alpha \le 1$, by (3.8), this implies $\mathcal{E}^{W,\psi}(\chi) \ge \mathcal{E}^\alpha(\chi)$. Equality holds whenever χ has the single path property and hence $m(\xi, t) = |\chi(\xi, t)|$.

In order to compute an approximate value of the weighted cost, fix any $\varepsilon > 0$ and let $\widehat\gamma_1, \ldots \widehat\gamma_\nu$ be the maximal ε-good paths. Consider the elementary paths γ_i constructed by the path splitting algorithm **(PSA)** at (4.7), and let $W_i^\varepsilon : [a_i, b_i] \mapsto [\varepsilon, +\infty[$ be the corresponding approximate weights constructed at (4.8)–(4.10). We claim that

$$\mathcal{E}^{W^\varepsilon,\psi}(\chi) \doteq \int_0^M \int_0^{\tau(\xi)} \frac{\psi(W^\varepsilon(\xi,t))}{m(\xi,t)}|\dot\chi(\xi,t)|\, dt\, d\xi = \sum_{i=1}^N \int_{a_i}^{b_i} \psi(W_i^\varepsilon(s))\, ds. \tag{4.19}$$

Indeed, recalling (4.12), denote by $\Omega_\varepsilon \subseteq [0, M]$ the set of particles such that $\tau_\varepsilon(\xi) > 0$. By the definition of approximate weights W^ε at (4.13), it follows

$$\int_0^M \int_0^{\tau(\xi)} \frac{\psi(W^\varepsilon(\xi,t))}{m(\xi,t)}|\dot\chi(\xi,t)|\, dt\, d\xi = \int_{\Omega_\varepsilon} \int_0^{\tau_\varepsilon(\xi)} \frac{\psi(W^\varepsilon(\xi,t))}{m(\xi,t)}|\dot\chi(\xi,t)|\, dt\, d\xi. \tag{4.20}$$

For each $\xi \in \Omega_\varepsilon$, define

$$s_\varepsilon(\xi) \doteq \int_0^{\tau_\varepsilon(\xi)} |\dot\chi(\xi,t)|\, dt.$$

To fix the ideas, assume that $\chi(\xi,\cdot)\Big|_{[0,\tau_\varepsilon(\xi)]} \simeq \widehat\gamma_i\Big|_{[0,s_\varepsilon(\xi)]}$ for some maximal ε-good path $\widehat\gamma_i$. Recalling (4.13), by a standard change of variable formula we obtain

$$\int_0^{\tau_\varepsilon(\xi)} \frac{\psi(W^\varepsilon(\xi,t))}{m(\xi,t)}|\dot\chi(\xi,t)|\, dt = \int_0^{s_\varepsilon(\xi)} \frac{\psi(\widehat W_i(s))}{\widehat m_i(s)}\, ds. \tag{4.21}$$

For each $s > 0$ consider the set

$$\Omega_{i,k}(s) \doteq \Big\{\xi \in [0,\, M];\ \chi(\xi,\cdot)\Big|_{[0,t]} \simeq \widehat\gamma_i\Big|_{[0,s]} \text{ for some } t > 0,\ t_{i,k-1} < s \le t_{i,k}\Big\}. \tag{4.22}$$

Splitting the integral in (4.21) over the disjoint intervals $]t_{i,k-1}, t_{i,k}]$ considered at (4.5), one obtains

$$\int_0^{s_\varepsilon(\xi)} \frac{\psi(\widehat W_i(s))}{\widehat m_i(s)}\, ds = \sum_k \int_{t_{i,k-1}}^{t_{i,k}} \Big[\frac{\psi(\widehat W_i(s))}{\widehat m_i(s)} \mathbf{I}_{\Omega_{i,k}(s)}(\xi)\Big]\, ds, \tag{4.23}$$

where $\mathbf{I}_{\Omega_{i,k}(s)}(\cdot)$ is the indicator function of set $\Omega_{i,k}(s)$. Observing that

$$\int_{\Omega_\varepsilon} \left[\frac{\psi(\widehat{W}_i(s))}{\widehat{m}_i(s)}\mathbf{I}_{\Omega_{i,k}(s)}(\xi)\right] d\xi \;=\; \psi(\widehat{W}_i(s)),$$

we eventually obtain (4.19).

The next lemma shows that the family of approximating weight functions W^ε is monotonically increasing as $\varepsilon \downarrow 0$.

Lemma 4.8 *Let χ be an irrigation plan and let the approximate weights W^ε be defined as in (4.8)-(4.10). Then for any $0 < \varepsilon' < \varepsilon$ and $0 \le s < t$ one has*

$$W^\varepsilon(\xi,t) \;\le\; W^{\varepsilon'}(\xi,t). \tag{4.24}$$

Proof. To prove (4.24), let $\varepsilon' < \varepsilon$ and let $\tau_{\varepsilon'}(\xi) \ge \tau_\varepsilon(\xi)$ be the corresponding stopping times in (4.12). By construction, it trivially follows

$$W^\varepsilon(\xi,t) \;=\; W^{\varepsilon'}(\xi,t) \;=\; 0 \qquad \text{for all } t \ge \tau_{\varepsilon'}(\xi), \tag{4.25}$$

$$W^\varepsilon(\xi,t) \;=\; 0 \;\le\; W^{\varepsilon'}(\xi,t) \qquad \text{for all } t \in]\tau_\varepsilon(\xi),\, \tau_{\varepsilon'}(\xi)]. \tag{4.26}$$

To prove the inequality in (4.24) for $t \le \tau_\varepsilon(\xi)$, let $\widehat{\gamma}'_1, \ldots, \widehat{\gamma}'_{\nu'}$ be maximal ε'-good paths, and let $\gamma'_1, \ldots, \gamma'_{N'}$ be the corresponding elementary paths, generated by the algorithm **(PSA)**. By definition, the weights $W^{\varepsilon'}$ are obtained by induction, performing the steps (i)-(ii) at (4.8)–(4.10) for the elementary paths γ'_i.

Consider the functions

$$f_i^\varepsilon(w,s) \;=\; \begin{cases} f(w) & \text{if} \quad m_i(s) \ge \varepsilon, \\ 0 & \text{if} \quad m_i(s) < \varepsilon. \end{cases}$$

Performing the same inductive construction, but with f replaced by f_i^ε on each elementary path γ'_i, $i = 1, \ldots, N'$, we now recover exactly the weights W^ε. A comparison argument now yields (4.24), for all ξ, t. □

As a consequence, we have

Corollary 4.9 *Let χ be an irrigation plan which satisfies the assumption* **(A2)**. *Then the weighted irrigation cost in (4.17) is computed by*

$$\mathcal{E}^{W,\psi}(\chi) \;=\; \lim_{\varepsilon\to 0+} \int_0^M \int_0^{\tau_\varepsilon(\xi)} \frac{\psi(W^\varepsilon(\xi,t))}{m(\xi,t)} |\dot\chi(\xi,t)|\, dt d\xi\,. \tag{4.27}$$

5 Lower semicontinuity

The goal of this section is to establish the lower semicontinuity of the weighted cost functional $\mathcal{E}^{W,\psi}(\chi)$ w.r.t. pointwise convergence of the irrigation plans.

More precisely, consider a sequence of irrigation plans $\chi_n : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}^d$. We say that $\chi_n \to \chi$ *pointwise* if, for a.e. $\xi \in [0, M]$, as $n \to \infty$ one has the convergence

$$\chi_n(\xi, t) \;\to\; \chi(\xi, t) \qquad \text{uniformly for } t \text{ in compact intervals.} \tag{5.1}$$

In terms of the distance (3.1), this means

$$\lim_{n\to\infty} d(\chi_n(\xi, \cdot), \chi(\xi, \cdot)) \;=\; 0 \qquad \text{for a.e.} \quad \xi \in [0, M].$$

Theorem 5.1 *Consider a sequence* $(\chi_n)_{n\geq 1}$ *of irrigation plans, all satisfying the assumption* **(A2)***, pointwise converging to an irrigation plan* χ*. Assume that the functions* f, ψ *both satisfy the conditions in* **(A1)***. Then the corresponding weighted costs satisfy*

$$\mathcal{E}^{W,\psi}(\chi) \;\leq\; \liminf_{n\to\infty} \mathcal{E}^{W,\psi}(\chi_n). \tag{5.2}$$

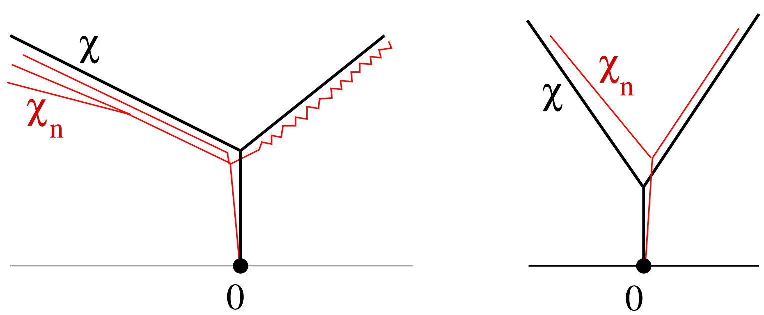

Figure 4: Left: two cases where the inequality (5.2) can be strict. (i) Paths in χ_n may remain separate, while in χ they all join together. (ii) Paths in χ_n may converge to a path in χ with strictly smaller length. Right: two cases where the weighted irrigation costs satisfy $\mathcal{E}^{W,\psi}(\chi_n) < \mathcal{E}^{W,\psi}(\chi)$. (i) The paths in χ_n can be slightly shorter than those in χ. (ii) Paths in χ_n may remain joined together for a slightly longer time than those in χ. However, these differences vanish asymptotically, as $n \to \infty$.

Toward a proof, some preliminary results will be needed.

Lemma 5.2 *Let* $\gamma : [a, b] \mapsto \mathbb{R}^d$ *be a Lipschitz path, and let* $\varepsilon > 0$*. Then there exists* $\delta > 0$ *such that, for any Lipschitz path* $\gamma^\dagger : [a, b] \mapsto \mathbb{R}^d$ *which satisfies*

$$|\gamma^\dagger(s) - \gamma(s)| \;\leq\; \delta \qquad\qquad \textit{for all } s \in [a + \delta,\, b - \delta],$$

the length of $\gamma^\dagger$ *is bounded below by*

$$\int_{a+\delta}^{b-\delta} |\dot\gamma^\dagger(s)|\, ds \;\geq\; (1 - \varepsilon) \int_a^b |\dot\gamma(s)|\, ds. \tag{5.3}$$

Proof. This is an immediate consequence of the lower semicontinuity of the path length. □

In the forthcoming analysis, it will be convenient to use a distance between two paths which is independent of their parameterization. For this purpose, following [6] we introduce

Definition 5.3 (Parameterization-free distance among paths). *Given two continuous paths $\varphi_i : [0,\, S_i] \mapsto \mathbb{R}^d$, $i = 1, 2$, the distance $\delta(\varphi_1, \varphi_2)$ is defined as*

$$\delta(\varphi_1, \varphi_2) \;\doteq\; \inf_{\eta_1,\eta_2} \max_{t\in[0,1]} \Big|\varphi_1(\eta_1(t) - \varphi_2(\eta_2(t))\Big|\,, \tag{5.4}$$

where the infimum is taken over all couples of continuous, nondecreasing, surjective maps $\eta_i : [0, 1] \mapsto [0,\, S_i]$.

As shown in [6], one has

i) $\delta(\varphi_1, \varphi_2) \;=\; \delta(\varphi_2, \varphi_1) \;\geq\; 0\,,$

ii) $\delta(\varphi_1, \varphi_2) \;=\; 0$ if and only if $\varphi_1 \;\simeq\; \varphi_2$, in the sense of Definition 3.3 ,

iii) $\delta(\varphi_1, \varphi_3) \;\leq\; \delta(\varphi_1, \varphi_2) + \delta(\varphi_2, \varphi_3)\,.$

The proof of the following lemma is elementary, but the conclusion turns out to be crucial in the proof of lower semicontinuity of the irrigation cost.

Lemma 5.4 *Let $\gamma_i : [0,\, \ell_i] \mapsto \mathbb{R}^d$, $i = 1, 2$, be two paths parametrized by arc-length. Assume that they bifurcate at some time $0 \leq \tau < \min\{\ell_1, \ell_2\}$, i.e.*

$$\tau \;=\; \sup\Big\{t \geq 0\,;\quad \gamma_1(s) = \gamma_2(s) \quad \textit{for all } s \in [0, t]\Big\}.$$

Then for any $h > 0$, there exists $\sigma > 0$ such that

$$\delta\Big(\gamma_1\Big|_{[0,s]}, \gamma_2\Big|_{[0,t]}\Big) \;\geq\; \sigma, \qquad \textit{for all } s \in [\tau + h,\, \ell_1],\, t \in [0,\, \ell_2]\,. \tag{5.5}$$

Proof. The map $(s, t) \mapsto \delta(\gamma_1|_{[0,s]},\, \gamma_2|_{[0,t]})$ is continuous and strictly positive on the compact domain $[\tau + h,\, \ell_1] \times [0,\, \ell_2]$. Hence it has a strictly positive minimum. □

In Lemma 3.11 we compared the weight $w(s)$ along a single path γ with a sum of weights $\sum_i w_i(s)$ along a family of distinct paths γ_i. The next lemma extends this result to a more general configuration where the paths γ_i are not necessarily disjoint, as shown in Fig. 5.

More precisely, consider an irrigation plan χ containing finitely many maximal paths $\widehat{\gamma}_j$: $[0, T] \mapsto \mathbb{R}^d$, $j = 1, \ldots, \nu$, all parameterized by arc-length and all with the same length T. Let $\widehat{m}_j : [0, T] \mapsto \mathbb{R}^d$ be the (non-increasing) multiplicity function along $\widehat{\gamma}_j$, defined as in (4.3), and consider weights

$$\widehat{W}_j(T) \;\geq\; \widehat{m}_j(T) \;>\; 0, \tag{5.6}$$

arbitrarily assigned at the terminal point of each maximal path. In turn, these data determine the weight functions along all paths. Namely, let $\gamma_i : [a_i, b_i] \mapsto \mathbb{R}^d, 1 \leq i \leq N$ be the corresponding elementary paths, constructed by the Path Splitting Algorithm **(PSA)**. By backward induction we can now construct the weights W_i along each elementary path, in a similar way as in (4.8)–(4.10).

- For every index i such that $b_i = T$, the weight $W_i : [a_i, b_i] \mapsto \mathbb{R}_+$ along the elementary path γ_i is computed by solving

$$w(t) \;=\; \int_t^{b_i} f(w(s))\,ds + m_i(t) + [\widehat{W}_{j(i)}(T) - m_i(T)]\,, \qquad\qquad t \in\,]a_i, T]. \tag{5.7}$$

  Here $\widehat{\gamma}_{j(i)}$ is the unique maximal path that contains γ_i as its restriction to $[a_i, b_i] = [a_i, T]$.

- If $b_i < T$, the weight $W_i : [a_i, b_i] \mapsto \mathbb{R}_+$ along the elementary path γ_i is then defined to be the solution of

$$w(t) \;=\; \int_t^{b_i} f(w(s))\,ds + m_i(t) + \left[\sum_{k\in\mathcal{O}(i)} W_k(a_k+) - \sum_{k\in\mathcal{O}(i)} m_k(a_k+)\right], \qquad t \in\,]a_i, b_i]. \tag{5.8}$$

  As in (4.10), here the summations range over all elementary paths γ_k that originate from the tip of γ_i.

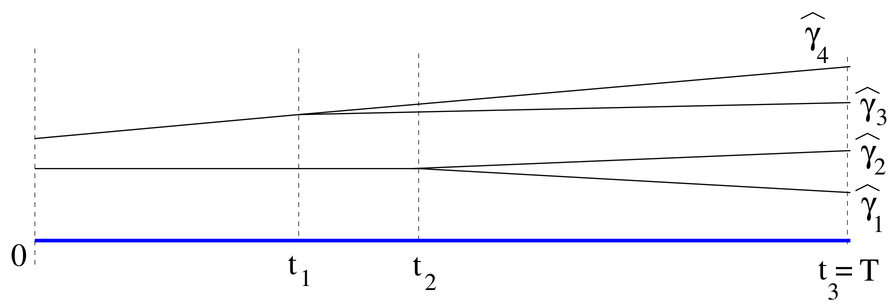

Figure 5: The two configurations compared in Lemma 5.5. For every $t \in [0, T]$, the sum of the weight functions $W_i(t)$ along a family of maximal paths $\widehat{\gamma}_i$ is compared with a single weight $W(t)$, satisfying the ODE (5.10).

Lemma 5.5 *Let the weights $W_i : [a_i, b_i] \mapsto \mathbb{R}_+$ be constructed as above. Given any constant $\widehat{W}$ such that*

$$0 \;<\; \widehat{W} \;\leq\; \sum_{j=1}^{\nu} \widehat{W}_j(T)\,, \tag{5.9}$$

let $W : [0, T] \mapsto \mathbb{R}$ be the solution to the backward Cauchy problem

$$\dot{W}(t) \;=\; -\,f(W(t)), \qquad\qquad W(T) \;=\; \widehat{W}. \tag{5.10}$$

Then for all $t \in\,]0, T]$ one has

$$W(t) \;\leq\; \sum_{i\in I(t)} W_i(t)\,, \tag{5.11}$$

where $I(t)$ denotes the set of indices $i \in \{1, \ldots, N\}$ such that $a_i < t \leq b_i$. As a consequence,

$$\int_0^T \psi(W(t))\,dt \;\leq\; \sum_{i=1}^{N} \int_{a_i}^{b_i} \psi(W_i(t))\,dt\,. \tag{5.12}$$

Proof. Let $0 < t_1 < \cdots < t_q = T$ be the times where two or more maximal paths bifurcate. The proof will achieved by backward induction on $p = 1, 2, \ldots, q$.

1. For $t \in]t_{q-1}, t_q] =]t_{q-1}, T]$, the above definition implies $I(t) = I(T)$. By (5.9) it follows

$$\sum_{i \in I(T)} W_i(T) \;\geq\; \widehat{W}\,. \tag{5.13}$$

For each $i \in I(T)$, $t \in]t_{q-1}, T]$, by (5.7) it follows

$$W_i(t) \;=\; \int_t^{t_q} f(W_i(s))\,ds + W_i(T) + [m_i(t) - m_i(T)] \tag{5.14}$$

On the other hand,

$$W(t) \;=\; \int_t^{t_q} f(W(s))\,ds + \widehat{W}\,. \tag{5.15}$$

Because of (5.13) we can apply Lemma 3.11 and conclude that

$$\sum_{i \in \mathcal{I}(T)} W_i(t) \;\geq\; W(t), \qquad \text{for all } t \in]t_{q-1}, T]. \tag{5.16}$$

2. Next, assume that that the inequality (5.11) has been proved for all $t \in]t_p, T]$, for some $1 \leq p < q$. We claim that it also holds for $t \in]t_{p-1}, t_p]$.

Indeed, since the solution W of (5.10) is continuous while all weights W_i are non-increasing, the inductive assumption yields

$$\sum_{i \in \mathcal{I}(t_p)} W_i(t_p) \;\geq\; W(t_p)\,. \tag{5.17}$$

For each $i \in I(t_p)$, $t \in]t_{p-1}, t_p]$, we then have

$$\begin{aligned} W_i(t) \;&=\; \int_t^{t_p} f(W_i(s))\,ds + W_i(t_p) + [m_i(t) - m_i(t_p)]\;, \\ W(t) \;&=\; \int_t^{t_p} f(W(s))\,ds + W(t_p) \end{aligned} \tag{5.18}$$

Because of (5.17) we can again apply Lemma 3.11 and conclude

$$\sum_{i \in \mathcal{I}(t_p)} W_i(t) \;\geq\; W(t), \qquad \text{for all } t \in (t_{p-1},\, t_p]. \tag{5.19}$$

By induction on p, this yields a proof of (5.11).

3. Since ψ satisfies the assumption **(A2)**, from (5.11) it follows

$$\begin{aligned} \sum_{i=1}^{N} \int_{a_i}^{b_i} \psi(W_i(t))\,dt \;&=\; \sum_{p=1}^{q} \sum_{i \in I(t_p)} \int_{t_{p-1}}^{t_p} \psi(W_i(t))\,dt \\ \geq\; \sum_{p=1}^{q} \int_{t_{p-1}}^{t_p} \psi(W(t))\,dt \;&=\; \int_0^T \psi(W(t))\,dt \end{aligned} \tag{5.20}$$

Hence (5.12) holds. □

Remark 5.6 In Lemma 5.5 we assumed that all maximal paths $\widehat{\gamma}_j$ had the same length T. The same conclusions (5.11)-(5.12) remain valid if each maximal path $\widehat{\gamma}_j : [0, T_j] \mapsto \mathbb{R}^d$ is defined on an interval of length $T_j \geq T$, replacing (5.9) with

$$\widehat{W} \leq \sum_{j=1}^{\nu} \widehat{W}_j(T_j). \tag{5.21}$$

To prove this, it suffices to consider the restriction of each $\widehat{\gamma}_j$ to the sub-interval $[0, T]$, and observe that (5.21) implies (5.9), because the weight functions are non-increasing,

After these preliminaries, we are now ready to give a proof of the main result of this section, in several steps.

Proof of Theorem 5.1.

1. Without loss of generality, we can assume that all paths $\chi_n(\xi, \cdot)$ are parameterized by arc-length. As a consequence, for each $\xi \in [0, M]$, the limit paths $\chi(\xi, \cdot)$ will be 1-Lipschitz, but not necessarily parameterized by arc-length.

Fix any $\varepsilon_0 > 0$. Let τ_{ε_0} be the corresponding stopping time as in (4.12), and define the truncated irrigation plan

$$\chi^{\varepsilon_0}(\xi, t) \doteq \begin{cases} \chi(\xi, t) & \text{if} \quad t \leq \tau_{\varepsilon_0}(\xi), \\ \chi(\xi, \tau_{\varepsilon_0}(\xi)) & \text{if} \quad t > \tau_{\varepsilon_0}(\xi). \end{cases} \tag{5.22}$$

Using Corollary 4.9, the theorem will be proved by showing that

$$\mathcal{E}^{W,\psi}(\chi^{\varepsilon_0}) \leq \liminf_{n\to\infty} \mathcal{E}^{W,\psi}(\chi_n). \tag{5.23}$$

2. For each $\xi \in [0, M]$, in order to re-parameterize the limit path $\chi(\xi, \cdot)$ in terms of arc-length, let

$$s(\xi, t) \doteq \int_0^t |\dot{\chi}(\xi, r)| \, dr. \tag{5.24}$$

A left-continuous inverse of $s(\xi, \cdot)$, taking values in $\mathbb{R}_+ \cup \{+\infty\}$, can be defined as

$$\eta(\xi, s) \doteq \inf\Big\{t \geq 0;\ s(\xi, t) = s\Big\}. \tag{5.25}$$

The map

$$s \mapsto \chi(\xi, \eta(\xi, s)) \tag{5.26}$$

now provides the arc-length parameterization of $\chi(\xi, \cdot)$. We observe that, for each s, the map $\xi \mapsto \eta(\xi, s)$ is measurable. Moreover, since $|\dot{\chi}(\xi, t)| \leq 1$, one has

$$\eta(\xi, s_2) - \eta(\xi, s_1) \geq s_2 - s_1 \qquad \text{for all } 0 \leq s_1 < s_2\,. \tag{5.27}$$

3. Next, let $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_\nu$ be the maximal ε_0-good paths for the irrigation plan χ. As before, we assume that each $\widehat{\gamma}_j : [0, \widehat{\ell}_j] \mapsto \mathbb{R}^d$ is parameterized by arc-length. For $s \in]0, \widehat{\ell}_j]$, let

$$\widehat{\Omega}_j(s) \doteq \left\{\xi \in [0, M];\quad \chi(\xi, \cdot)\Big|_{[0,t]} \simeq \widehat{\gamma}_j\Big|_{[0,s]} \ \text{ for some } t > 0\right\}$$

be the set of particles whose trajectory follows the path $\widehat{\gamma}_j$, at least up to the point $\widehat{\gamma}_j(s)$.

Implementing the algorithm **(PSA)** described at (4.5)–(4.7), these maximal paths can be split into finitely many elementary paths $\gamma_1, \ldots, \gamma_N$. By construction, each $\gamma_i : [a_i, b_i] \mapsto \mathbb{R}^d$, $1 \le i \le N$ is the restriction of some $\widehat{\gamma}_j$ to a subinterval $[a_i, b_i]$. We then define

$$\Omega_i(s) \doteq \widehat{\Omega}_j(s) \qquad \text{for all } s \in [a_i, b_i]. \tag{5.28}$$

The multiplicity function $m_i : [a_i, b_i] \mapsto \mathbb{R}_+$ along the elementary path γ_i is then computed by

$$m_i(s) = \text{meas}(\Omega_i(s)), \qquad s \in [a_i, b_i]. \tag{5.29}$$

According to (4.19), the approximate weighted irrigation cost is computed by a sum over all elementary paths:

$$E^{W,\psi}(\chi^{\varepsilon_0}) = \sum_{i=1}^{N} \int_{a_i}^{b_i} \psi(W_i^{\varepsilon_0}(s))\, ds, \tag{5.30}$$

where the weights $W_i^{\varepsilon_0}$ are determined as in (4.8)–(4.10).

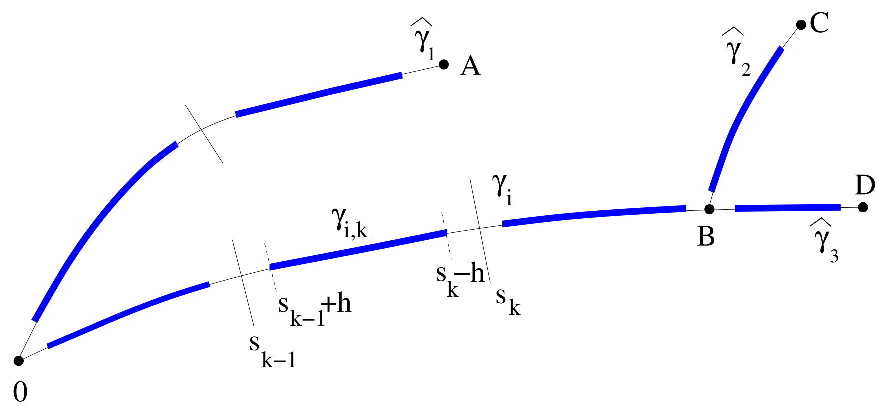

Figure 6: Proving the lower semicontinuity of the weighted irrigation cost, steps 3-4. Here $\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3$ are maximal ε_0-good paths of χ, while $\gamma_i : [a_i, b_i] \mapsto \mathbb{R}^d$, with endpoints $0, B$, is an elementary path produced by the algorithm **(PSA)**. The path γ_i is further partitioned, taking subintervals $[s_{k-1}, s_k]$ of length $\le \delta$. We then approximate the multiplicity $m_i(s)$ with a piecewise constant function $\widetilde{m}_i$, as in (5.32) and replace f with f^h as in (5.33). By choosing the constants $\delta, \delta_0, h > 0$ sufficiently small, the new weight $\widetilde{W}_i$ determined by (5.35) can be kept arbitrarily close to the original weight $W_i^{\varepsilon_0}$.

4. We claim that it is possible to replace the multiplicity functions m_i by strictly smaller piecewise constant functions $\widetilde{m}_i$, producing a very small change in the weights $W_i^{\varepsilon_0}$. More precisely, for each $i \in \{1, \ldots, N\}$, choose $\delta > 0$ and insert the times (see Fig. 6)

$$a_i = s_0 < s_1 < \cdots < s_{n(i)} = b_i, \tag{5.31}$$

so that $s_k - s_{k-1} \le \delta$ for every $k = 1, \ldots, n(i)$. For a given $\delta_0 > 0$, with $\delta_0 < \varepsilon_0$, we then define the piecewise constant function

$$\widetilde{m}_i(t) = m_i(s_k) - \delta_0 \qquad \text{for all } t \in]s_{k-1}, s_k]. \tag{5.32}$$

Since $s \mapsto m_i(s) \in [\varepsilon_0, +\infty[$ is non-increasing, we clearly have $0 < \widetilde{m}_i(t) < m(t)$ for all $t \in]a_i, b_i]$. Next, given another constant $h > 0$, with $h << \delta$, we define

$$f^h(t,\omega) \doteq \begin{cases} f(\omega) & \text{if} \quad t \in [s_{k-1} + h,\ s_k - h], \quad 1 \le k \le n(i), \\ 0 & \text{otherwise.} \end{cases} \tag{5.33}$$

We claim that, for any $\varepsilon > 0$, one can choose the above constants $\delta, \delta_0, h > 0$ small enough so that, replacing the multiplicities m_i with $\widetilde{m}_i$, and replacing f with f^h, the corresponding weights $\widetilde{W}_i$ satisfy

$$\|W_i^{\varepsilon_0} - \widetilde{W}_i\|_{\mathbf{L}^1([a_i,b_i])} < \varepsilon, \qquad |W_i^{\varepsilon_0}(a_i+) - \widetilde{W}_i(a_i+)| < \varepsilon. \tag{5.34}$$

Indeed, recalling (1.5) consider first the case $i \in \mathcal{I}_1$, so that γ_i is one of the outer-most branches. Then the weight $W_i^{\varepsilon_0}$ is obtained by solving (4.8), while $\widetilde{W}_i$ provides a solution to

$$w(t) = \int_t^{b_i} f^h(s, w(s))\, ds + \widetilde{m}_i(t), \qquad t \in]a_i, b_i]. \tag{5.35}$$

By choosing $\delta, \delta_0, h > 0$ sufficiently small, we can make the differences

$$\|m_i - \widetilde{m}_i\|_{\mathbf{L}^1([a_i,b_i])}, \qquad |m_i(a_i+) - \widetilde{m}_i(a_i+)|, \qquad \text{meas}\left([a_i, b_i] \setminus \bigcup_{k=1}^{n(i)} [s_{k-1} + h, s_k - h]\right)$$

as small as we like. The estimate (5.34) thus follows from part (iii) of Lemma 3.10.

The case $i \in \mathcal{I}_p$ for $p > 1$ is proved in the same way, by induction on p.

In view of (5.23) and (5.30), to prove the theorem it thus suffices to show that, for any given $\delta, \delta_0, h > 0$, the corresponding weights $\widetilde{W}_i$ satisfy

$$\sum_{i=1}^N \int_{a_i}^{b_i} \psi(\widetilde{W}_i(s))\, ds \leq \liminf_{n\to\infty} \mathcal{E}^{W,\psi}(\chi_n). \tag{5.36}$$

5. Consider again the arrival time $\xi \mapsto \tau(\xi)$ introduced in Definition 2.1. For any $\varepsilon > 0$, by Corollary 3.2 there is a compact set $\Omega_\varepsilon \subseteq [0, M]$, with

$$\text{meas}\,([0, M] \setminus \Omega_\varepsilon) < \varepsilon, \tag{5.37}$$

on which that map $\tau(\cdot)$ is continuous. Hence

$$\max_{\xi\in\Omega_\varepsilon} \tau(\xi) \leq \kappa \tag{5.38}$$

for some constant κ. By (5.1) and Egoroff's theorem, by slightly shrinking the compact set Ω_ε, we can assume that (5.37) still holds, together with

$$\lim_{n\to\infty} \sup_{\xi\in\Omega_\varepsilon} \|\chi(\xi, \cdot) - \chi_n(\xi, \cdot)\|_{\mathbf{L}^\infty([0,\kappa])} = 0. \tag{5.39}$$

In addition, calling $\tau^n(\xi)$ the smallest time τ such that $\chi_n(\xi, \cdot)$ is constant for $t \geq \tau$, by further shrinking Ω_ε we can also assume

$$\liminf_{n\to\infty} \inf_{\xi\in\Omega_\varepsilon} [\tau^n(\xi) - \tau(\xi)] \geq 0. \tag{5.40}$$

Indeed, since

$$\liminf_{n\to\infty} \tau^n(\xi) \geq \tau(\xi), \tag{5.41}$$

it follows that the non-decreasing sequence

$$\widehat{\tau}^n(\xi) \ \doteq \ \inf_{k\geq n} \tau^k(\xi)$$

converges to a limit

$$\lim_{n\to\infty} \widehat{\tau}^n(\xi) \ = \ \tau^\infty(\xi) \ \geq \ \tau(\xi)$$

for a.e. $\xi \in [0, M]$. Again by Egoroff's theorem we can choose a large subset $\Omega_\varepsilon \subset [0, M]$ where the pointwise convergence is uniform. This yields (5.40).

Furthermore, since each χ_n satisfies the assumption **(A2)**, we can choose $\varepsilon_n > 0$ small enough so that the following holds. Defining the stopping time

$$\tau^n_{\varepsilon_n}(\xi) \ \doteq \ \sup\{t \geq 0\,; \quad m_n(\xi, t) \geq \varepsilon_n\}, \tag{5.42}$$

by possibly further shrinking the set Ω_ε in (5.37) one has

$$\tau^n_{\varepsilon_n}(\xi) \ \geq \ \tau^n(\xi) - \frac{h}{2} \qquad \text{for all } \xi \in \Omega_\varepsilon, \quad n \geq 1\,. \tag{5.43}$$

6. Let $\widehat{\gamma}'_1, \ldots, \widehat{\gamma}'_{\nu'}$ be the maximal ε_n-good paths in χ_n, and let $\gamma'_1, \ldots, \gamma'_{N'}$ be the elementary paths constructed by the algorithm **(PSA)**. As in step **3**, for each $\widehat{\gamma}'_j : [0, \hat{\ell}'_j] \mapsto \mathbb{R}^d$ we define

$$\begin{aligned} \widehat{\Omega}'_j(s) \ &\doteq \ \Big\{\xi \in [0, M];\ \chi_n(\xi, \cdot)\Big|_{[0,t]} \simeq \widehat{\gamma}'_j\Big|_{[0,s]} \text{ for some } t > 0\Big\} \\ &= \ \Big\{\xi \in [0, M];\ \chi_n(\xi, t) = \widehat{\gamma}'_j(t), \text{ for all } t \in [0, s]\Big\}. \end{aligned} \tag{5.44}$$

This is the set of particles whose trajectory follows the maximal path $\widehat{\gamma}'_j$, at least up to time s. Notice that the last identity holds because $\widehat{\gamma}'_j$ and $\chi_n(\xi, \cdot)$ are both parameterized by arc-length. By construction, each elementary path $\gamma'_i : [a'_i, b'_i] \mapsto \mathbb{R}^d$, $1 \leq i \leq N'$ is the restriction of some $\widehat{\gamma}'_j$ to a subinterval $[a'_i, b'_i]$. We then define

$$\Omega'_i(s) \ \doteq \ \widehat{\Omega}'_j(s) \qquad \text{for all } s \in \ [a'_i, b'_i]. \tag{5.45}$$

7. Now consider a particle $\xi \in \Omega_i(s_k) \cap \Omega_\varepsilon$, so that the path $t \mapsto \chi(\xi, t)$ reaches the point $\gamma_i(s_k)$ at some time $t = \eta(\xi, s_k)$. This implies $\tau(\xi) \geq \eta(\xi, s_k)$. Hence by (5.40) we have

$$\tau^n(\xi) \ > \ \eta(\xi, s_k) - \frac{h}{2}$$

for all n large enough. In turn, choosing $\varepsilon_n > 0$ sufficiently small, by (5.43) it follows

$$\tau^n_{\varepsilon_n}(\xi) \ \geq \ \tau^n(\xi) - \frac{h}{2} \ > \ \eta(\xi, s_k) - h \ \geq \ \eta(\xi, s_k - h).$$

Otherwise stated, by further slightly shrinking the compact set Ω_ε in (5.37), for any $h > 0$ we can thus achieve the implication

$$\xi \ \in \ \Omega_i(s_k) \cap \Omega_\varepsilon \qquad \Longrightarrow \qquad \eta(\xi, s_k - h) \ < \ \tau^n_{\varepsilon_n}(\xi), \tag{5.46}$$

for all n sufficiently large.

8. We observe that two particles $\xi, \widetilde{\xi}$, which have the same trajectory in the irrigation plan χ, may be sent along different paths by the irrigation plan χ_n. To account for this fact, recalling (5.28) and (5.45), for a fixed $n \geq 1$ we define

$$A_i^j(s_k) \ \doteq \ \left\{\xi \in \Omega_\varepsilon \cap \Omega_i(s_k);\ \chi_n(\xi,t) \ = \ \widehat{\gamma}'_j(t),\ \text{for all } 0 \leq t \leq \eta(\xi, s_k - h)\right\}. \qquad (5.47)$$

In other words, $A_i^j(s_k)$ is the set of particles $\xi \in \Omega_\varepsilon$ such that:

- By the irrigation plan χ they are moved along the ε_0-good elementary path γ_i, at least up to the point $\gamma_i(s_k)$.
- By the irrigation plan χ_n they are moved along the ε_n-good maximal path $\widehat{\gamma}'_j$, at least up to point $\widehat{\gamma}'_j(\eta(\xi, s_k - h))$.

Using Lusin's theorem and by possibly shrinking the compact domain $\Omega_\varepsilon \subseteq [0, M]$, in addition to (5.37) we can assume that, restricted to each $A_i^j(s_k)$, the two maps

$$\eta(\cdot, s_k - h) : A_i^j(s_k) \mapsto \mathbb{R}_+, \qquad\qquad \eta(\cdot, s_{k-1} + h) : A_i^j(s_k) \mapsto \mathbb{R}_+$$

are continuous.

9. The set of paths

$$\gamma_{i,k} \ \doteq \ \gamma_i\Big|_{[s_{k-1},\, s_k]} \qquad (5.48)$$

comes with an obvious partial ordering. Namely, we define

$$(i,k) \ \preceq \ (i^\dagger, k^\dagger) \qquad (5.49)$$

if the two elementary paths $\gamma_i, \gamma_{i^\dagger}$ for the irrigation plan χ are both contained in some ε_0-good maximal path $\widehat{\gamma}_j$, and moreover $s_k \leq s_{k^\dagger}$.

As shown in Fig. 7, to each portion $\gamma_{i,k}$ of the elementary path γ_i in the irrigation plan χ we shall associate a family $\{\gamma_l^\sharp\}$ of paths in the irrigation plan χ_n, and compare the corresponding costs. For this purpose, assuming $A_i^j(s_k) \neq \emptyset$, we define

$$s_{k+}^{i,j} \ \doteq \ \inf_{\xi \in A_i^j(s_k)} \eta(\xi, s_k - h), \qquad\qquad s_{k-}^{i,j} \ \doteq \ \inf_{\xi \in A_i^j(s_k)} \eta(\xi, s_{k-1} + h). \qquad (5.50)$$

Notice that, by (5.27), one has

$$s_{k+}^{i,j} - s_{k-}^{i,j} \ \geq \ s_k - s_{k-1} - 2h. \qquad (5.51)$$

For each i, j, k such that $A_i^j(s_k)$ is non-empty, we now consider all the paths $\gamma_l^\sharp : [a_l^\sharp,\, b_l^\sharp] \mapsto \mathbb{R}^d$, obtained as follows. Consider all the ε_n-good elementary paths $\gamma'_p : [a'_p, b'_p] \mapsto \mathbb{R}^d$ of χ_n. which are contained in the maximal path $\widehat{\gamma}'_j$. We then take $\gamma_l^\sharp$ to be the restriction of γ'_p to the subinterval

$$[a_l^\sharp,\, b_l^\sharp] \ \doteq \ [a'_p, b'_p] \cap [s_{k-}^{i,j}\,, s_{k+}^{i,j}]. \qquad (5.52)$$

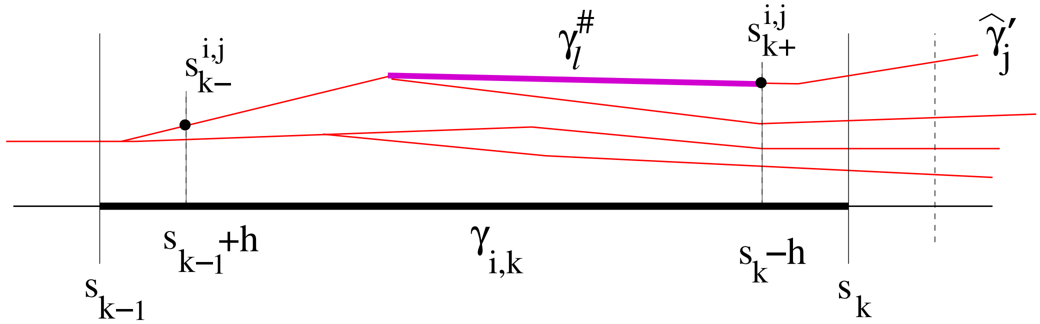

Figure 7: To compare the cost of the irrigation plans χ and χ_n, to the portion of the ε_0-good elementary path $\gamma_i : [s_{k-1}+h, s_k-h] \mapsto \mathbb{R}^d$ we associate a family of ε_n-good paths $\gamma^\sharp_l$ in χ_n.

Call $\Gamma_{i,k}$ the collection of all such paths $\gamma^\sharp_l$, as j varies among all the maximal ε_n-good paths of χ_n, with $A^j_i(s_k) \neq \emptyset$.

10. Let $m^\sharp_l : [a^\sharp_l, b^\sharp_l] \mapsto \mathbb{R}_+$ be the multiplicity of the path $\gamma^\sharp_l$ in the irrigation plan χ_n. We claim that, choosing $\delta_0 = \varepsilon$ in (5.32), for all n large enough the piecewise constant multiplicity $\widetilde{m}_i$ defined at (5.32) satisfies

$$\widetilde{m}_i(t) \;=\; \widetilde{m}_i(s_k - h) \;\doteq\; m_i(s_k) - \varepsilon \;<\; \sum_{\gamma^\sharp_l \in \Gamma_{i,k},\; b^\sharp_l = s^{i,j}_{k+}} m^\sharp_l(b^\sharp_l) \tag{5.53}$$

for all $t \in]s_{k-1}, s_k]$. Indeed, in view of (5.46)-(5.47), for each $\xi \in \Omega_i(s_k) \cap \Omega_\varepsilon$, there is some maximal path $\widehat{\gamma}'_j$ such that

$$\chi_n(\xi, t) \;=\; \widehat{\gamma}'_j(t), \qquad \text{for all } t \in [0,\, \eta(\xi, s_k - h)]. \tag{5.54}$$

By (5.50) we know that $\eta(\xi, s_k - h) \geq s^{i,j}_{k+}$. Hence (5.54) implies $\xi \in \widehat{\Omega}'_j(s^{i,j}_{k+})$. Therefore

$$\widetilde{m}_i(s_k - h) \;=\; \text{meas}(\Omega_i(s_k)) - \varepsilon \;<\; \text{meas}\Big(\Omega_i(s_k) \cap \Omega_\varepsilon\Big) \;<\; \sum_{\gamma^\sharp_l \in \Gamma_{i,k},\; b^\sharp_l = s^{i,j}_{k+}} m^\sharp_l(b^\sharp_l). \tag{5.55}$$

11. Toward a proof of (5.36), a key observation is the following. If two particles $\xi, \widetilde{\xi}$ are sent by χ along two maximal paths $\widehat{\gamma}_i, \widehat{\gamma}_j$ which bifurcate at a some time τ_{ij}, then, for all n large enough, the irrigation plan χ_n must send these two particles along distinct paths as well. In this step we prove a precise estimate in this direction.

Let $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_\nu$ be the maximal ε_0-good paths for the irrigation plan χ. For any given $h > 0$, by Lemma 5.4 one can find $\sigma > 0$ such that

$$\delta\left(\widehat{\gamma}_i\Big|_{[0,s]}, \widehat{\gamma}_j\Big|_{[0,t]}\right) \;\geq\; \sigma\,, \qquad \text{for all } i \neq j\,,\;\; s \in [\tau_{ij} + h,\, \hat{\ell}_i],\;\; t \in [0,\, \hat{\ell}_j]\,. \tag{5.56}$$

Here τ_{ij} is the time where the maximal paths $\widehat{\gamma}_i$ and $\widehat{\gamma}_j$ bifurcate, as defined at (4.4).

On the other hand, by (5.39), for all n sufficiently large one has

$$\sup_{\xi \in \Omega_\varepsilon} \|\chi(\xi, \cdot) - \chi_n(\xi, \cdot)\|_{\mathbf{L}^\infty([0,\kappa])} \;<\; \frac{\sigma}{3}\,, \tag{5.57}$$

where σ is the constant in (5.56).

Consider two particles $\xi, \widetilde{\xi} \in \Omega_\varepsilon$ which are sent by χ along the two distinct maximal paths $\widehat{\gamma}_i, \widehat{\gamma}_j$. More precisely, recalling the definition (5.44), assume that for some $h > 0$

$$\xi \in \widehat{\Omega}_i(\tau_{ij} + h) \cap \Omega_\varepsilon, \qquad\qquad \widetilde{\xi} \in \widehat{\Omega}_j(\tau_{ij} + h) \cap \Omega_\varepsilon\,,$$

Without loss of generality, assume

$$T \doteq \eta(\xi, \tau_{ij} + h) \leq \eta(\widetilde{\xi}, \tau_{ij} + h).$$

Recalling the notation used at (5.24), we can now find $\tau \doteq s(\widetilde{\xi}, T) \leq \hat{\ell}_j$, such that by (5.56) and (5.57)

$$\begin{aligned}
\sigma &\leq \delta\Big(\widehat{\gamma}_i\Big|_{[0,\tau_{ij}+h]}, \widehat{\gamma}_j\Big|_{[0,\tau]}\Big) = \delta\Big(\chi(\xi,\cdot)\Big|_{[0,T]}, \chi(\widetilde{\xi},\cdot)\Big|_{[0,T]}\Big) \\
&\leq \|\chi(\xi,\cdot) - \chi_n(\xi,\cdot)\|_{\mathbf{L}^\infty([0,T])} + \delta\Big(\chi_n(\xi,\cdot)\Big|_{[0,T]}, \chi_n(\widetilde{\xi},\cdot)\Big|_{[0,T]}\Big) \\
&\qquad + \|\chi(\widetilde{\xi},\cdot) - \chi_n(\widetilde{\xi},\cdot)\|_{\mathbf{L}^\infty([0,T])} \\
&\leq \delta\Big(\chi_n(\xi,\cdot)\Big|_{[0,T]}, \chi_n(\widetilde{\xi},\cdot)\Big|_{[0,T]}\Big) + \frac{2\sigma}{3}\,.
\end{aligned} \tag{5.58}$$

This proves that the two paths $\chi_n(\xi,\cdot)$ and $\chi_n(\widetilde{\xi},\cdot)$, which are parameterized by arc-length, cannot coincide over the entire interval $[0,T]$.

12. We are finally ready to prove (5.36). Let $\varepsilon > 0$ be given. Since the weights $\widetilde{W}_i$ are uniformly bounded, by choosing $h > 0$ small enough for every $i = 1, \ldots, N$ we achieve

$$\int_{a_i}^{b_i} \psi(\widetilde{W}_i(s))\,ds - \sum_{k=1}^{n(i)} \int_{s_{k-1}+h}^{s_k-h} \psi(\widetilde{W}_i(s))\,ds < \frac{\varepsilon}{N}\,. \tag{5.59}$$

Since $\varepsilon > 0$ is arbitrary, to prove (5.36), it is thus suffices to show that

$$\sum_{i=1}^{N} \sum_{k=1}^{n(i)} \int_{s_{k-1}+h}^{s_k-h} \psi(\widetilde{W}_i(s))\,ds \leq \liminf_{n\to\infty} \mathcal{E}^{W,\psi}(\chi_n). \tag{5.60}$$

As shown in step **9**, there is a map

$$\gamma_{i,k} \mapsto \Gamma_{i,k} \doteq \{\gamma_l^\sharp\,;\; l = l(i,j,k)\}, \tag{5.61}$$

which associates to the portion of elementary path $\gamma_{i,k}$ of χ a corresponding family of ε_n-good paths of χ_n, as in Fig. 7. Using the ordering (5.49), by induction on (i,k) we will show that

$$\int_{s_{k-1}+h}^{s_k-h} \psi(\widetilde{W}_i(s))\,ds \leq \sum_{\gamma_l^\sharp \in \Gamma_{i,k}} \int_{a_l^\sharp}^{b_l^\sharp} \psi(W_l^\sharp(s))\,ds, \tag{5.62}$$

for every (i,k). By showing that paths $\gamma_l^\sharp$ belonging to distinct families $\Gamma_{i,k}, \Gamma_{i^\dagger,k^\dagger}$ are disjoint, we will conclude

$$\sum_{i=1}^{N} \sum_{k=1}^{n(i)} \int_{s_{k-1}+h}^{s_k-h} \psi(\widetilde{W}_i(s))\,ds \leq \sum_{i=1}^{N} \sum_{k=1}^{n(i)} \sum_{\gamma_l^\sharp \in \Gamma_{i,k}} \int_{a_l^\sharp}^{b_l^\sharp} \psi(W_l^\sharp(s))\,ds \leq \mathcal{E}^{W,\psi}(\chi_n), \tag{5.63}$$

for every $n \geq 1$ sufficiently large. This will prove (5.60), and hence (5.36).

13. In this step we prove our claim that paths $\gamma_l^\sharp$ belonging to distinct families $\Gamma_{i,k}$, $\Gamma_{i^\dagger,k^\dagger}$ are disjoint. Assume $(i,k) \neq (i^\dagger, k^\dagger)$. Two cases can occur.

CASE 1: the elementary paths $\gamma_i, \gamma_{i^\dagger}$ are not contained in the same maximal ε_0-good path of χ.

In this case, there exists two distinct maximal paths $\widehat{\gamma}_p, \widehat{\gamma}_q$, which bifurcate at time

$$\tau_{pq} \;\leq\; \min\{s_{k-1},\, s_{k^\dagger-1}\} \tag{5.64}$$

and such that

$$\gamma_i(t) \;=\; \widehat{\gamma}_p(t), \quad \text{for all } t \in [s_{k-1}, s_k], \qquad \gamma_{i^\dagger}(t) \;=\; \widehat{\gamma}_q(t), \quad \text{for all } t \in [s_{k^\dagger-1}, s_{k^\dagger}]. \tag{5.65}$$

If now $\xi \in A_i^j(s_k)$ and $\widetilde{\xi} \in A_{i^\dagger}^{j^\dagger}(s_{k^\dagger})$, by (5.47) there exists two ε_n-good maximal paths for χ_n such that

$$\begin{aligned} \chi_n(\xi,t) \;&=\; \widehat{\gamma}'_j(t), \qquad \text{for all } 0 \leq t \leq \eta(\xi, s_k - h),\\ \chi_n(\widetilde{\xi},t) \;&=\; \widehat{\gamma}'_{j^\dagger}(t), \qquad \text{for all } 0 \leq t \leq \eta(\widetilde{\xi}, s_{k^\dagger} - h). \end{aligned} \tag{5.66}$$

By (5.64) and the analysis in step **11**, the two paths $\chi_n(\xi,\cdot)$ and $\chi_n(\widetilde{\xi},\cdot)$ must bifurcate before time $\eta(\xi, s_{k-1}+h) \wedge \eta(\widetilde{\xi}, s_{k^\dagger-1}+h)$. Here and in the sequel we use the notation $a \wedge b \doteq \min\{a,b\}$. Calling $\tau'_{jj^\dagger}$ the time where the two maximal paths $\widehat{\gamma}'_j$ and $\widehat{\gamma}'_{j^\dagger}$ bifurcate, by (5.66) and (5.50) one obtains

$$\tau'_{jj^\dagger} \;\leq\; \eta(\xi, s_{k-1}+h) \wedge \eta(\widetilde{\xi}, s_{k^\dagger-1}+h) \;\leq\; s_{k-}^{i,j} \wedge s_{k^\dagger -}^{i^\dagger, j^\dagger}\,. \tag{5.67}$$

If now $A_i^j(s_k)$ is nonempty, by construction the path $\gamma_l^\sharp \in \Gamma_{i,k}$, which is contained in the maximal path $\widehat{\gamma}'_j$, is defined for $s \in [s_{k-}^{i,j}, s_{k+}^{i,j}]$. Similarly, $A_{i^\dagger}^{j^\dagger}(s_{k^\dagger})$ is nonempty, the path $\gamma_{l^\dagger}^\sharp \in \Gamma_{i^\dagger,k^\dagger}$ which is contained in $\widehat{\gamma}'_{j^\dagger}$ will be defined for $s \in [s_{k^\dagger-}^{i^\dagger,j^\dagger}, s_{k^\dagger+}^{i^\dagger,j^\dagger}]$.

Since the two maximal ε_n-good paths $\widehat{\gamma}'_j$ and $\widehat{\gamma}'_{j^\dagger}$ already bifurcate at the time (5.67), the two paths $\gamma_l^\sharp$ and $\gamma_{l^\dagger}^\sharp$ are disjoint. By the above argument, we conclude that the two families $\Gamma_{i,k}$ and $\Gamma_{i^\dagger,k^\dagger}$ consist of distinct paths.

CASE 2: with the partial ordering (5.49) one has $(i,k) \preceq (i^\dagger, k^\dagger)$.

This implies that there exists a maximal ε_0-good path $\widehat{\gamma}_j$ in χ, such that

$$\gamma_i(t) \;=\; \widehat{\gamma}_j(t), \quad \text{for all } t \in [s_{k-1}, s_k], \qquad \gamma_{i^\dagger}(t) \;=\; \widehat{\gamma}_j(t), \quad \text{for all } t \in [s_{k^\dagger-1}, s_{k^\dagger}], \tag{5.68}$$

and moreover $s_k < s_{k^\dagger}$. For each fixed maximal ε_n-good path $\widehat{\gamma}'_j$ in χ_n, there are two cases:

- $A_{i^\dagger}^j(s_{k^\dagger})$ is nonempty. By (5.47) and (5.68) we thus have $A_{i^\dagger}^j(s_{k^\dagger}) \subseteq A_i^j(s_k)$. Hence

$A_i^j(s_k)$ is nonempty as well. By the definition (5.50) one has

$$\begin{aligned} s^{i^\dagger,j}_{k^\dagger -} - s^{i,j}_{k+} &= \inf_{\xi\in A^j_{i^\dagger}(s_{k^\dagger})} \eta(\xi, s_{k^\dagger -1}+h) - \inf_{\xi\in A^j_i(s_k)} \eta(\xi, s_k - h) \\ &\geq \inf_{\xi\in A^j_{i^\dagger}(s_{k^\dagger})} \big[\eta(\xi, s_{k^\dagger -1}+h) - \eta(\xi, s_k-h)\big] \\ &\geq \inf_{\xi\in A^j_{i^\dagger}(s_{k^\dagger})} [\eta(\xi, s_k+h) - \eta(\xi, s_k-h)] \ \geq\ 2h. \end{aligned} \tag{5.69}$$

In this case, for every path $s\mapsto \gamma^\sharp_{l^\dagger}(s)$ in $\Gamma_{i^\dagger,k^\dagger}$, which is contained in $\widehat{\gamma}'_j$, the arc-length parameter ranges in $[s^{i^\dagger,j}_{k^\dagger -}, s^{i^\dagger,j}_{k^\dagger +}]$. On the other hand, for every path $s\mapsto \gamma^\sharp_l(s)$ in $\Gamma_{i,k}$, which is contained in $\widehat{\gamma}'_j$, the time parameter ranges in $[s^{i,j}_{k-}, s^{i,j}_{k+}]$. By (5.69) these two paths are disjoint.

- $A^j_{i^\dagger}(s_{k^\dagger})$ is empty. By construction, this implies that every path $\gamma^\sharp_{l^\dagger} \in \Gamma_{i^\dagger,k^\dagger}$ is not contained in the maximal path $\widehat{\gamma}'_j$. Thus, if $\gamma^\sharp_l \in \Gamma_{i,k}$ is contained in $\widehat{\gamma}'_j$, $\gamma^\sharp_l$ is disjoint from all the paths in $\Gamma_{i^\dagger,k^\dagger}$.

Since the above analysis applies to each maximal path $\widehat{\gamma}'_j$ in χ_n, we conclude that when $(i,k)\preceq(i^\dagger,k^\dagger)$, the two families $\Gamma_{i,k}$ and $\Gamma_{i^\dagger,k^\dagger}$ consist of disjoint paths.

14. As before, let $\gamma_1,\ldots,\gamma_N$ be the elementary ε_0-good paths in χ. The weights $\widetilde{W}_i$ are then constructed along each γ_i by the same inductive procedure as in (4.8)–(4.10), for $i\in\mathcal{I}_p$, $p=1,2,\ldots$ We recall that $\mathcal{I}_p$ are the sets of indices introduced at (1.5).

Toward a proof of (5.60) we claim that, for any i, $1\leq k\leq n(i)$,

$$\sum_{\gamma^\sharp_l\in\Gamma_{i,k}} \int_{a^\sharp_l}^{b^\sharp_l} \psi(W^\sharp_l(s))\,ds \ \geq\ \int_{s_{k-1}+h}^{s_k-h} \psi(\widetilde{W}_i(s))\,ds, \tag{5.70}$$

$$\sum_{\gamma^\sharp_l\in\Gamma_{i,k},\ a^\sharp_l = s^{i,j}_{k-}} W^\sharp_l(a^\sharp_l) \ \geq\ \widetilde{W}_i(s_{k-1}+h). \tag{5.71}$$

The above inequalities will be proved first for $i\in\mathcal{I}_1$ (i.e., for the outer-most branches), then inductively for $i\in\mathcal{I}_2,\mathcal{I}_3,\ldots$

We begin by considering an elementary path γ_i with $i\in\mathcal{I}_1$. We compare the weight $\widetilde{W}_i$ along γ_i with the sum of weights along the corresponding ε_n-good paths $\gamma^\sharp_l$ of χ_n. On the last subinterval $[s_{n(i)-1}+h,\, s_{n(i)}-h]$ of γ_i, by (5.51)-(5.53) the assumptions in Lemma 5.5 are satisfied. From (5.11)-(5.12) we thus have

$$\sum_{\gamma^\sharp_l\in\Gamma_{i,n(i)}} \int_{a^\sharp_l}^{b^\sharp_l} \psi(W^\sharp_l(s))\,ds \ \geq\ \int_{s_{n(i)-1}+h}^{s_{n(i)}-h} \psi(\widetilde{W}_i(s))\,ds\,, \tag{5.72}$$

$$\sum_{\gamma^\sharp_l\in\Gamma_{i,n(i)},\ a^\sharp_l = s^{i,j}_{n(i)-}} W^\sharp_l(a^\sharp_l) \ \geq\ \widetilde{W}_i(s_{n(i)-1}+h). \tag{5.73}$$

Now consider the previous interval $[s_{n(i)-2}+h,\, s_{n(i)-1}-h]$. By (5.32)-(5.33) it follows

$$\widetilde{W}_i(s_{n(i)-1}-h) \;=\; \widetilde{W}_i(s_{n(i)-1}+h)+\widetilde{m}_i(s_{n(i)-1}-h)-\widetilde{m}_i(s_{n(i)-1}+h). \tag{5.74}$$

Hence by (5.53) and (5.73) one has

$$\begin{aligned}\sum_{\gamma_l^\sharp\in\Gamma_{i,n(i)-1},\; b_l^\sharp=s^{i,j}_{(n(i)-1)+}} W_l^\sharp(b_l^\sharp) \;&\geq\; \sum_{\gamma_{l'}^\sharp\in\Gamma_{i,n(i)},\; a_{l'}^\sharp=s^{i,j}_{n(i)-}} W_{l'}^\sharp(a_{l'}^\sharp) \\ &\qquad+\widetilde{m}_i(s_{n(i)-1}-h)-\widetilde{m}_i(s_{n(i)-1}+h) \\ &\geq\; \widetilde{W}_i(s_{n(i)-1}-h).\end{aligned} \tag{5.75}$$

By (5.51) and (5.75) we can again apply Lemma 5.5 on $\Gamma_{i,n(i)-1}$ and the restriction of γ_i on $[s_{n(i)-2}+h,\, s_{n(i)-1}-h]$. By similar arguments we prove (5.70)-(5.71) for each $i\in\mathcal{I}_1, 1\leq k\leq n(i)$.

15. Next, assume that (5.70)-(5.71) have been proved for all $i\in\mathcal{I}_1\cup\cdots\cup\mathcal{I}_{p-1}$.We claim that these same inequalities also hold for all $i\in\mathcal{I}_p$.

Indeed, consider an elementary path γ_i with $i\in\mathcal{I}_p$. Along this path, consider the last subinterval, with $s\in[s_{n(i)-1}+h,\, s_{n(i)}-h]$. Recalling the construction of the weight $\widetilde{W}_i$ at (4.9)-(4.10) and (5.32)-(5.33), we obtain

$$\begin{aligned}\widetilde{W}_i(s_{n(i)}-h) \;&=\; \widetilde{W}_i(s_{n(i)}) \;=\; \sum_{k\in\mathcal{O}(i)}\widetilde{W}_k(a_k+)+\left[\widetilde{m}_i(s_{n(i)})-\sum_{k\in\mathcal{O}(i)}\widetilde{m}_k(a_k+)\right] \\ &=\; \sum_{k\in\mathcal{O}(i)}\widetilde{W}_k(a_k+h)+\left[\widetilde{m}_i(s_{n(i)}-h)-\sum_{k\in\mathcal{O}(i)}\widetilde{m}_k(a_k+h)\right].\end{aligned} \tag{5.76}$$

$$\begin{aligned}\sum_{\gamma_l^\sharp\in\Gamma_{i,n(i)},\; b_l^\sharp=s^{i,j}_{n(i)+}} W_l^\sharp(b_l^\sharp) \;&\geq\; \sum_{k\in\mathcal{O}(i)}\;\sum_{\gamma_{l'}^\sharp\in\Gamma_{k,1},\; a_{l'}^\sharp=s^{k,j}_{1-}} W_{l'}^\sharp(a_{l'}^\sharp) \\ &\qquad+\left[\widetilde{m}_i(s_{n(i)}-h)-\sum_{k\in\mathcal{O}(i)}\widetilde{m}_k(a_k+h)\right].\end{aligned} \tag{5.77}$$

For each $k\in\mathcal{O}(i)$, the inductive assumption (5.73) yields

$$\sum_{\gamma_{l'}^\sharp\in\Gamma_{k,1},\; a_{l'}^\sharp=s^{k,j}_{1-}} W_{l'}^\sharp(a_{l'}^\sharp) \;\geq\; \widetilde{W}_k(a_k+h). \tag{5.78}$$

Hence, by (5.76)–(5.78),

$$\sum_{\gamma_l^\sharp\in\Gamma_{i,n(i)},\; b_l^\sharp=s^{i,j}_{n(i)+}} W_l^\sharp(b_l^\sharp) \;\geq\; \widetilde{W}_i(s_{n(i)}-h). \tag{5.79}$$

Thanks to (5.79) and (5.51), we can use again Lemma 5.5 and conclude (5.72)-(5.73). By backward induction on $k=n(i),n(i)-1,\ldots,1$, we then achieve the proof of (5.70)-(5.71) as in step **14**.

16. By induction on $\mathcal{I}_p$, $p = 1, 2, \ldots$, we conclude that the inequalities (5.70) hold for every $i = 1, \ldots, N$ and every $k = 1, \ldots, n(i)$. In turn, since the families of paths $\Gamma_{i,k}$ are all disjoint from each other, from (5.70) we obtain (5.63). As remarked in step **12**, this implies the lower semicontinuity of the weighted irrigation cost. □

6 Weights depending on the inclination of the branches

Aim of this section is to extend the previous results to the case where the right hand side of the ODE in (1.2) also depends on the inclination of the branch. More precisely, if $s \mapsto \gamma(s)$ is a parameterization of the branch, we replace (1.2) with

$$W'(s) \;=\; -\,f(\dot\gamma(s), W(s)). \tag{6.1}$$

Concerning the function $f : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$, we shall assume

(A3) *The function $f = f(v, W)$ is continuous w.r.t. both variables. For each $v \in \mathbb{R}^d$, the map $W \mapsto f(v, W)$ satisfies the same conditions as in (1.3), namely*

$$f(v,0) \;=\; 0, \qquad f_W(v,W) \;>\; 0, \qquad f_{WW}(v,W) \;\le\; 0 \qquad \textit{for all } W > 0. \tag{6.2}$$

For each $W > 0$, the map $v \mapsto f(v, W)$ is convex and positively homogeneous, namely

$$f(rv, W) \;=\; r\,f(v,W) \qquad \textit{for all } r \ge 0. \tag{6.3}$$

An example of a function satisfying **(A3)** is

$$f(v,W) \;=\; \left(|v_1| + \sqrt{v_1^2 + v_2^2}\right) W^\beta,$$

where $0 < \beta \le 1$.

Let now $\chi : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}^d$ be an irrigation plan. When f also depends on v, the weight functions $W(\xi, t)$ can be constructed following exactly the same procedure described in Section 4. The only difference is that, for each elementary path $\gamma_i : [a_i, b_i] \mapsto \mathbb{R}^d$, the formulas (4.8)-(4.9) are now replaced respectively by

$$w(t) \;=\; \int_t^{b_i} f(\dot\gamma_i(s), w(s))\,ds + m_i(t), \qquad\qquad t \in \,]a_i, b_i], \tag{6.4}$$

$$w(t) \;=\; \int_t^{b_i} f(\dot\gamma_i(s), w(s))\,ds + m_i(t) + \overline{w}_i\,, \qquad\qquad t \in \,]a_i, b_i]. \tag{6.5}$$

Here the upper dot denotes a derivative w.r.t. the parameter s along the arc. We observe that all conclusions of Lemma 3.10 remain valid if (3.21) is replaced by

$$w(t) \;=\; \int_t^T f(s, w(s))\,ds + m(t) \qquad\qquad \text{for all} \;\; t \in [0, T], \tag{6.6}$$

assuming that $f : [0, T] \times [\varepsilon, +\infty[$ is measurable w.r.t. t and Lipschitz continuous w.r.t. w.

Relying on the assumptions **(A3)**, the lower semicontinuity of the weighted irrigation cost proved in Theorem 5.1 can now be extended to this more general case.

Theorem 6.1 *Consider a sequence $(\chi_n)_{n\geq 1}$ of irrigation plans, all satisfying the assumption* **(A2)***, pointwise converging to an irrigation plan χ. Assume that the function ψ satisfies the conditions in* **(A1)***, while f satisfies* **(A3)***. Then the corresponding weighted costs satisfy*

$$\mathcal{E}^{W,\psi}(\chi) \ \leq \ \liminf_{n\to\infty} \mathcal{E}^{W,\psi}(\chi_n). \tag{6.7}$$

Proof. We shall follow step by step all the arguments in the proof of Theorem 5.1, and indicate only the modifications which are needed to cover this more general case.

Steps **1–3** and **5–13** do not make any reference to the function f, and thus remain valid without any change.

In step **4** we considered an approximate family of weights $\widetilde{W}_i$ yielding almost the same cost as the original ones. That construction must here be somewhat refined, approximating all ε_0-good paths in χ with polygonal lines. That step is now replaced by

4′. By the properties of f, there exist constants L, κ such that

$$|f(v,w_1) - f(v,w_2)| \ \leq \ L\,|w_1 - w_2|, \qquad \text{for all } w_1, w_2 > \varepsilon_0\,, \quad |v| \leq 1, \tag{6.8}$$

$$|f(v,w)| \ \leq \ \kappa \qquad \text{for all } w \leq w_{max}, \quad |v| \leq 1, \tag{6.9}$$

Here w_{max} denotes the maximum weight over all ε_0-good paths of χ.

Let $\gamma_1, \ldots, \gamma_N$ be the elementary ε_0-good paths in the irrigation plan χ, determined by the Path Splitting Algorithm **(PSA)**, and let $\varepsilon > 0$ be given. By choosing $\delta, \delta_0 > 0$ sufficiently small, the following holds.

For each $\gamma_i : [a_i, b_i] \mapsto \mathbb{R}^d$, consider any set of intermediate times s_k as in (5.31) with

$$s_k - s_{k-1} \ < \ \delta_0 \qquad \text{for all } k \in \{1, \ldots, n(i)\}\,.$$

Define the piecewise constant multiplicity function $\widetilde{m}_i(t)$ as in (5.32). Next, let $J_i \subset [a_i, b_i]$ by any measurable subset with $meas(J_i) \leq \delta_0$ and define

$$\tilde{f}_i(t,\omega) \ \doteq \ \begin{cases} f(\dot{\gamma}_i(t), \omega) - \delta_0 & \text{if} \quad t \notin J_i, \\ \qquad 0 & \text{if} \quad t \in J_i\,. \end{cases} \tag{6.10}$$

Then the corresponding weights $\widetilde{W}_i$ still satisfy (5.34).

In the present case, an additional approximation will be useful. Namely, we refine the partition (5.31) of the interval $[a_i, b_i]$ by inserting points

$$a_i \ = \ \tau_0 \ < \ \tau_1 \ < \ \cdots \ < \ \tau_{m(i)} \ = \ b_i\,, \tag{6.11}$$

and replace γ_i with a path $\gamma_i^\diamondsuit$ which is affine on each sub-interval $[\tau_{j-1}, \tau_j]$ and satisfies

$$\gamma_i(\tau_j) \ = \ \gamma^\diamondsuit(\tau_j) \qquad \text{for all } j.$$

Then we choose $h > 0$ small enough and set

$$J_i \ \doteq \ \bigcup_j \ [\tau_j - h, \tau_j + h].$$

By choosing the partition (6.11) sufficiently fine, and $h > 0$ sufficiently small, we can achieve

$$\kappa L \cdot \sup_j |\tau_j - \tau_{j-1}| \;<\; \frac{\delta_0}{2}\,. \tag{6.12}$$

Setting

$$m_i^\diamondsuit(s) \;\doteq\; m_i(s) - \delta_0\,, \tag{6.13}$$

the weights $W^\diamondsuit$ now satisfy

$$\frac{d}{dt}[W^\diamondsuit(t) - m^\diamondsuit(t)] \;=\; -\tilde f(\dot\gamma^\diamondsuit(t), W^\diamondsuit(t)). \tag{6.14}$$

If $\delta_0, h > 0$ are chosen small enough, then the corresponding weight functions $W_i^\diamondsuit : [a_i, b_i] \mapsto \mathbb{R}_+$ satisfy the analogue of (5.34), namely

$$\|W_i^{\varepsilon_0} - W_i^\diamondsuit\|_{\mathbf{L}^1([a_i,b_i])} \;<\; \varepsilon, \qquad\qquad |W_i^{\varepsilon_0}(a_i+) - W_i^\diamondsuit(a_i+)| \;<\; \varepsilon\,. \tag{6.15}$$

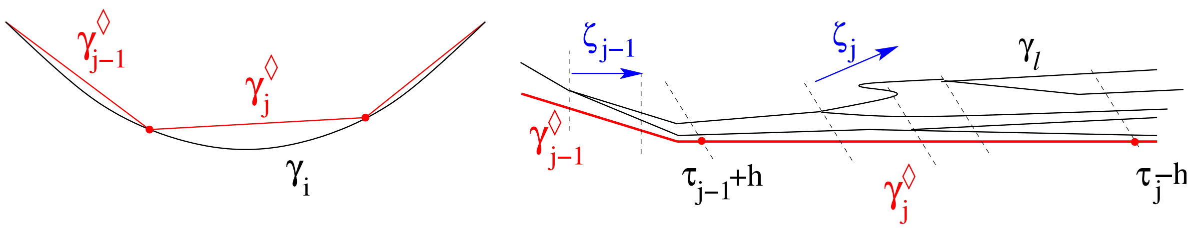

Figure 8: Left: approximating an ε_0-good path γ_i in the irrigation plan χ with a polygonal $\gamma^\diamondsuit$. Right: the weight $W^\diamondsuit$ along the segment with endpoints $\gamma^\diamondsuit(\tau_{j-1}+h')$, $\gamma^\diamondsuit(\tau_j - h')$ is compared with the sum of weights along corresponding ε_n-good paths $\gamma_l^\sharp$ of χ_n. Differently from the case illustrated in Fig. 7, we now compare weights at points having the same inner product with the vector ζ_j, introduced at (6.18).

Toward a proof of Theorem 6.1, the heart of the matter is to achieve the inequalities (5.70)-(5.71) in steps **14-15**. Since Lemma 5.5 no longer applies, a different argument must now be developed. The last three steps **14–16** in the proof of Theorem 5.1 can be replaced by the steps below.

14′. We wish to compare the weights $W_i^\diamondsuit$ with a sum of the weights along the corresponding ε_n-good elementary paths in the approximating irrigation plans χ_n. This will be done separately on each subinterval $[\tau_{j-1}, \tau_j]$, where the tangent vector $\dot\gamma^\diamondsuit(t) = v_j$ is constant.

As in step **9** of the previous proof, in connection with $\gamma_i\Big|_{[\tau_{j-1},\tau_j]}$ we can determine a family $\Gamma_{i,j}$ of ε_n-good paths $\gamma_\ell : [a_\ell, b_\ell] \mapsto \mathbb{R}^d$ in the irrigation plan χ_n, which approach γ_i as $n \to \infty$ (see Fig. 8, right). By construction, the corresponding weight and multiplicity functions $W_\ell, m_\ell : [a_\ell, b_\ell] \mapsto \mathbb{R}$ are non-increasing and satisfy

$$\frac{d}{ds}[W_\ell(s) - m_\ell(s)] \;=\; -f(\dot\gamma_\ell(s), W_\ell(s)). \tag{6.16}$$

As remarked in the previous sections, each elementary path γ_ℓ of χ_n is contained in some maximal path $\widehat\gamma_q : [a_q, b_q] \mapsto \mathbb{R}^d$. The set of elementary paths is partially ordered by setting

$$\gamma_\ell \;\prec\; \gamma_r$$

if γ_ℓ and γ_r are contained in the same maximal path, and $b_\ell \leq a_r$. The set of paths which bifurcate from the tip of γ_ℓ is defined as

$$\mathcal{O}(\ell) \doteq \{r\,;\ \ \gamma_\ell \prec \gamma_r \ \text{ and } \ b_\ell = a_r\}.$$

At the endpoint of γ_ℓ, by construction we have

$$W_\ell(b_\ell) - m_\ell(b_\ell) = \sum_{r\in\mathcal{O}(\ell)} [W_r(a_r) - m_r(a_r)]. \tag{6.17}$$

Of course, the above definition here implies $a_r = b_\ell$.

15′. Throughout this step we fix an elementary path γ_i of the limit irrigation plan χ. To shorten notation, we shall thus drop the index i and simply write $\gamma^\diamondsuit = \gamma_i^\diamondsuit$, $W^\diamondsuit = W_i^\diamondsuit$, etc. To establish a comparison we observe that, by the convexity and positive homogeneity of the map $v \mapsto f(v, W^\diamondsuit(\tau_j))$, there exists a vector $\zeta_j \in \mathbb{R}^d$ such that

$$f(v_j, W^\diamondsuit(\tau_j)) = \langle v_j, \zeta_j\rangle, \qquad\qquad f(v, W^\diamondsuit(\tau_j)) \geq \langle v, \zeta_j\rangle \quad \text{for all } v \in \mathbb{R}^d. \tag{6.18}$$

Notice that, if f is smooth, then $\zeta_j = \nabla_v f(v_j, W^\diamondsuit(\tau_j))$ is simply the gradient of f w.r.t. the variable $v \in \mathbb{R}^d$. If f is convex but not smooth, then ζ_j can be any sub-gradient.

By construction, for $t \in [\tau_{j-1}, \tau_j]$ the derivative $\dot\gamma^\diamondsuit(t) = v_j$ is constant. From (6.14), using (6.10) and then (6.8)-(6.9) and (6.12), one obtains

$$\begin{aligned} \frac{d}{dt}[W^\diamondsuit(t) - m^\diamondsuit(t)] &= -\tilde f(v_j, W^\diamondsuit(t)) \\ &\geq -f(v_j, W^\diamondsuit(\tau_j)) + \delta_0 - \kappa L|\tau_j - t| \geq -\langle v_j, \zeta_j\rangle + \frac{\delta_0}{2}. \end{aligned} \tag{6.19}$$

Let $\gamma_\ell : [a_\ell, b_\ell] \mapsto \mathbb{R}^d$ be any ε_n-good path in χ_n. For $t \in [\tau_{j-1} + h, \tau_j - h]$ we define

$$s_\ell(t) \doteq \inf\Big\{s \geq a_\ell\,;\ \ \langle \zeta_j, \gamma_\ell(s)\rangle \geq \langle \zeta_j, \gamma^\diamondsuit(t)\rangle\Big\}, \tag{6.20}$$

and consider the set of indices

$$I(t) \doteq \{\ell\,;\ \ a_\ell < s_\ell(t) < b_\ell\}.$$

By an approximation argument, we can assume that each γ_ℓ is a polygonal, with $\langle\dot\gamma_\ell(t), \zeta_j\rangle \neq 0$ for a.e. $t \in [a_\ell, b_\ell]$. This implies

$$\frac{d}{dt}s_\ell(t) = \frac{\langle v_j, \zeta_j\rangle}{\langle\dot\gamma_\ell(s_\ell(t)), \zeta_j\rangle} > 0 \tag{6.21}$$

for all except finitely many times t. We can now estimate

$$\begin{aligned} \frac{d}{dt}\sum_{\ell\in I(t)} [W_\ell(s_\ell(t)) - m_\ell(s_\ell(t))] &= -\sum_{\ell\in I(t)} \frac{\langle v_j, \zeta_j\rangle}{\langle\dot\gamma_\ell(s_\ell(t)), \zeta_j\rangle}\cdot f(\dot\gamma_\ell(s_\ell(t)), W_\ell(s_\ell(t))) \\ &\leq -\sum_{\ell\in I(t)} \langle v_j, \zeta_j\rangle \frac{f(\dot\gamma_\ell(s_\ell(t)), W^\diamondsuit(\tau_j))}{\langle\dot\gamma_\ell(s_\ell(t)), \zeta_j\rangle}\cdot \min\left\{\frac{W_\ell(s_\ell(t))}{W^\diamondsuit(\tau_j)}, 1\right\} \\ &\leq -\sum_{\ell\in I(t)} \langle v_j, \zeta_j\rangle \min\left\{\frac{W_\ell(s_\ell(t))}{W^\diamondsuit(\tau_j)}, 1\right\}. \end{aligned} \tag{6.22}$$

Here the first identity follows from (6.4)-(6.5) and (6.21), while the second inequality is a consequence of (6.18) and of the concavity of the map $W \mapsto f(v, W)$. The third inequality follows from (6.18). Notice that, at points where two or more paths bifurcate, by (6.17) the sum $\sum_{\ell \in I(t)} [W_\ell(t) - m_\ell(t)]$ remains continuous. However, this sum will have downward jumps at points where one of the maps $t \mapsto s_\ell(t)$ is discontinuous.

16′. We are now ready to describe the comparison argument that replaces the estimates in step **15** of the proof of Theorem 5.1. As in the previous proof, for each large n we can identify a finite family of ε_n-good paths γ_ℓ in χ_n which converge to the elementary path γ_i of χ. In particular, for $n \geq 1$ large enough, we can assume

$$\sum_\ell m_\ell(s_\ell(t)) \ \geq \ m^\diamondsuit(t) \qquad \text{for all } t \in [\tau_{j-1} + h\,,\ \tau_j - h], \quad j = 1, 2, \ldots, m(i). \tag{6.23}$$

By backward induction, assume that

$$[W^\diamondsuit(\tau_j + h) - m^\diamondsuit(\tau_j + h)] \ \leq \sum_{\ell \in I(\tau_j + h)} [W_\ell(s_\ell(\tau_j + h)) - m_\ell(s_\ell(\tau_j + h)]. \tag{6.24}$$

Since $\tilde{f} = 0$ for $t \in [\tau_j - h, \tau_j + h]$ while all differences $W_\ell - m_\ell$ are non-increasing, this immediately yields

$$[W^\diamondsuit(\tau_j - h) - m^\diamondsuit(\tau_j - h)] \ = \ [W^\diamondsuit(\tau_j + h) - m^\diamondsuit(\tau_j + h)] \ \leq \sum_{\ell \in I(\tau_j - h)} [W_\ell(s_\ell(\tau_j - h)) - m_\ell(s_\ell(\tau_j - h)]. \tag{6.25}$$

For $t \in [\tau_{j-1} + h\,,\ \tau_j - h]$, we claim that the quantity

$$\Phi(t) \ \doteq \ [W^\diamondsuit(t) - m^\diamondsuit(t)] - \sum_{\ell \in I(t)} [W_\ell(s_\ell(t)) - m_\ell(s_\ell(t))]$$

remains non-positive. Indeed, comparing the derivatives in (6.19) and (6.22) one obtains

$$\frac{d}{dt}\Phi(t) \ \geq \ -\langle v_j, \zeta_j \rangle + \frac{\delta_0}{2} + \sum_{\ell \in I(t)} \langle v_j, \zeta_j \rangle \min\left\{ \frac{W_\ell(s_\ell(t))}{W^\diamondsuit(\tau_j)}\,,\, 1 \right\}. \tag{6.26}$$

Set

$$t^* \ \doteq \ \sup\{t \leq \tau_j\,;\ \ \Phi(t) > 0\}.$$

If $t^* > \tau_{j-1} + h$, to derive a contradiction we will show that

$$\left.\frac{d\Phi}{dt}\right|_{t=t^*} \ > \ 0. \tag{6.27}$$

Toward this goal we observe that, by continuity, $\Phi(t^*) = 0$. Since the map $t \mapsto W^\diamondsuit(t)$ is decreasing, by (6.23) we obtain

$$W^\diamondsuit(\tau_j) - \sum_\ell W_\ell(s_\ell(t^*)) \ \leq \ W^\diamondsuit(t^*) - \sum_\ell W_\ell(s_\ell(t^*)) \ = \ m^\diamondsuit(t^*) - \sum_\ell m_\ell(s_\ell(t^*)) \ \leq \ 0. \tag{6.28}$$

By (6.26) this implies

$$\left.\frac{d\Phi}{dt}\right|_{t=t^*} \ \geq \ -\langle v_j, \zeta_j \rangle + \frac{\delta_0}{2} + \sum_{\ell \in I(t^*)} \langle v_j, \zeta_j \rangle \min\left\{ \frac{W_\ell(s_\ell(t^*))}{W^\diamondsuit(\tau_j)}\,,\, 1 \right\} \ \geq \ \frac{\delta_0}{2}. \tag{6.29}$$

We thus conclude that $\Phi(t) \leq 0$ for all $t \geq \tau_{j-1}$. This achieves the key inductive step, showing that the inequality (6.24) remains valid with j replaced by $j-1$.

The remainder of the proof follows the same arguments used for Theorem 5.1. □

Remark 6.2 By a minor modification of the previous arguments, the above results can be further extended to the case where $f = f(x, v, W)$ depends continuously also on the variable $x \in \mathbb{R}^d$.

7 Optimal weighted irrigation plans

Given a positive, bounded Radon measure μ on $\mathbb{R}^d$, we define

$$\mathcal{I}^{W,\psi}(\mu) \doteq \inf_{\chi} \mathcal{E}^{W,\psi}(\chi).$$

where the infimum is taken over all irrigation plans for the measure μ. Relying on the lower semicontinuity of the weighted irrigation cost, proved in Theorems 5.1 and 6.1, we can now prove the existence of an optimal irrigation plan.

Theorem 7.1 *Let μ be a positive, bounded Radon measure on $\mathbb{R}^d$. Let f satisfy the assumptions in* **(A3)** *while ψ satisfies* **(A1)**. *If μ admits an irrigation plan with bounded weighted cost, then there exists an irrigation plan with minimum weighted cost.*

Proof. Let $M = \mu(\mathbb{R}^d)$ and let $(\chi_n)_{n\geq 1}$ be a minimizing sequence of irrigation plans for μ, so that

$$\lim_{n\to\infty} \mathcal{E}^{W,\psi}(\chi_n) = \mathcal{I}^{W,\psi}(\mu). \tag{7.1}$$

Since $f \geq 0$, by construction the weights are larger than the corresponding multiplicities. Namely, for every ξ, t and $n \geq 1$ one has $W_n(\xi, t) \geq m_n(\xi, t)$. Since the costs in (7.1) are bounded, we deduce

$$\int_0^M \int_0^{+\infty} |\dot{\chi}_n(\xi, t)|\, dt\, d\xi \leq C$$

for some constant C and all $n \geq 1$.

By the sequential compactness of traffic plans (see for example Proposition 3.27 in [1]), we can extract a subsequence $(\chi_{n_j})_{j\geq 1}$ pointwise converging to an irrigation plan χ. The lower semicontinuity result proved in Theorem 6.1 yields

$$\mathcal{E}^{W,\psi}(\chi) \leq \liminf_{n\to\infty} \mathcal{E}^{W,\psi}(\chi_n) = \mathcal{I}^{W,\psi}(\mu). \tag{7.2}$$

Hence χ achieves the minimum weighted cost. □

We conclude by proving the lower semicontinuity of the weighted irrigation cost, w.r.t. weak convergence of measures.

Theorem 7.2 *Let f satisfy the assumptions in* **(A3)** *while ψ satisfies* **(A1)**. *Let $(\mu_n)_{n\geq 1}$ be a sequence of bounded positive measures, with uniformly bounded supports, weakly converging to μ. Then*

$$\mathcal{I}^{W,\psi}(\mu) \ \leq \ \liminf_{n\to\infty} \mathcal{I}^{W,\psi}(\mu_n). \tag{7.3}$$

Proof. Without loss of generality, one can assume

$$\liminf_{n\to\infty} \ \mathcal{I}^{W,\psi}(\mu_n) \ \doteq \ K \ < \ +\infty\,. \tag{7.4}$$

Let χ_n an optimal irrigation plan of μ_n, so that $\mathcal{E}^{W,\psi}(\chi_n) = \mathcal{I}^{W,\psi}(\mu_n)$ for every $n \geq 1$. As in the previous proof, by sequential compactness we can extract a subsequence $(\chi_{n_j})_{j\geq 1}$ pointwise converging to an irrigation plan χ. A standard argument shows that χ provides an irrigation plan for the measure μ. Using Theorem 5.1 we conclude

$$\mathcal{I}^{W,\psi}(\mu) \ \leq \ \mathcal{E}^{W,\psi}(\chi) \ \leq \ \liminf_{n\to\infty} \ \mathcal{E}^{W,\psi}(\chi_n) \ = \ \liminf_{n\to\infty} \mathcal{I}^{W,\psi}(\mu_n). \tag{7.5}$$

□

Acknowledgment. This research was partially supported by NSF grant DMS-1714237, "Models of controlled biological growth".

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