A 2-Dimensional Shape Optimization Problem for Tree Branches

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Abstract

The paper is concerned with a shape optimization problem, where the functional to be maximized describes the total sunlight collected by a distribution of tree leaves, minus the cost for transporting water and nutrient from the base of trunk to all the leaves. In a 2-dimensional setting, the solution is proved to be unique, and explicitly determined.

Keywords: shape optimization, sunlight functional, branched transport.

MSC: 49Q10, 49Q20.

1 Introduction

In the recent papers [7, 9] two functionals were introduced, measuring the amount of light collected by the leaves, and the amount of water and nutrients collected by the roots of a tree. In connection with a ramified transportation cost [1, 14, 18], these lead to various optimization problems for tree shapes.

Quite often, optimal solutions to problems involving a ramified transportation cost exhibit a fractal structure [2, 3, 4, 12, 15, 16, 17]. In the present note we analyze in more detail the optimization problem for tree branches proposed in [7], in the 2-dimensional case. In this simple setting, the unique solution can be explicitly determined. Instead of being fractal, its shape reminds of a solar panel.

The present analysis was partially motivated by the goal of understanding phototropism, i.e., the tendency of plant stems to bend toward the source of light. Our results indicate that this behavior cannot be explained purely in terms of maximizing the amount of light collected by the leaves (Fig. 1). Apparently, other factors must have played a role in the evolution of this trait, such as the competition among different plants. See [6] for some results in this direction.

The remainder of this paper is organized as follows. In Section 2 we review the two functionals defining the shape optimization problem and state the main results. Proofs are then worked out in Sections 3 to 5. Finally, in Section 6 we show the sharpness of the assumptions used in Theorem 2.8, and discuss various possible extensions.

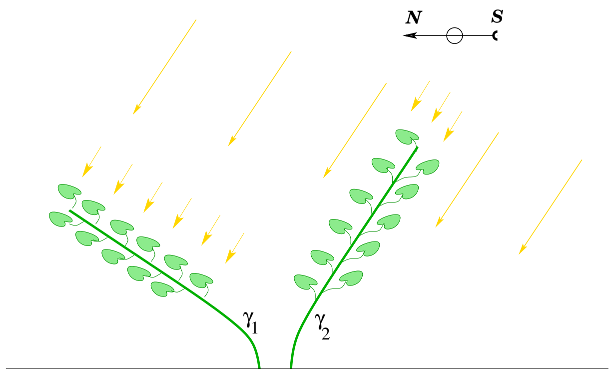

Figure 1: A stem γ_1 perpendicular to the sun rays is optimally shaped to collect the most light. For the stem γ_2 bending toward the light source, the upper leaves put the lower ones in shade.

2 Statement of the main results

We begin by reviewing the two functionals considered in [7, 9].

2.1 A sunlight functional

Let μ be a positive, bounded Radon measure on $\mathbb{R}^d_+ \doteq \{(x_1, x_2, \ldots, x_d)\,;\ x_d \geq 0\}$. Thinking of μ as the density of leaves on a tree, we seek a functional $\mathcal{S}(\mu)$ describing the total amount of sunlight absorbed by the leaves. Fix a unit vector

$$\mathbf{n} \ \in \ S^{d-1} \ \doteq \ \{x \in \mathbb{R}^d\,;\ \ |x| = 1\},$$

and assume that all light rays come parallel to $\mathbf{n}$. Call $E^\perp_\mathbf{n}$ the $(d-1)$-dimensional subspace perpendicular to $\mathbf{n}$ and let $\pi_\mathbf{n} : \mathbb{R}^d \mapsto E^\perp_\mathbf{n}$ be the perpendicular projection. Each point $\mathbf{x} \in \mathbb{R}^d$ can thus be expressed uniquely as

$$\mathbf{x} \ = \ \mathbf{y} + s\mathbf{n} \tag{2.1}$$

with $\mathbf{y} \in E^\perp_\mathbf{n}$ and $s \in \mathbb{R}$.

On the perpendicular subspace $E^\perp_\mathbf{n}$ consider the projected measure $\mu^\mathbf{n}$, defined by setting

$$\mu^\mathbf{n}(A) \ = \ \mu\Big(\big\{x \in \mathbb{R}^d\,;\ \ \pi_\mathbf{n}(x) \in A\big\}\Big). \tag{2.2}$$

Call $\Phi^\mathbf{n}$ the density of the absolutely continuous part of $\mu^\mathbf{n}$ w.r.t. the $(d-1)$-dimensional Lebesgue measure on $E^\perp_\mathbf{n}$.

Definition 2.1 *The total amount of sunlight from the direction* $\mathbf{n}$ *captured by a measure* μ *on* $\mathbb{R}^d$ *is defined as*

$$\mathcal{S}^{\mathbf{n}}(\mu) \doteq \int_{E_{\mathbf{n}}^{\perp}} \Big(1 - \exp\big\{-\Phi^{\mathbf{n}}(y)\big\}\Big)\, dy\,. \tag{2.3}$$

More generally, given an integrable function $\eta \in \mathbf{L}^1(S^{d-1})$*, the total sunlight absorbed by* μ *from all directions is defined as*

$$\mathcal{S}^{\eta}(\mu) \doteq \int_{S^{d-1}} \left(\int_{E_{\mathbf{n}}^{\perp}} \Big(1 - \exp\big\{-\Phi^{\mathbf{n}}(y)\big\}\Big)\, dy \right) \eta(\mathbf{n})\, d\mathbf{n}\,. \tag{2.4}$$

In the formula (2.4), $\eta(\mathbf{n})$ accounts for the intensity of light coming from the direction $\mathbf{n}$.

Remark 2.2 According to the above definition, the amount of sunlight $\mathcal{S}^{\mathbf{n}}(\mu)$ captured by the measure μ only depends on its projection $\mu^{\mathbf{n}}$ on the subspace perpendicular to $\mathbf{n}$. In particular, if a second measure $\widetilde{\mu}$ is obtained from μ by shifting some of the mass in a direction parallel to $\mathbf{n}$, then $\mathcal{S}^{\mathbf{n}}(\widetilde{\mu}) = \mathcal{S}^{\mathbf{n}}(\mu)$.

2.2 Optimal irrigation patterns

Consider a positive Radon measure μ on $\mathbb{R}^d$ with total mass $M = \mu(\mathbb{R}^d)$, and let $\Theta = [0, M]$. We think of $\xi \in \Theta$ as a Lagrangian variable, labeling a water particle.

Definition 2.3 *A measurable map*

$$\chi : \Theta \times \mathbb{R}_+ \mapsto \mathbb{R}^d \tag{2.5}$$

is called an **admissible irrigation plan** *for the measure* μ *if*

(i) For every $\xi \in \Theta$*, the map* $t \mapsto \chi(\xi, t)$ *is Lipschitz continuous. More precisely, for each* ξ *there exists a stopping time* $T(\xi)$ *such that, calling*

$$\dot{\chi}(\xi, t) = \frac{\partial}{\partial t}\,\chi(\xi, t)$$

the partial derivative w.r.t. time, one has

$$\big|\dot{\chi}(\xi, t)\big| = \begin{cases} 1 & \text{for a.e. } t \in \big[0, T(\xi)\big], \\ 0 & \text{for } t > T(\xi). \end{cases} \tag{2.6}$$

(ii) At time $t = 0$ *all particles are at the origin:* $\chi(\xi, 0) = \mathbf{0}$ *for all* $\xi \in \Theta$.

(iii) The push-forward of the Lebesgue measure on $[0, M]$ *through the map* $\xi \mapsto \chi(\xi, T(\xi))$ *coincides with the measure* μ*. In other words, for every open set* $A \subset \mathbb{R}^d$ *there holds*

$$\mu(A) = \text{meas}\Big(\{\xi \in \Theta\,;\ \ \chi(\xi, T(\xi)) \in A\}\Big). \tag{2.7}$$

One may think of $\chi(\xi, t)$ as the position of the water particle ξ at time t.

To define the corresponding transportation cost, we first compute how many particles travel through a point $x \in \mathbb{R}^d$. This is described by

$$|x|_\chi \doteq \text{meas}\Big(\big\{\xi \in \Theta\,;\quad \chi(\xi,t) = x \quad \text{for some} \quad t \geq 0\big\}\Big). \tag{2.8}$$

We think of $|x|_\chi$ as the *total flux going through the point* x. Following [13, 14], we consider

Definition 2.4 *For a given* $\alpha \in [0,1]$*, the total cost of the irrigation plan* χ *is*

$$\mathcal{E}^\alpha(\chi) \doteq \int_\Theta \left(\int_0^{T(\xi)} \big|\chi(\xi,t)\big|_\chi^{\alpha-1}\, dt \right) d\xi. \tag{2.9}$$

The α**-irrigation cost** *of a measure* μ *is defined as*

$$\mathcal{I}^\alpha(\mu) \doteq \inf_\chi \mathcal{E}^\alpha(\chi), \tag{2.10}$$

where the infimum is taken over all admissible irrigation plans for the measure μ*.*

Remark 2.5 Sometimes it is convenient to consider more general irrigation plans where, in place of (2.6), for a.e. $t \in [0, T(\xi)]$ the speed satisfies $|\dot\chi(\xi,t)| \leq 1$. In this case, the cost (2.9) is replaced by

$$\mathcal{E}^\alpha(\chi) \doteq \int_\Theta \left(\int_0^{T(\xi)} \big|\chi(\xi,t)\big|_\chi^{\alpha-1} \,|\dot\chi(\xi,t)|\, dt \right) d\xi. \tag{2.11}$$

Of course, one can always re-parameterize each trajectory $t \mapsto \chi(\xi,t)$ by arc-length, so that (2.6) holds. This does not affect the cost (2.11).

Remark 2.6 In the case $\alpha = 1$, the expression (2.9) reduces to

$$\mathcal{E}^\alpha(\chi) \doteq \int_\Theta \left(\int_{\mathbb{R}_+} |\dot\chi_t(\xi,t)|\, dt \right) d\xi = \int_\Theta [\text{total length of the path } \chi(\xi,\cdot)]\, d\xi\,.$$

Of course, this length is minimal if every path $\chi(\cdot,\xi)$ is a straight line, joining the origin with $\chi(\xi, T(\xi))$. Hence

$$\mathcal{I}^\alpha(\mu) \doteq \inf_\chi \mathcal{E}^\alpha(\chi) = \int_\Theta |\chi(\xi, T(\xi))|\, d\xi = \int |x|\, d\mu\,.$$

On the other hand, when $\alpha < 1$, moving along a path which is traveled by few other particles comes at a high cost. Indeed, in this case the factor $\big|\chi(\xi,t)\big|_\chi^{\alpha-1}$ becomes large. To reduce the total cost, it is thus convenient that many particles travel along the same path.

For the basic theory of ramified transport we refer to the monograph [1]. For future use, we recall that optimal irrigation plans satisfy

Single Path Property: *If $\chi(\xi,\tau) = \chi(\xi',\tau')$ for some $\xi,\xi' \in \Theta$ and $0 < \tau \le \tau'$, then*

$$\chi(\xi,t) \;=\; \chi(\xi',t) \qquad \textit{for all } t \in [0,\tau]. \tag{2.12}$$

Another property that will be repeatedly used in the sequel is the following.

Lemma 2.7 *Let χ be an admissible irrigation plan for the measure μ. Let $\mathbf{C} \subset \mathbb{R}^d$ be a closed convex set containing the origin, and let $p_{\mathbf{C}} : \mathbb{R}^d \mapsto \mathbf{C}$ be the perpendicular projection. Consider the projected measure $\widetilde{\mu}$ supported on $\mathbf{C}$, obtained as the push-forward of μ by the map $p_{\mathbf{C}}$. Then the composed map $\widetilde{\chi}(\xi,t) \;=\; p_{\mathbf{C}}(\chi(\xi,t))$ is an admissible irrigation plan for the measure $\widetilde{\mu}$. Moreover, for every $\alpha \in [0,1]$ one has*

$$\mathcal{E}^\alpha(\widetilde{\chi}) \;\le\; \mathcal{E}^\alpha(\chi). \tag{2.13}$$

If $\widetilde{\mu} \neq \mu$, then the above inequality is strict.

Proof. The first statement is obvious. As in Lemma 5.15 in [1], the inequality (2.13) follows from the fact that, in the projected irrigation plan, the length of particle trajectories decreases while the multiplicity increases. Indeed,

$$\begin{aligned}
\mathcal{E}^\alpha(\widetilde{\chi}) \;&\doteq\; \int_\Theta \left(\int_0^{T(\xi)} \big|\widetilde{\chi}(\xi,t)\big|_{\widetilde{\chi}}^{\alpha-1} \left| \frac{d}{dt}\widetilde{\chi}(\xi,t)\right| dt \right) d\xi \\
&=\; \int_\Theta \left(\int_0^{T(\xi)} \big|(p_{\mathbf{C}}\circ\chi)(\xi,t)\big|_{p_{\mathbf{C}}\circ\chi}^{\alpha-1} \left| \frac{d}{dt}(p_{\mathbf{C}}\circ\chi)(\xi,t)\right| dt \right) d\xi \\
&\le\; \int_\Theta \left(\int_0^{T(\xi)} \big|\chi(\xi,t)\big|_{\chi}^{\alpha-1} \, |\dot\chi(\xi,t)|\, dt \right) d\xi \;=\; \mathcal{E}^\alpha(\chi).
\end{aligned}$$

□

2.3 The general optimization problem for branches.

Combining the two functionals (2.4) and (2.10), one can formulate an optimization problem for the shape of branches:

(OPB) Given a light intensity function $\eta \in \mathbf{L}^1(S^{d-1})$ and two constants $c > 0$, $\alpha \in [0,1]$, find a positive measure μ supported on R^d_+ that maximizes the payoff

$$\mathcal{S}^\eta(\mu) - c\,\mathcal{I}^\alpha(\mu). \tag{2.14}$$

2.4 Optimal branches in dimension $d = 2$.

We consider here the optimization problem for branches in the planar case $d = 2$. We assume that the sunlight comes from a single direction $\mathbf{n} = (\cos\theta_0, \sin\theta_0)$, so that the sunlight functional takes the form (2.3). Moreover, as irrigation cost we take (2.10), for some fixed $\alpha \in]0,1]$. For a given constant $c > 0$, this leads to the problem

$$\text{maximize:} \quad \mathcal{S}^{\mathbf{n}}(\mu) - c\mathcal{I}^{\alpha}(\mu), \tag{2.15}$$

over all positive measures μ supported on the half space $\mathbb{R}^2_+ \doteq \{x = (x_1, x_2)\,; \quad x_2 \geq 0\}$. To fix ideas, we shall assume that $0 \leq \theta_0 \leq \pi/2$. Our main goal is to prove that for this problem the "solar panel" configuration shown in Fig. 2 is optimal, namely:

Theorem 2.8 *In dimension $d = 2$, assume that $0 \leq \theta_0 \leq \pi/2$ and $1/2 \leq \alpha \leq 1$. Then the optimization problem (2.15) has a unique solution. The optimal measure is supported along two rays, namely*

$$\text{Supp}(\mu) \;\subset\; \Big\{(r\cos\theta, r\sin\theta)\,; \quad r \geq 0, \quad \textit{either } \theta = 0 \textit{ or } \theta = \theta_0 + \frac{\pi}{2}\Big\} \;\doteq\; \Gamma_0 \cup \Gamma_1\,. \tag{2.16}$$

When $0 < \alpha < 1/2$, the same conclusion holds if either $\theta_0 = 0$, or else the angle θ_0 satisfies

$$\sin\theta_0 \;\geq\; 1 - 2^{2\alpha - 1}. \tag{2.17}$$

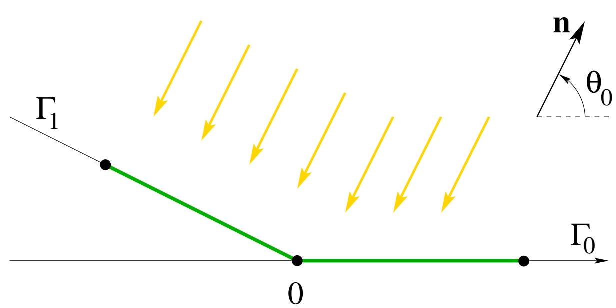

Figure 2: When the light rays impinge from a fixed direction $\mathbf{n}$, the optimal distribution of leaves is supported on the two rays Γ_0 and Γ_1.

In the case $\alpha = 1$ the result is straightforward. Indeed, for any measure μ we can consider its projection $\widetilde{\mu}$ on $\Gamma_0 \cup \Gamma_1$, obtained by shifting the mass in the direction parallel to the vector $\mathbf{n}$. In other words, for $x \in \mathbb{R}^2$ call $\phi^{\mathbf{n}}(x)$ the unique point in $\Gamma_0 \cup \Gamma_1$ such that $\phi^{\mathbf{n}}(x) - x$ is parallel to $\mathbf{n}$. Then let $\widetilde{\mu}$ be the push-forward of the measure μ w.r.t. $\phi^{\mathbf{n}}$. Since this projection satisfies $|\phi^{\mathbf{n}}(x)| \leq |x|$ for every $x \in \mathbb{R}^2_+$, the transportation cost decreases. On the other hand, by Remark 2.2 the sunlight captured remains the same. We conclude that

$$\mathcal{S}^{\mathbf{n}}(\widetilde{\mu}) - c\mathcal{I}^1(\widetilde{\mu}) \;\geq\; \mathcal{S}^{\mathbf{n}}(\mu) - c\mathcal{I}^1(\mu),$$

with strict inequality if μ is not supported on $\Gamma_0 \cup \Gamma_1$.

In the case $0 < \alpha < 1$, the result is not so obvious. A proof of Theorem 2.8 will be worked out in Sections 3 and 4.

Having proved that the optimal measure μ is supported on the two rays $\Gamma_0 \cup \Gamma_1$, the density of μ w.r.t. one-dimensional measure can then be determined using the necessary conditions derived in [6]. Indeed, the density u_1 of μ along the ray Γ_1 provides a solution to the scalar optimization problem

$$\text{maximize:}\quad \mathcal{J}_1(u) \;\doteq\; \int_0^{+\infty} \big(1-e^{-u(s)}\big)\,ds - c\int_0^{+\infty}\left(\int_s^{+\infty} u(r)\,dr\right)^{\alpha} ds\,, \tag{2.18}$$

among all non-negative functions $u:\mathbb{R}_+\mapsto\mathbb{R}_+$. Here s is the arc-length variable along Γ_1. Similarly, the density u_0 of μ along the ray Γ_0 provides a solution to the problem

$$\text{maximize:}\quad \mathcal{J}_0(u) \;\doteq\; \int_0^{+\infty} \sin\theta_0\,\big(1-e^{-u(s)/\sin\theta_0}\big)\,ds - c\int_0^{+\infty}\left(\int_s^{+\infty} u(r)\,dr\right)^{\alpha} ds\,. \tag{2.19}$$

We write (2.18) in the form

$$\text{maximize:}\quad \mathcal{J}_1(u) \;\doteq\; \int_0^{+\infty} \Big[\big(1-e^{-u(s)}\big) - cz^{\alpha}\Big]ds\,, \tag{2.20}$$

subject to

$$\dot z \;=\; -\,u, \qquad z(+\infty) \;=\; 0. \tag{2.21}$$

The necessary conditions for optimality (see for example [8, 11]) now yield

$$u(s) \;=\; \operatorname*{argmax}_{\omega\geq 0}\Big\{-e^{-\omega} - \omega q(s)\Big\} \;=\; -\ln q(s), \tag{2.22}$$

where the dual variable q satisfies

$$\dot q \;=\; c\alpha z^{\alpha-1}, \qquad\qquad q(0)=0. \tag{2.23}$$

Notice that, by (2.22), $u>0$ only if $q<1$. Combining (2.21) with (2.23) one obtains an ODE for the function $q\mapsto z(q)$, with $q\in[0,1]$. Namely

$$\frac{dz(q)}{dq} \;=\; \frac{z^{1-\alpha}\ln q}{c\alpha}, \qquad\qquad z(1)=0. \tag{2.24}$$

This equation admits the explicit solution

$$z(q) \;=\; c^{-1/\alpha}\,[1+q\ln q - q]^{1/\alpha}\,. \tag{2.25}$$

Inserting (2.25) in (2.23), we obtain an implicit equation for $q(s)$:

$$s \;=\; \frac{1}{\alpha c^{1/\alpha}}\int_0^{q(s)} [1+t\ln t - t]^{\frac{1-\alpha}{\alpha}}\,dt. \tag{2.26}$$

In turn, the density $u(s)$ of the optimal measure μ along Γ_1, as a function of the arc-length s, is recovered from (2.22). Notice that this measure is supported only on an initial interval $[0,\ell_1]$, determined by

$$\ell_1 \;=\; \frac{1}{\alpha c^{1/\alpha}}\int_0^1 [1+s\ln s - s]^{\frac{1-\alpha}{\alpha}}\,ds.$$

In particular, the total mass M_1 along the ray Γ_1 is computed setting $q=0$ in (2.25), namely

$$M_1 \;=\; \int_0^{\ell_1} u(s)\,ds \;=\; z(0) \;=\; c^{-1/\alpha}. \tag{2.27}$$

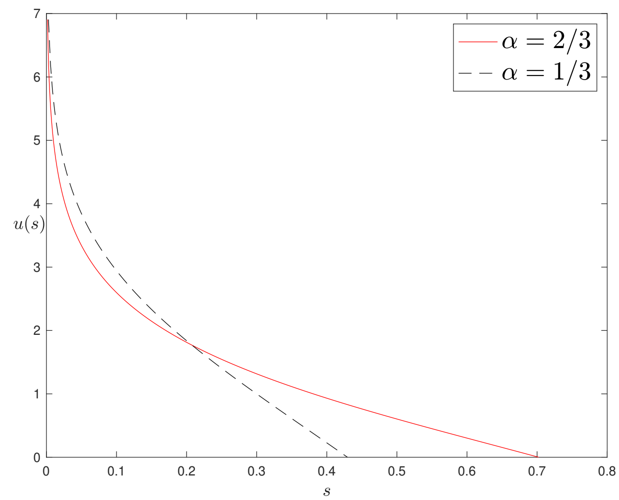

Figure 3: Density profile $u(s)$ for $s \in [0, \ell_1]$ along the ray Γ_1 for $c = 1$ and $\alpha = 2/3, 1/3$.

The density of the optimal measure along the ray Γ_0 is computed in an entirely similar way. In this case, the equations (2.22) and (2.26) are replaced respectively by

$$u(s) \;=\; -\,(\sin\theta_0)\ln q(s),$$

$$s \;=\; \frac{(\sin\theta_0)^{\frac{1-\alpha}{\alpha}}}{\alpha c^{1/\alpha}} \int_0^{q(s)} \left[1 + t\ln t - t\right]^{\frac{1-\alpha}{\alpha}} dt.$$

Again, the condition $u(s) > 0$ implies $q(s) < 1$. Along Γ_0, the optimal measure μ is supported on an initial interval $[0, \ell_0]$, where

$$\ell_0 \;=\; \frac{(\sin\theta_0)^{\frac{1-\alpha}{\alpha}}}{\alpha c^{1/\alpha}} \int_0^1 \left[1 + s\ln s - s\right]^{\frac{1-\alpha}{\alpha}} ds.$$

The total mass M_0 along the ray Γ_0 is now computed by

$$M_0 \;=\; \int_0^{\ell_0} u(s)\,ds \;=\; \int_0^1 u(s(q))\frac{ds(q)}{dq}\,dq. \tag{2.28}$$

Inserting the expressions for u and $\frac{ds}{dq}$ along Γ_0 we find that the above integral equals

$$\int_0^1 (-\sin\theta_0 \ln q)\frac{(\sin\theta_0)^{\frac{1-\alpha}{\alpha}}}{\alpha c^{1/\alpha}}\,(1 + q\ln q - q)^{\frac{1-\alpha}{\alpha}}\,dq \;=\; -\left(\frac{\sin\theta_0}{c}\right)^{1/\alpha}\Big[(1 + q\ln q - q)^{1/\alpha}\Big]_0^1,$$

leading to

$$M_0 \;=\; \left(\frac{\sin\theta_0}{c}\right)^{1/\alpha}. \tag{2.29}$$

2.5 The case $\alpha = 0$.

In the analysis of the optimization problem **(OPB)**, the case $\alpha = 0$ stands apart. Indeed, the general theorem on the existence of an optimal shape proved in [7] does not cover this case.

When $\alpha = 0$, a measure μ is irrigable only if it is concentrated on a set of dimension ≤ 1. When this happens, in any dimension $d \geq 3$ we have $\mathcal{S}^\eta(\mu) = 0$ and the optimization problem is trivial. The only case of interest occurs in dimension $d = 2$. In the following, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$.

Theorem 2.9 *Let $\alpha = 0$, $d = 2$. Let $\eta \in \mathbf{L}^1(S^1)$ and define*

$$K \doteq \max_{|\mathbf{w}|=1} \int_{\mathbf{n}\in S^1} \Big|\langle \mathbf{w}, \mathbf{n}\rangle\Big| \eta(\mathbf{n})\, d\mathbf{n}. \tag{2.30}$$

(i) If $K > c$, then the optimization problem **(OPB)** *has no solution, because the supremum of all possible payoffs is $+\infty$.*

(ii) If $K \leq c$, then the maximum payoff is zero, which is trivially achieved by the zero measure.

A proof will be given in Section 5.

3 Properties of optimal branch configurations

In this section we consider the optimization problem (2.15) in dimension $d = 2$. As a step toward the proof of Theorem 2.8, some properties of optimal branch configurations will be derived.

By the result in [7] we know that an optimal measure μ exists and has bounded support, contained in $\mathbb{R}^2_+ \doteq \{(x_1, x_2)\,;\ \ x_2 \geq 0\}$. Call $M = \mu(\mathbb{R}^2_+)$ the total mass of μ and let $\chi : [0, M] \times \mathbb{R}_+ \mapsto \mathbb{R}^2_+$ be an optimal irrigation plan for μ.

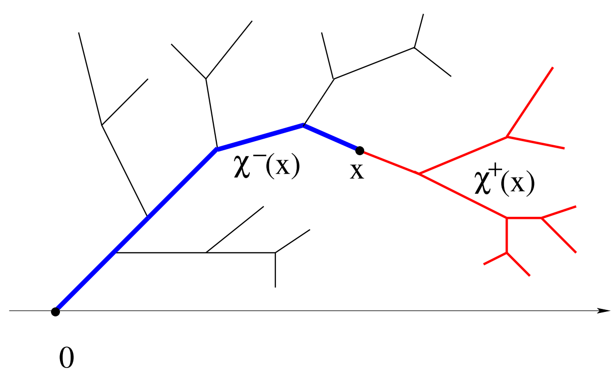

Figure 4: According to the definition (3.3), the set $\chi^-(x)$ is a curve joining the origin to the point x. The set $\chi^+(x)$ is a subtree, containing all paths that start from x.

Next, consider the set of all branches, namely

$$\mathcal{B} \doteq \{x \in \mathbb{R}^2_+\,;\ \ |x|_\chi > 0\}. \tag{3.1}$$

By the single path property, we can introduce a partial ordering among points in $\mathcal{B}$. Namely, for any $x, y \in \mathcal{B}$ we say that $x \preceq y$ if for any $\xi \in [0, M]$ we have the implication

$$\chi(t, \xi) = y \qquad \Longrightarrow \qquad \chi(t', \xi) = x \qquad \text{for some } \ t' \in [0, t]. \tag{3.2}$$

This means that all particles that reach the point y pass through x before getting to y.

For a given $x \in \mathcal{B}$ the subsets of points $y \in \mathcal{B}$ that precede or follow x are defined as

$$\chi^-(x) \doteq \{y \in \mathcal{B}\,;\;\; y \preceq x\}, \qquad\qquad \chi^+(x) \doteq \{y \in \mathcal{B}\,;\;\; x \preceq y\}, \tag{3.3}$$

respectively (see Fig. 4).

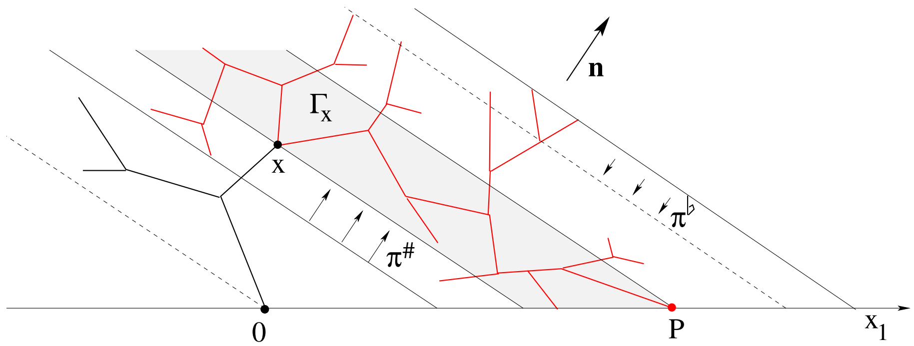

Figure 5: If the set $\chi^+(x)$ is not contained in the slab Γ_x (the shaded region), by taking the perpendicular projections $\pi^\sharp$ and $\pi^\flat$ we obtain another irrigation plan with strictly lower cost, which irrigates a new measure $\widetilde{\mu}$ gathering exactly the same amount of sunlight. Notice that here P is the point in the closed set $\overline{\chi^+(x)} \cap \mathbb{R}\mathbf{e}_1$ which has the largest inner product with $\mathbf{n}$.

We begin by deriving some properties of the sets $\chi^+(x)$. Introducing the unit vectors $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$, we denote by $\mathbb{R}\mathbf{e}_1$ the set of points on the x_1-axis. As before, $\mathbf{n} = (\cos\theta_0, \sin\theta_0)$ denotes the unit vector in the direction of the sunlight. Throughout the following, the closure of a set A is denoted by $\overline{A}$, while $\langle\cdot,\cdot\rangle$ denotes an inner product.

Lemma 3.1 *Let the measure μ provide an optimal solution to the problem (2.15), and let χ be an optimal irrigation plan for μ. Then, for every $x \in \mathcal{B}$, one has*

$$\chi^+(x) \;\subset\; \Gamma_x \;\doteq\; \Big\{y \in \mathbb{R}^2_+\,;\;\; \langle\mathbf{n}, y\rangle \;\in\; [a_x, b_x]\Big\}, \tag{3.4}$$

where $a_x \;\doteq\; \langle\mathbf{n}, x\rangle$, while b_x is defined as follows.

- *If $\overline{\chi^+(x)} \cap \mathbb{R}\mathbf{e}_1 = \emptyset$, then $b_x = a_x = \langle\mathbf{n}, x\rangle$.*
- *If $\overline{\chi^+(x)} \cap \mathbb{R}\mathbf{e}_1 \neq \emptyset$, then*

$$b_x \;=\; \max\;\{a_x, b'_x\}, \qquad\qquad b'_x \;\doteq\; \sup\Big\{\langle\mathbf{n}, z\rangle\,;\;\; z \in \overline{\chi^+(x)} \cap \mathbb{R}\mathbf{e}_1\Big\}.$$

Proof. The right-hand side of (3.4) is illustrated in Fig. 5. To prove the lemma, consider the set of all particles that pass through x, namely

$$\Theta_x \;\doteq\; \big\{\xi \in [0, M]\,;\;\; \chi(\tau, \xi) = x \;\text{ for some }\; \tau \geq 0\big\}.$$

1. We first show that, by the optimality of the solution,

$$\langle\mathbf{n}\,,\,\chi(\xi, t)\rangle \;\geq\; a_x \qquad\qquad \text{for all }\; \xi \in \Theta_x\,,\; t \geq \tau. \tag{3.5}$$

Indeed, consider the perpendicular projection on the half plane

$$\pi^\sharp : \mathbb{R}^2 \mapsto S^\sharp \doteq \{y \in \mathbb{R}^2 ;\ \langle \mathbf{n}, y\rangle \ \geq\ a_x\}.$$

Define the projected irrigation plan

$$\chi^\sharp(t,\xi) \doteq \begin{cases} \pi^\sharp \circ \chi(t,\xi) & \text{if } \ \xi \in \Theta_x\,, \quad t \geq \tau, \\ \chi(t,\xi) & \text{otherwise.} \end{cases}$$

Then the new measure $\mu^\sharp$ irrigated by $\chi^\sharp$ is still supported on $\mathbb{R}^2_+$ and has exactly the same projection on $E_{\mathbf{n}}^\perp$ as μ. Hence it gathers the same amount of sunlight. However, if the two irrigation plans do not coincide a.e., then the cost of $\chi^\sharp$ is strictly smaller than the cost of χ, contradicting the optimality assumption.

2. Next, we show that

$$\langle \mathbf{n}\,,\, \chi(\xi,t)\rangle \ \leq\ b_x \qquad\qquad \text{for all } \ \xi \in \Theta_x \ \ t \geq \tau. \tag{3.6}$$

Indeed, call

$$b'' \doteq \sup\ \Big\{\langle \mathbf{n}, z\rangle\,;\quad z \in \chi^+(x)\Big\}.$$

If $b'' \leq b_x$, we are done. In the opposite case, by a continuity and compactness argument we can find $\delta > 0$ such that the following holds. Introducing the perpendicular projection on the half plane

$$\pi^\flat : \mathbb{R}^2 \mapsto S^\flat \doteq \{y \in \mathbb{R}^2 ;\ \langle \mathbf{n}, y\rangle \ \leq\ b'' - \delta\},$$

one has

$$\big\{\pi^\flat(y)\,;\quad y \in \chi^+(x)\big\} \ \subseteq\ \mathbb{R}^2_+\,. \tag{3.7}$$

Similarly as before, define the projected irrigation plan

$$\chi^\flat(t,\xi) \doteq \begin{cases} \pi^\flat \circ \chi(t,\xi) & \text{if } \ \xi \in \Theta_x\,, \quad t \geq \tau, \\ \chi(t,\xi) & \text{otherwise.} \end{cases}$$

Then the new measure $\mu^\flat$ irrigated by $\chi^\flat$ is supported on $\mathbb{R}^2_+ \cap S^\flat$ and has exactly the same projection on $E_{\mathbf{n}}^\perp$ as μ. Hence it gathers the same amount of sunlight. However, if the two irrigation plans do not coincide a.e., then the cost of $\chi^\flat$ is strictly smaller than the cost of χ, contradicting the optimality assumption. This completes the proof of the Lemma. □

Based on the previous lemma, we now consider the set

$$\mathcal{B}^* \doteq \{x \in \mathcal{B}\,;\ \ \overline{\chi^+(x)} \cap \mathbb{R}\mathbf{e}_1 \neq \emptyset\}. \tag{3.8}$$

It will be convenient to rotate coordinates by an angle of $\pi/2-\theta_0$, and choose new coordinates (z_1, z_2) oriented as in Fig. 6. In these new coordinates, the direction of sunlight becomes vertical, while the positive x_1-axis corresponds to the line

$$\mathbf{S} \doteq \big\{(z_1,z_2)\,;\quad z_1 \geq 0\,, \quad z_2 = -\lambda z_1\big\}, \qquad \text{where} \quad \lambda = \tan\theta_0\,. \tag{3.9}$$

Calling $\big(z_1(\xi,t), z_2(\xi,t)\big)$ the corresponding coordinates of the point $\chi(\xi,t)$, from Lemma 3.1 we immediately obtain

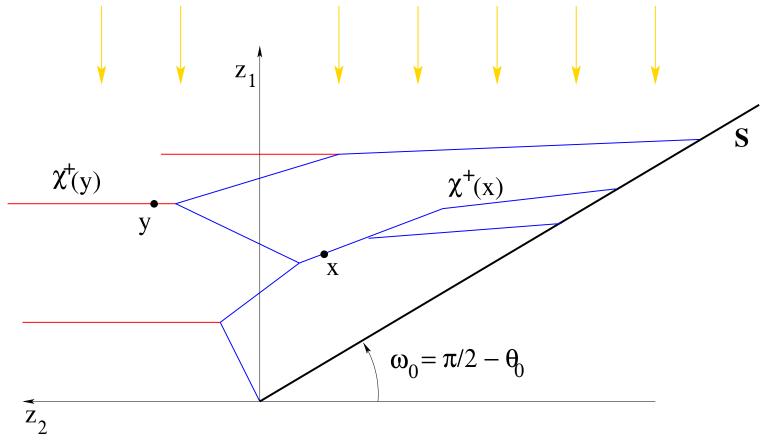

Figure 6: After a rotation of coordinates, the sunlight comes from the vertical direction. Here the blue lines correspond to the set $\mathcal{B}^*$ in (3.8).

Lemma 3.2 *Let χ be an optimal irrigation plan for a solution to (2.15). Then*

(i) For every $\xi \in [0, M]$, the map $t \mapsto z_1(\xi, t)$ is non-decreasing.

(ii) If $\bar z = (\bar z_1, \bar z_2) \notin \mathcal{B}^*$, then $\chi^+(\bar z)$ is contained in a horizontal line. Namely,

$$\chi^+(\bar z) \subset \{(\bar z_1, s)\,; \quad s \in \mathbb{R}\}. \tag{3.10}$$

To make further progress, we define

$$z_1^{\max} \doteq \sup\{z_1\,;\ (z_1,\, z_2) \in \mathcal{B}^*\}.$$

Moreover, on the interval $[0, z_1^{max}[$ we consider the function

$$\varphi(z_1) \doteq \sup\{s\,;\ \ (z_1, s) \in \mathcal{B}^*\}. \tag{3.11}$$

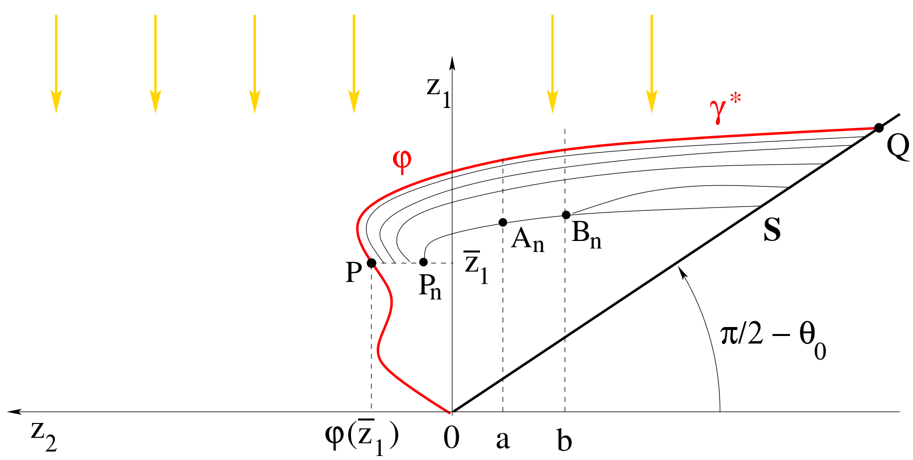

Figure 7: The construction used in the proof of Lemma 3.3.

Lemma 3.3 *For every $z_1 \in [0, z_1^{max}[$, the supremum $\varphi(z_1)$ is attained as a maximum.*

Proof. 1. Assume that, on the contrary, for some $\bar{z}_1$ the supremum is not a maximum. In this case, as shown in Fig. 7, there exist a sequence of points $P_n \to P$ with $P_n = (\bar{z}_1, s_n)$, $P = (\bar{z}_1, \bar{z}_2)$, $s_n \uparrow \bar{z}_2$. Here $P_n \in \mathcal{B}^*$ for every $n \geq 1$ but $P \notin \mathcal{B}^*$. Without loss of generality, we can assume that all points P_n lie on distinct branches (i.e., there is no couple $m \neq n$ such that $P_m \prec P_n$ or $P_n \prec P_m$). Otherwise, we could group all these points into finitely many horizontal branches. But since every horizontal branch intersects the horizontal line through P in a closed interval, this would already imply that the supremum in (3.11) is attained.

2. Choose two values a, b such that

$$-\lambda \bar{z}_1 \;<\; b \;<\; a \;<\; \varphi(\bar{z}_1).$$

By construction, for every $n \geq 1$ the set $\overline{\chi^+(P_n)}$ intersects $\mathbf{S}$. Therefore we can find points

$$P_n \;\prec\; A_n \;\prec\; B_n$$

all in $\mathcal{B}^*$, with

$$A_n \;=\; (t_n, a), \qquad B_n \;=\; (t'_n, b), \qquad\qquad \bar{z}_1 \leq t_n \leq t'_n \leq z_1^{max}\,.$$

3. Since the total mass M is finite, we have

$$\sum_{n \geq 1} |A_n|_\chi \;\leq\; M \;\doteq\; \mu(\mathbb{R}^2_+).$$

We can thus find N large enough so that the amount of particles $\varepsilon_N \;\doteq\; |A_N|_\chi$ going through A_N is so small that

$$c(b-a)\alpha\, \varepsilon_N^{\alpha-1} \;>\; 1. \tag{3.12}$$

Consider the modified transport plan $\widetilde{\chi}$, obtained from χ by removing all particles that go through the point B_N. More precisely, $\widetilde{\chi}$ is the restriction of χ to the domain

$$\widetilde{\Theta} \;\doteq\; \Theta \setminus \{\xi\,;\;\; \chi(\xi, \tau) = B_N \quad \text{for some } \tau \geq 0\}.$$

Let $\widetilde{\mu}$ be the measure irrigated by $\widetilde{\chi}$.

Calling $\sigma_0 > 0$ the total amount of particles going through B_N, since $\widetilde{\mu} \leq \mu$, the total amount of sunlight gathered by the measure $\widetilde{\mu}$ satisfies

$$\mathcal{S}^{\mathbf{n}}(\mu) - \mathcal{S}^{\mathbf{n}}(\widetilde{\mu}) \;\leq\; (\mu - \widetilde{\mu})(\mathbb{R}^2) \;=\; \sigma_0. \tag{3.13}$$

We now estimate the reduction in the transportation cost, achieved by replacing μ with $\widetilde{\mu}$. Let $\gamma : [s_A, s_B] \mapsto \mathbb{R}^2$ be an arclength parameterization of the branch from A_N to B_N. Along this arc, when all the particles reaching B_N are removed, the multiplicity (2.8) decreases from $|\gamma(s)|_\chi$ to $|\gamma(s)|_\chi - \sigma_0$. The transportation cost through γ is reduced in the amount

$$\begin{aligned}
&\int_{s_A}^{s_B} |\gamma(s)|_\chi^\alpha ds - \int_{s_A}^{s_B} \big(|\gamma(s)|_\chi - \sigma_0\big)^\alpha ds \\
&\qquad \geq\; (s_B - s_A)\alpha \sup_s |\gamma(s)|_\chi^{\alpha-1} \cdot \sigma_0 \;\geq\; (b-a)\,\alpha \varepsilon_N^{\alpha-1} \sigma_0\,.
\end{aligned}$$

This yields

$$\mathcal{I}^\alpha(\widetilde{\mu}) \;\leq\; \mathcal{E}^\alpha(\widetilde{\mu}) \;\leq\; \mathcal{I}^\alpha(\mu) - (b-a)\,\alpha\varepsilon_N^{\alpha-1}\sigma_0\,. \tag{3.14}$$

If (3.12) holds, combining (3.13)-(3.14) one obtains

$$\mathcal{S}^{\mathbf{n}}(\mu) - c\,\mathcal{I}^\alpha(\mu) \;<\; \mathcal{S}^{\mathbf{n}}(\widetilde{\mu}) - c\,\mathcal{I}^\alpha(\widetilde{\mu}).$$

Hence the measure μ is not optimal. This contradiction proves the lemma. □

By the previous result, the graph of φ is contained in one single maximal trajectory of the transport plan χ. As in Figure 8, we denote this curve by γ, which provides the left boundary of the set $\mathcal{B}^*$.

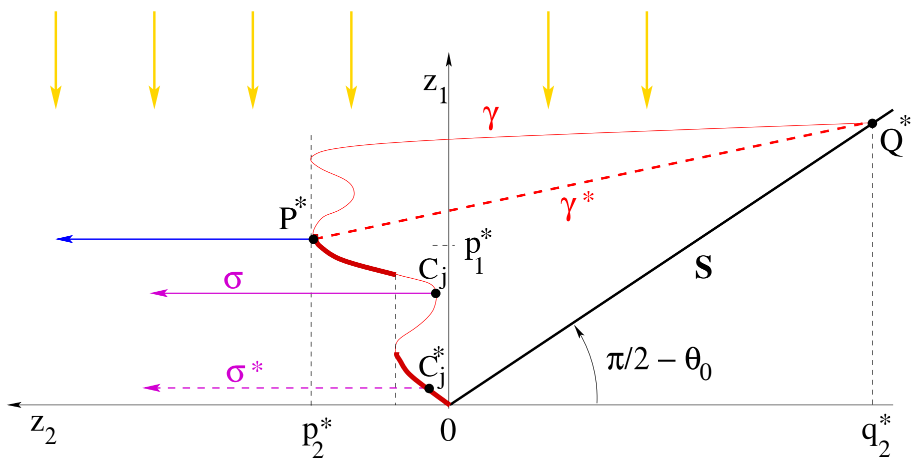

Figure 8: The thick portions of the curve γ are the only points where a left bifurcation can occur. If a horizontal branch σ bifurcates from C_j, all the mass on this branch can be shifted downward to another branch σ^* bifurcating from C_j^*. Furthermore, if some portion of the path γ between P^* and Q lies above the segment γ^* joining these two points, we can take a projection of γ on γ^*. In both cases, the transportation cost is strictly reduced.

Along the curve γ, we now consider the set of points $C_j = (z_{1,j}, z_{2,j})$ where some horizontal branch bifurcates on the left. A property of such points is given below.

Lemma 3.4 *In the above setting, for every j, one has*

$$\varphi(s) \;<\; z_{2,j} \qquad \textit{for all } s < z_{1,j}\,. \tag{3.15}$$

Proof. If (3.15) fails, there exists another point $C_j^* = (z_{1,j}^*, z_{2,j})$ along the curve γ, with $z_{1,j}^* < z_{1,j}$. We can now replace the measure μ by another measure $\widetilde{\mu}$ obtained as follows. All the mass lying on the horizontal half-line $\{(z_{1,j}, s)\,;\;\; s \geq z_{2,j}\}$ is shifted downward on the half-line $\{(z_{1,j}^*, s)\,;\;\; s \geq z_{2,j}\}$. Since the functional $\mathcal{S}^{\mathbf{n}}$ is invariant under vertical shifts, we have $\mathcal{S}^{\mathbf{n}}(\widetilde{\mu}) = \mathcal{S}^{\mathbf{n}}(\mu)$. However, the transportation cost is strictly smaller: $\mathcal{I}^\alpha(\widetilde{\mu}) < \mathcal{I}^\alpha(\mu)$. This contradicts the optimality of μ. □

Next, as shown in Fig. 8, we consider a point $P^* = (p_1^*, p_2^*) \in \gamma$ where the component z_2 achieves its maximum, namely

$$p_2^* \;=\; \max\{z_2\,;\;\; (z_1, z_2) \in \gamma\} \;\geq\; 0. \tag{3.16}$$

Notice that such a maximum exists because γ is a continuous curve, starting at the origin. If this maximum is attained at more than one point, we choose the one with smallest z_1-coordinate, so that

$$p_1^* \;=\; \min\{z_1\,;\quad (z_1, p_2^*) \in \gamma\}. \tag{3.17}$$

Moreover, call

$$q_2^* \;\doteq\; \inf\{z_2\,;\quad (z_1, z_2) \in \mathrm{Supp}(\mu)\},$$

and let $Q^* = (q_1^*, q_2^*) \in \mathbf{S}$ be the point on the ray $\mathbf{S}$ whose second coordinate is q_2^*. Recalling the notation of Lemma 3.1, we note that $q_1^* = b_x$ for $x = (0,0)$. We claim that, by the optimality of the solution, all paths of the irrigation plan χ must lie within the convex set

$$\Sigma^* \;\doteq\; \{(z_1, z_2)\,;\quad z_1 \in [0, q_1^*], \quad z_2 \geq q_2^*\}.$$

Otherwise, call $\pi^* : \mathbb{R}^2 \mapsto \Sigma^*$ the perpendicular projection on the convex set Σ^*, and let μ^* be the push-forward of μ by the map π^*. By Lemma 2.7 the composed map

$$\chi^*(\xi, t) \;\doteq\; \pi^*\big(\chi(\xi, t)\big)$$

is an irrigation plan for μ^* and satisfies $\mathcal{E}^\alpha(\chi^*) < \mathcal{E}^\alpha(\chi)$. Hence

$$\mathcal{S}^{\mathbf{n}}(\mu^*) \;=\; \mathcal{S}^{\mathbf{n}}(\mu), \qquad \mathcal{I}^\alpha(\mu^*) \;\leq\; \mathcal{E}^\alpha(\chi^*) \;<\; \mathcal{E}^\alpha(\chi) \;=\; \mathcal{I}^\alpha(\mu),$$

contradicting the optimality assumption.

By a projection argument we now show that, in an optimal solution, all the particle paths remain below the segment γ^* with endpoints P^* and Q^*.

Lemma 3.5 *In the above setting, let*

$$\gamma^* \;=\; \big\{(z_1, z_2)\,;\; z_1 = a + bz_2\,, \qquad z_2 \in [q_2^*, p_2^*]\big\}$$

be the segment with endpoints P^*, Q^*. If

$$(\xi, t) \mapsto \chi(\xi, t) \;=\; \big(z_1(\xi, t), z_2(\xi, t)\big) \tag{3.18}$$

is an optimal irrigation plan for the problem (2.15), then for a.e. $\xi \in \Theta$ we have the implication

$$z_2(\xi, t) \;\in\; [q_2^*, p_2^*] \qquad \Longrightarrow \qquad z_1(\xi, t) \;\leq\; a + b\, z_2(\xi, t). \tag{3.19}$$

Proof. 1. It suffices to show that the maximal curve γ lies below γ^*. If this is not the case, consider the set of particles which go through the point P^* and then move to the right of P^*, namely

$$\Omega^* \;=\; \Big\{\xi \in [0, M]\,;\quad \chi(\xi, t^*) = P^* \;\text{ for some } t^* \geq 0, \quad z_2(\xi, t) < p_2^* \;\text{ for } t > t^*\Big\}. \tag{3.20}$$

Notice that, by the single path property (see Section 7.1 in [1]), all these particles follow the same path from the origin to P^*. Hence the length t^* of this path is the same for all $\xi \in \Omega^*$.

2. Consider the convex region below γ^*, defined by

$$\Sigma \doteq \Big\{(z_1, z_2)\,;\quad 0 \leq z_1 \leq a + bz_2\,, \qquad z_2 \in [q_2^*, p_2^*]\Big\}.$$

Let $\pi : \mathbb{R}^2 \mapsto \Sigma$ be the perpendicular projection. Then the irrigation plan

$$\chi^\dagger(\xi, t) \doteq \begin{cases} \pi\Big(\chi(\xi, t)\Big) & \text{if } \xi \in \Omega^*,\ t > t^*, \\ \chi(\xi, t) & \text{otherwise,} \end{cases} \tag{3.21}$$

has total cost strictly smaller than χ. Indeed, for all x and a.e. ξ, t we have

$$\big|\pi(x)\big|_{\chi^\dagger} \geq |x|_\chi\,, \qquad \big|\dot\chi^\dagger(\xi, t)\big| \leq \big|\dot\chi(\xi, t)\big|. \tag{3.22}$$

Notice that, in (3.22), equality can hold for a.e. ξ, t only in the case where $\chi = \chi^\dagger$.

3. We now observe that the perpendicular projection on Σ can decrease the z_2-component. As a consequence, the measures μ and $\mu^\dagger$ irrigated by χ and $\chi^\dagger$ may have a different projections on the z_2 axis. If this happens, we may have $\mathcal{S}^{\mathbf{n}}(\mu) \neq \mathcal{S}^{\mathbf{n}}(\mu^\dagger)$.

To address this issue, we observe that all particles $\xi \in \Omega^*$ satisfy $\chi^\dagger(\xi, t^*) = \chi(\xi, t^*) = P^*$. In terms of the z_1, z_2 coordinates, this implies

$$z_2^\dagger(\xi, t^*) = z_2(\xi, t^*) = p_2^*, \qquad z_2^\dagger(\xi, T(\xi)) \leq z_2(\xi, T(\xi)) < p_2^*. \tag{3.23}$$

By continuity, for each $\xi \in \Omega^*$ we can find a stopping time $\tau(\xi) \in [t^*, T(\xi)]$ such that

$$z_2^\dagger(\xi, \tau(\xi)) = z_2(\xi, T(\xi)).$$

Call $\widetilde{\chi}$ the truncated irrigation plan, such that

$$\widetilde{\chi}(\xi, t) \doteq \begin{cases} \chi^\dagger(\xi, t) & \text{if } \xi \in \Omega^*,\ t \leq \tau(\xi), \\ \chi(\xi, \tau(\xi)) & \text{if } \xi \in \Omega^*,\ t \geq \tau(\xi), \\ \chi(\xi, t) & \text{if } \xi \notin \Omega^*. \end{cases} \tag{3.24}$$

By construction, the measures μ and $\widetilde{\mu}$ irrigated by χ and $\widetilde{\chi}$ have exactly the same projections on the z_2 axis. Hence $\mathcal{S}^{\mathbf{n}}(\widetilde{\mu}) = \mathcal{S}^{\mathbf{n}}(\mu)$. On the other hand, the corresponding costs satisfy

$$\mathcal{E}^\alpha(\widetilde{\chi}) \leq \mathcal{E}^\alpha(\chi^\dagger) < \mathcal{E}^\alpha(\chi).$$

This contradicts optimality, thus proving the lemma. □

4 Proof of Theorem 2.8

In this section we give a proof of Theorem 2.8. We recall that the functional (2.15) to be maximized is the difference between a payoff, i.e. the sunlight $\mathcal{S}^{\mathbf{n}}(\mu)$ absorbed by the measure μ, and the ramified transportation cost $c\mathcal{I}^\alpha(\mu)$. Together with the measure μ, at various steps of the proof we shall construct a second measure $\widetilde{\mu}$, obtained by shifting part of the mass in a direction parallel to $\mathbf{n}$. As in Remark 2.2, this will not change the sunlight gathered: $\mathcal{S}^{\mathbf{n}}(\widetilde{\mu}) = \mathcal{S}^{\mathbf{n}}(\mu)$. On the other hand, is the irrigation cost of $\widetilde{\mu}$ is strictly smaller: $\mathcal{I}^\alpha(\widetilde{\mu}) < \mathcal{I}^\alpha(\mu)$, we shall conclude that μ is not optimal.

As shown in Fig. 8, let $P^* = (p_1^*, p_2^*)$ be the point defined at (3.16). We consider two cases:

(i) $P^* = 0 \in \mathbb{R}^2$,

(ii) $P^* \neq 0$.

Assume that case (i) occurs. Then, by Lemma 3.4, the only branch that can bifurcate to the left of γ must lie on the z_2-axis. Moreover, by Lemma 3.5, the path γ cannot lie above the segment with endpoints P^*, Q^*. Therefore, the restriction of the measure μ to the half space $\{z_2 \leq 0\}$ is supported on the line $\mathbf{S}$. Combining these two facts we achieve the conclusion of the theorem.

The remainder of the proof will be devoted to showing that the case (ii) cannot occur, because it would contradict the optimality of the solution.

To illustrate the heart of the matter, we first consider the elementary configuration shown in Fig. 9, left, where all trajectories are straight lines. Water is first transported from the origin to the point P^*. Then, an amount $\sigma > 0$ is moved horizontally to the point Q, while an amount $\kappa > 0$ is moved to P_1. This yields a transport plan χ, which irrigates the measure μ consisting of a mass σ at Q and a mass κ at P_1.

Next, as shown in Fig. 9, right, we consider a point P along the segment $0P^*$. A new transport plan $\widetilde{\chi}$ is defined, where water is first transported from the origin to P. Then, an amount σ is moved horizontally to a point $\widetilde{Q}$ located along the same vertical line as Q. The remaining amount κ is moved in a straight line from P to P_1. Notice that the new transport plan $\widetilde{\chi}$ now irrigates a measure $\widetilde{\mu}$ consisting of a mass σ at $\widetilde{Q}$ and a mass κ at P_1.

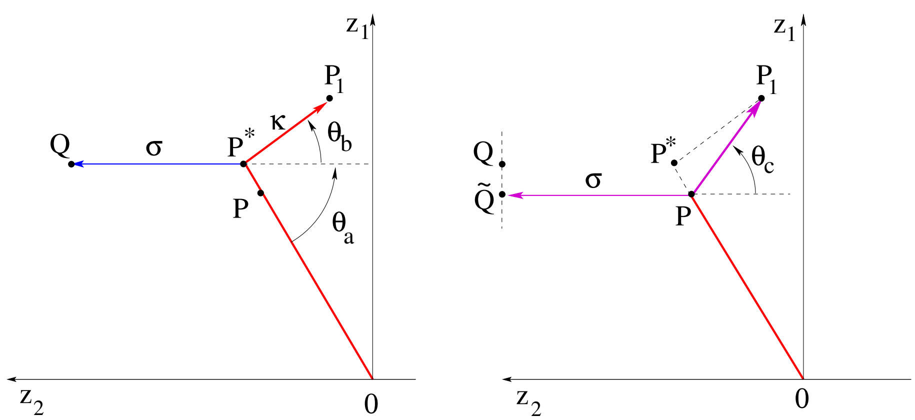

Figure 9: Left: an irrigation plan for a measure μ with two masses at Q and at P_1. Right: an irrigation plan for a modified measure $\widetilde{\mu}$ with two masses at $\widetilde{Q}$ and at P_1. The lengths of the segments PP^* and P^*P_1 will be denoted by ℓ_a, ℓ_b, respectively.

To fix ideas, we denote the lengths of the segments PP^* and P^*P_1 as

$$\ell_a \;=\; |P - P^*|, \qquad\qquad \ell_b \;=\; |P_1 - P^*|. \tag{4.1}$$

The angles between these segments and a horizontal line will be denoted by θ_a, θ_b, respectively. The next lemma provides a comparison between the costs of the two irrigation plans χ and $\widetilde{\chi}$.

Lemma 4.1 *Let $\sigma \geq 0$, $\kappa > 0$ be given, together with angles $\theta_a \in [0, \pi/2]$ and $\theta_b \in [0, \pi/2[$. Let $\chi, \widetilde{\chi}$ be the irrigation plans defined above, as shown in Fig. 9.*

(i) If $\alpha \geq 1/2$, then there exists $\varepsilon > 0$ such that $\ell_a/\ell_b \leq \varepsilon$ implies

$$\mathcal{E}^\alpha(\widetilde{\chi}) \;<\; \mathcal{E}^\alpha(\chi). \tag{4.2}$$

(ii) If $0 < \alpha < 1/2$, and if θ_b satisfies the additional bound

$$\cos\theta_b \;>\; 1 - 2^{2\alpha-1}, \tag{4.3}$$

then there exists $\varepsilon > 0$ such that $\ell_a/\ell_b \leq \varepsilon$ implies (4.2).

Proof. 1. To compute the difference between the quantities in (4.2), notice that the old transportation cost along PP^* and P^*P_1,

$$(\kappa+\sigma)^\alpha \ell_a + \kappa^\alpha \ell_b$$

is replaced by the new cost

$$\kappa^\alpha \sqrt{\ell_a^2 + \ell_b^2 - 2\ell_a\ell_b\cos(\theta_a+\theta_b)} + \sigma^\alpha \ell_a \cos\theta_a\,. \tag{4.4}$$

Notice that the last term in (4.4) accounts for the fact that an amount σ of particles need to cover a longer horizontal distance, traveling along the segment $P\widetilde{Q}$ instead of P^*Q.

The difference in the cost is thus expressed by the function

$$\begin{aligned} f(\ell_a,\ell_b) &= \mathcal{E}^\alpha(\chi) - \mathcal{E}^\alpha(\widetilde{\chi}) \\ &= (\kappa+\sigma)^\alpha \ell_a - \sigma^\alpha \ell_a \cos\theta_a + \kappa^\alpha \left[\ell_b - \sqrt{\ell_a^2+\ell_b^2-2\ell_a\ell_b\cos(\theta_a+\theta_b)}\right]. \end{aligned}$$

2. Introducing the variables

$$\varepsilon \;=\; \frac{\ell_a}{\ell_b}\,, \qquad \ell \;=\; \ell_b, \qquad \varepsilon\ell \;=\; \ell_a,$$

we obtain

$$\begin{aligned} f(\varepsilon\ell,\ell) &= \ell\left[\varepsilon(\kappa+\sigma)^\alpha - \varepsilon\sigma^\alpha\cos\theta_a + \kappa^\alpha\Big(1-\sqrt{1+\varepsilon^2-2\varepsilon\cos(\theta_a+\theta_b)}\Big)\right]. \\ &= \varepsilon\ell\Big[(\kappa+\sigma)^\alpha - \sigma^\alpha\cos\theta_a + \kappa^\alpha\cos(\theta_a+\theta_b) + \mathcal{O}(1)\cdot\varepsilon\Big]. \end{aligned} \tag{4.5}$$

Setting

$$\lambda \;=\; \frac{\sigma}{\kappa+\sigma} \;\in\; [0,1[\,,$$

we are thus led to study the function

$$F(\lambda,\theta_a,\theta_b) \;\doteq\; 1 - \lambda^\alpha\cos\theta_a + (1-\lambda)^\alpha\cos(\theta_a+\theta_b). \tag{4.6}$$

and find conditions which imply the positivity of F.

3. The function F in (4.6) can be written in terms of an inner product:

$$\begin{aligned} F(\lambda,\theta_a,\theta_b) &= 1 - \cos\theta_a\,[\lambda^\alpha - (1-\lambda)^\alpha\cos\theta_b] - \sin\theta_a(1-\lambda)^\alpha\sin\theta_b \\ &= 1 - \Big\langle\, (\cos\theta_a, \sin\theta_a)\,,\ \Big(\lambda^\alpha - (1-\lambda)^\alpha\cos\theta_b\,,\ (1-\lambda)^\alpha\sin\theta_b\Big)\Big\rangle. \end{aligned} \tag{4.7}$$

To prove that $F > 0$ it thus suffices to show that the second vector on the right hand side of (4.7) has length smaller than one, namely

$$\lambda^{2\alpha} + (1-\lambda)^{2\alpha} - 2\lambda^\alpha(1-\lambda)^\alpha \cos\theta_b \;<\; 1.$$

This inequality holds provided that

$$\cos\theta_b \;>\; \frac{\lambda^{2\alpha} + (1-\lambda)^{2\alpha} - 1}{2\lambda^\alpha(1-\lambda)^\alpha}\,. \tag{4.8}$$

Two cases must be considered. If $\alpha \geq 1/2$, then

$$\lambda^{2\alpha} + (1-\lambda)^{2\alpha} \;\leq\; 1 \qquad \text{for all } \lambda \in [0,1].$$

Hence (4.8) trivially holds for all $\theta_b < \pi/2$.

On the other hand, if $\alpha < 1/2$, consider the function

$$g(\lambda) \;\doteq\; \frac{\lambda^{2\alpha} + (1-\lambda)^{2\alpha} - 1}{2\lambda^\alpha(1-\lambda)^\alpha} \;=\; 1 + \frac{\big(\lambda^\alpha - (1-\lambda)^\alpha\big)^2 - 1}{2\lambda^\alpha(1-\lambda)^\alpha}\,.$$

We observe that, for $0 \leq \alpha \leq \frac{1}{2}$, one has

$$0 \;\leq\; g(\lambda) \;\leq\; g\Big(\frac{1}{2}\Big) \;=\; 1 - 2^{2\alpha-1}, \tag{4.9}$$

while

$$\lim_{\lambda\to 0+} g(\lambda) \;=\; \lim_{\lambda\to 1-} g(\lambda) \;=\; 0.$$

From (4.9) it now follows that the condition (4.3) guarantees that (4.8) holds, hence $F \geq 0$, as required.

Summarizing the previous analysis, for any $\lambda \in\,]0,1[$ and $\theta_a \in [0,\pi/2]$, we have proved:

(i) *When $\alpha \geq 1/2$, one has $F(\lambda,\theta_a,\theta_b) > 0$ for all $\theta_b \in [0,\pi/2[$.*

(ii) *When $0 < \alpha < 1/2$ one has $F(\lambda,\theta_a,\theta_b) > 0$ provided that θ_b satisfies the additional bound (4.3).*

4. Combining (4.5) with (4.6), we obtain

$$f(\theta_a,\theta_b) \;=\; \ell_a(\kappa+\sigma)^\alpha \left[F(\lambda,\theta_a,\theta_b) + \mathcal{O}(1)\cdot\frac{\ell_a}{\ell_b}\right]. \tag{4.10}$$

By the previous step, in both cases (i) and (ii) the right hand side of (4.10) is strictly positive provided that the ratio ℓ_a/ℓ_b is sufficiently small. This yields (4.2). □

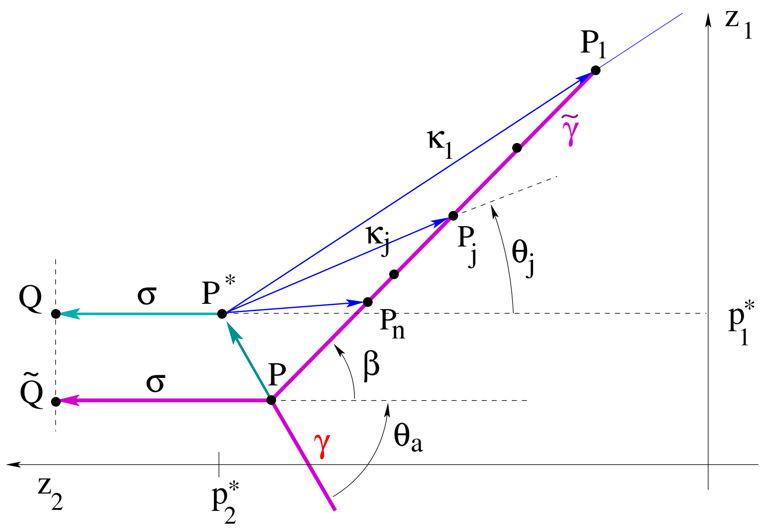

Figure 10: A more general configuration, considered in Lemma 4.2.

We now consider a more general irrigation plan χ, shown in Fig. 10. Water is transported from the origin along a straight path γ, up to the point P^*. Then the flux is split into a finite number of straight paths. One goes horizontally to the left, with flux $\sigma \geq 0$, reaching a point Q. The other paths go to the right, with fluxes $\kappa_1, \ldots, \kappa_n > 0$, at angles

$$0 \leq \theta_n < \cdots < \theta_2 < \theta_1, \tag{4.11}$$

until they reach points $P_1, \ldots, P_n$. This provides an irrigation plan for the measure concentrating a mass σ at the point Q, and masses $\kappa_1, \ldots, \kappa_n$ at the points $P_1, \ldots, P_n$. As shown in Fig. 10, we assume that all points P_i lie on the same straight line $\widetilde{\gamma}$, which intersects γ at a point P.

We compare this configuration with a modified irrigation plan $\widetilde{\chi}$ defined as follows. First, the plan $\widetilde{\chi}$ moves all the mass from the origin along the straight line γ up to the point P. Then an amount of mass σ is moved horizontally to the left, until it reaches a point $\widetilde{Q}$ on the same vertical line as Q. The remaining mass $\kappa = \kappa_1 + \cdots + \kappa_n$ is moved along the segment $\widetilde{\gamma}$, until it reaches the various points $P_1, \ldots, P_n$. Notice that $\widetilde{\chi}$ is an irrigation plan for a measure $\widetilde{\mu}$ which concentrates a mass σ at the point $\widetilde{Q}$, and masses $\kappa_1, \ldots, \kappa_n$ at the points $P_1, \ldots, P_n$. As shown in Fig. 10, we call $\theta_a \in [0, \pi/2]$ the angle between γ and a horizontal line, and let $\beta \in [0, \pi/2[$ be the angle between $\widetilde{\gamma}$ and a horizontal line.

Lemma 4.2 *Let the masses $\sigma \geq 0$ and $\kappa_1, \ldots, \kappa_n > 0$ be given, together with angles $\theta_a \in [0, \pi/2]$ and $\theta_i \in [0, \pi/2[$ as in (4.11). Let $\chi, \widetilde{\chi}$ be the irrigation plans defined above, as shown in Fig. 10.*

(i) If $\alpha \geq 1/2$, then there exists $\varepsilon > 0$ such that $0 < \beta - \theta_1 < \varepsilon$ implies

$$\mathcal{E}^\alpha(\chi) - \mathcal{E}^\alpha(\widetilde{\chi}) > 0. \tag{4.12}$$

(ii) If $0 < \alpha < 1/2$, and if θ_1 satisfies the additional bound

$$\cos\theta_1 > 1 - 2^{2\alpha-1}, \tag{4.13}$$

then there exists $\varepsilon > 0$ such that $0 < \beta - \theta_1 < \varepsilon$ implies (4.12).

Proof. 1. The left hand side of (4.12), describing the difference between the old and the new transportation cost, can be expressed as

$$|P-P^*|\left(\sigma+\sum_{j=1}^n \kappa_j\right)^\alpha + \sum_{j=1}^n \kappa_j^\alpha |P^*-P_j| - \sigma^\alpha \cos\theta_a |P-P^*| - \sum_{j=1}^n \left(\sum_{i=1}^j \kappa_i\right)^\alpha |P_{j+1}-P_j|, \quad (4.14)$$

where, for notational convenience, we set $P_{n+1} \doteq P$. According to (4.14) we can write

$$\mathcal{E}^\alpha(\chi) - \mathcal{E}^\alpha(\widetilde{\chi}) \;=\; A + S_n\,, \quad (4.15)$$

where

$$A \;\doteq\; |P-P^*|\left[\left(\sigma+\sum_{j=1}^n \kappa_j\right)^\alpha - \sigma^\alpha \cos\theta_a\right] + \left(\sum_{j=1}^n \kappa_j\right)^\alpha \Big(|P^*-P_1| - |P-P_1|\Big), \quad (4.16)$$

$$S_n \;=\; \sum_{j=1}^n \kappa_j^\alpha |P^*-P_j| - \Big(\sum_{j=1}^n \kappa_j\Big)^\alpha \Big(|P^*-P_1| - |P_{n+1}-P_1|\Big) - \sum_{j=1}^n \Big(\sum_{i=1}^j \kappa_i\Big)^\alpha |P_{j+1}-P_j|. \quad (4.17)$$

2. Notice that the quantity A in (4.16) would describe the difference in the costs if all the mass $\kappa = \kappa_1 + \cdots + \kappa_n$ were flowing through the point P_1. We claim that

$$A \;\geq\; |P-P^*|\left[(\sigma+\kappa)^\alpha - \sigma^\alpha \cos\theta_a + \kappa^\alpha \cos(\theta_a+\theta_1) - \frac{\kappa^\alpha}{2}\frac{|P-P^*|}{|P_1-P^*|}\right]. \quad (4.18)$$

Indeed, the last two terms within the square brackets in (4.18) are derived from

$$\begin{aligned} |P^*-P_1| - |P-P_1| \;&=\; |P^*-P_1|\left[1 - \sqrt{1 - 2\frac{|P-P^*|}{|P^*-P_1|}\cos(\theta_a+\theta_1) + \frac{|P-P^*|^2}{|P^*-P_1|^2}}\right] \\ &\geq\; |P^*-P_1|\left[1 - \left(1 - \frac{|P-P^*|}{|P^*-P_1|}\cos(\theta_a+\theta_1) + \frac{|P-P^*|^2}{2|P^*-P_1|^2}\right)\right]. \end{aligned}$$

Using Lemma 4.1, in both cases (i) and (ii) we can now choose $\varepsilon' > 0$ small enough such that, if

$$\frac{|P-P^*|}{|P_1-P^*|} \;<\; \varepsilon', \quad (4.19)$$

then the right hand side of (4.18) is strictly positive. It now suffices to observe that, given all the angles $\theta_a, \theta_1, \ldots, \theta_n$, by choosing $\varepsilon > 0$ small enough one achieves the implication

$$\beta - \theta_1 \;<\; \varepsilon \qquad \Longrightarrow \qquad \frac{|P-P^*|}{|P_1-P^*|} \;<\; \varepsilon'. \quad (4.20)$$

In turn, this implies the strict inequality

$$A \;>\; 0. \quad (4.21)$$

3. To complete the proof of the lemma, it remains to prove that $S_n \geq 0$. This will be proved by induction on n. Starting from (4.17) and using the inequalities

$$|P_n - P_1| \;\leq\; |P^*-P_1|, \qquad \Big(\sum_{i=1}^n \kappa_i\Big)^\alpha \;\leq\; \kappa_n^\alpha + \Big(\sum_{i=1}^{n-1} \kappa_i\Big)^\alpha,$$

we obtain

$$
\begin{aligned}
S_n &= \sum_{j=1}^{n} \kappa_j^\alpha |P^* - P_j| - \Big(\sum_{j=1}^{n} \kappa_j\Big)^\alpha \underbrace{\big(|P^* - P_1| - |P_n - P_1|\big)}_{\geq 0} - \sum_{j=1}^{n-1} \Big(\sum_{i=1}^{j} \kappa_i\Big)^\alpha |P_{j+1} - P_j| \\
&\geq \kappa_n^\alpha |P^* - P_n| + \sum_{j=1}^{n-1} \kappa_j^\alpha |P^* - P_j| - \Big(\kappa_n^\alpha + \Big(\sum_{j=1}^{n-1} \kappa_j\Big)^\alpha\Big) \big(|P^* - P_1| - |P_n - P_1|\big) \\
&\quad -\Big(\sum_{i=1}^{n-1} \kappa_i\Big)^\alpha |P_n - P_{n-1}| - \sum_{j=1}^{n-2} \Big(\sum_{i=1}^{j} \kappa_i\Big)^\alpha |P_{j+1} - P_j| \\
&= \sum_{j=1}^{n-1} \kappa_j^\alpha |P^* - P_j| - \Big(\sum_{j=1}^{n-1} \kappa_j\Big)^\alpha \big(|P^* - P_1| - |P_{n-1} - P_1|\big) - \sum_{j=1}^{n-2} \Big(\sum_{i=1}^{j} \kappa_i\Big)^\alpha |P_{j+1} - P_j| \\
&\quad +\kappa_n^\alpha \Big(|P^* - P_n| + |P_n - P_1| - |P^* - P_1|\Big) \\
&= S_{n-1} + \kappa_n^\alpha \Big(|P^* - P_n| - |P^* - P_1| + |P_n - P_1|\Big) \;\geq\; S_{n-1}\,,
\end{aligned}
\tag{4.22}
$$

where in the second equality we have used $|P_{n-1} - P_1| = |P_n - P_1| - |P_n - P_{n-1}|$. Repeating this same argument, by induction we obtain

$$
S_n \;\geq\; S_{n-1} \;\geq\; \cdots \;\geq S_1\,.
$$

Observing that

$$
S_1 \;=\; \kappa_1^\alpha |P^* - P_1| - \kappa_1^\alpha \Big(|P^* - P_1| - |P_2 - P_1|\Big) - \kappa_1^\alpha |P_2 - P_1| \;=\; 0,
$$

the proof of the lemma is completed. □

Remark 4.3 As soon as all the masses $\sigma, \kappa_1, \ldots, \kappa_n$ and all the angles $\theta_a, \beta, \theta_1, \ldots, \theta_n$ have been assigned, the difference between the two irrigation costs in (4.12) is a positive homogeneous function of the distance $|P_1 - P^*|$. We can thus replace (4.12) with the inequality

$$
\mathcal{E}^\alpha(\chi) - \mathcal{E}^\alpha(\widetilde{\chi}) \;>\; c_0\,|P_1 - P^*|, \tag{4.23}
$$

for some $c_0 > 0$ depending on all the above constants. Notice that, by continuity, the bound (4.23) remains valid if θ_a is replaced by some other angle θ_a', with $|\theta_a' - \theta_a|$ sufficiently small.

4.1 Completion of the proof.

Let μ be an optimal measure, maximizing the functional (2.15), and let $\chi : \Theta \times \mathbb{R}_+ \mapsto \mathbb{R}^2$ be an optimal irrigation plan for μ. According to (2.6), we assume that all paths are parameterized by arc-length.

As remarked at the beginning of this section, a proof of Theorem 2.8 can be achieved by showing that, for an optimal solution, the point $P^* = (p_1^*, p_2^*)$ introduced at (3.16) must coincide with the origin. Throughout the following we shall thus assume $p_2^* > 0$ and derive a contradiction.

1. Call

$$\Theta^* \doteq \{\xi \in \Theta\,;\ \chi(\xi, t^*) = P^* \quad \text{for some } t^* > 0\} \tag{4.24}$$

the set of particles that move through P^*. Notice that, by the single path property, there exists a unique path $\gamma : [0, t^*] \mapsto \mathbb{R}^2$ such that

$$\gamma(0) = \mathbf{0}, \qquad \gamma(t^*) = P^*, \qquad \chi(\xi, t) = \gamma(t) \qquad \text{for all } \chi \in \Theta^*,\ t \in [0, t^*]. \tag{4.25}$$

As a consequence, in (4.24) the time t^* is the same for all $\xi \in \Theta^*$.

Within the set Θ^* of all particles reaching P^*, we distinguish the ones which proceed to the left or to the right of P^*. Namely

$$\Theta^* = \Theta_l \cup \Theta_r\,. \tag{4.26}$$

Here Θ_l denotes the set of particles that, after reaching P^*, move along the horizontal line $\{(z_1, z_2);\ \ z_1 = p_1^*,\ z_2 > p_2^*\}$ to the left of P^*. Moreover, $\Theta_r = \Theta^* \setminus \Theta_l$ is the set of particles which, after reaching P^*, move to the right. For all $\xi \in \Theta_r$ and $t > t^*$, we thus have

$$\chi(t, \xi) \in \{(z_1, z_2)\,;\ z_1 \geq p_1^*,\ z_2 \leq p_2^*\}. \tag{4.27}$$

For future use, we denote

$$\sigma \doteq \text{meas}(\Theta_l), \qquad \kappa \doteq \text{meas}(\Theta_r). \tag{4.28}$$

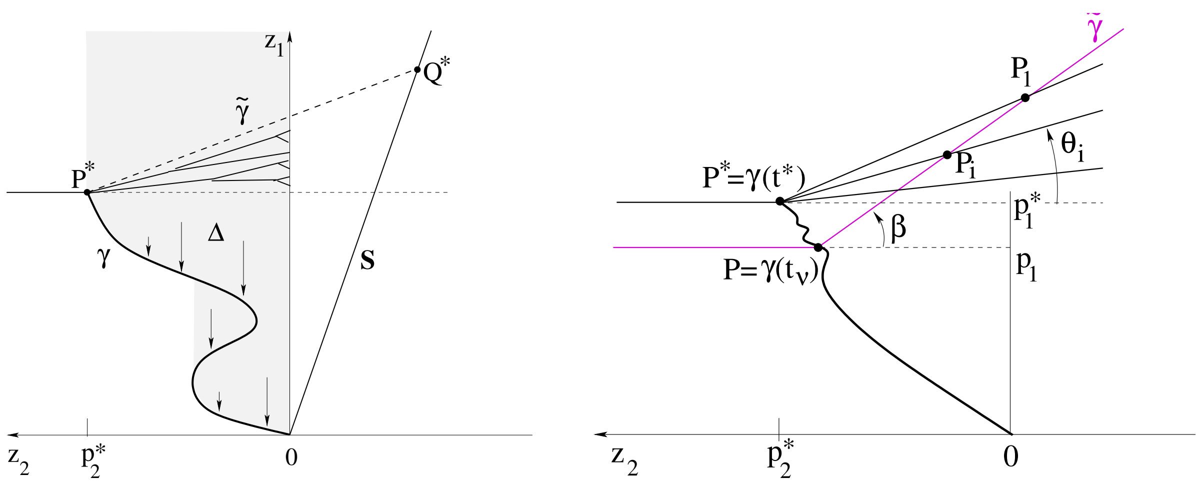

Figure 11: Left: in the shaded region Δ above the curve γ, the measure μ cannot concentrate any mass. Otherwise, by shifting this mass downward until it hits a point on γ, we would obtain a second measure $\widetilde{\mu}$ which gathers the same amount of sunlight, but has a lower irrigation cost. As a consequence, by the interior regularity property, the flow out of P^* is locally supported on a finite number of line segments. Right: the construction used in steps **4**–**6** of the proof of Theorem 2.8.

2. In connection with the path γ at (4.25), consider the set (the shaded region in Fig. 11)

$$\Delta \doteq \Big\{(z_1, z_2)\,;\ \ \text{there exists } \widehat{z}_1 < z_1 \text{ and } t \in [0, t^*] \text{ such that } \ (\widehat{z}_1, z_2) = \gamma(t)\Big\}. \tag{4.29}$$

We claim that the measure μ cannot concentrate any mass on the open set Δ. Indeed, if $\mu(\Delta) > 0$, then we consider the measure $\widehat{\mu}$ obtained by vertically shifting all the mass in Δ

until it touches some point on the curve Γ. More precisely, let $\phi : \Delta \mapsto \{\gamma(t); t \in [0, t^*]\}$ be a measurable map such that $\phi(z_1, z_2) = (\widehat{z}_1, z_2)$, with $\widehat{z}_1$ as in (4.29). Let $\widehat{\mu}$ be the push-forward of the measure μ by the map ϕ. This new measure μ would then satisfy

$$\mathcal{S}^{\mathbf{n}}(\widehat{\mu}) \;=\; \mathcal{S}^{\mathbf{n}}(\mu), \qquad \mathcal{I}^{\alpha}(\widehat{\mu}) \;<\; \mathcal{I}^{\alpha}(\mu),$$

contradicting the optimality of μ.

3. The previous argument shows that there are no sinks inside Δ. Hence all particles $\xi \in \Theta_r$ continue to move to the right of P^*, eventually crossing the z_1-axis. For $\xi \in \Theta_r$ we can thus define the stopping time

$$\tau(\xi) \;\doteq\; \min\{t > t^*\,;\; \chi(\xi, t) = (z_1, 0) \;\text{ for some } z_1 \geq p_1^*\},$$

and introduce the measure $\overline{\mu}$, supported on the z_1-axis, defined by

$$\overline{\mu}(V) \;=\; \text{meas}\Big\{\xi \in \Theta_r\,;\;\; \chi(\xi, \tau(\xi)) \in V\Big\}.$$

We observe that the restriction of χ to the set

$$\big\{(\xi, t)\,;\; \xi \in \Theta_r\,,\; t \in [t^*, \tau(\xi)]\big\}$$

yields an optimal transport plan from a point mass located at P^* to the measure $\overline{\mu}$.

By the interior regularity property [1, 19], outside a neighborhood of the support of $\overline{\mu}$, the optimal transport plan is supported on a finite union of line segments. In particular, restricted to the set $\{z_2 > p_2^*/2\}$, all paths $\chi(\xi, \cdot)$, with $\xi \in \Theta_r$, $t > t^*$ are contained within finitely many line segments.

4. By the previous step, within a neighborhood of P^*, all particle paths which move out of P^* are contained in finitely many straight lines starting at P^*.

Adopting the same notation used in Lemma 4.2, we call $\sigma = \text{meas}(\Theta_l)$ the amount of mass which moves horizontally to the left of P^*. As in (4.11), we call $\theta_1, \ldots, \theta_n$ the angles formed by the line segments to the right of P^* with a horizontal line (see Fig. 11, right). The fluxes along these line segments are denoted by $\kappa_1, \ldots, \kappa_n$. We introduce the decomposition

$$\Theta_r \;=\; \Theta_1 \cup \cdots \cup \Theta_n\,, \tag{4.30}$$

where Θ_i denotes the set of particles $\xi \in \Theta_r$ that move along the i-th segment. Notice that, with this notation, one has $\kappa_i = \text{meas}(\Theta_i)$.

We recall that, by Lemma 3.5, all particle paths $\chi(\xi, t)$, $\xi \in \Theta_r$, lie below the segment γ^* with endpoints P^*, Q^*. This implies that all angles $\theta_1, \ldots, \theta_n$ are strictly smaller than $\frac{\pi}{2} - \theta_0$. By the assumption (2.17), we are led to consider two cases.

CASE 1: $1/2 \leq \alpha \leq 1$ and $0 \leq \theta_1 < \pi/2$,

CASE 2: $0 < \alpha < 1/2$ and

$$\cos\theta_1 \;>\; \cos\Big(\frac{\pi}{2} - \theta_0\Big) \;=\; \sin\theta_0 \;\geq\; 1 - 2^{2\alpha - 1}. \tag{4.31}$$

5. In this step, under the assumption that $p_2^* > 0$, we construct a competing measure $\widetilde{\mu}$. Using Lemma 4.2, this will eventually allow us to conclude that the measure μ is not optimal.

For $t < t^*$, consider the unit vector

$$\mathbf{w}(t) \;=\; \frac{\gamma(t) - \gamma(t^*)}{|\gamma(t) - \gamma(t^*)|}.$$

By compactness, there exists an increasing sequence $t_\nu \to t^*-$ such that

$$\lim_{\nu\to\infty} \mathbf{w}(t_\nu) \;=\; \overline{\mathbf{w}}, \tag{4.32}$$

for some unit vector $\overline{\mathbf{w}} \;=\; (\overline{w}_1, \overline{w}_2)$, with $\overline{w}_1 \leq 0$, $\overline{w}_2 \leq 0$. Call $\theta_a \in [0, \pi/2]$ the angle between $\overline{\mathbf{w}}$ and a horizontal line.

In connection with the masses $\sigma, \kappa_1, \ldots, \kappa_n$ and the angles $\theta_1, \ldots, \theta_n$ defined above, we now choose an angle $\beta > \theta_1$, sufficiently close to θ_1, so that the conclusion of Lemma 4.2 holds. In particular, by Remark 4.3 the inequality (4.23) holds.

Next, we choose $\nu \geq 1$ sufficiently large (its precise value will be determined later), and consider the point $P = (p_1, p_2) = \gamma(t_\nu)$. Again referring to Fig. 11, right, we denote by $\widetilde{\gamma}$ the straight line through P, forming an angle β with a horizontal line. As in Lemma 4.2, we denote by $P_1, \ldots, P_n$ the points where $\widetilde{\gamma}$ intersects the line segments through P^*, forming angles $0 \leq \theta_n < \cdots < \theta_1$ with a horizontal line.

A new measure $\widetilde{\mu}$ and a new irrigation plan $\widetilde{\chi}$ are now defined as follows.

- The measure $\widetilde{\mu}$ is obtained from μ by vertically shifting all the mass located on the horizontal line to the left of P^* downward on the horizontal line to the left of P. More precisely, $\widetilde{\mu}$ is the push-forward of μ by the map

$$\phi(z_1, z_2) \;=\; \begin{cases} (p_1, z_2) & \text{if } \; z_1 = p_1^*, \; z_2 > p_2^*, \\ (z_1, z_2) & \text{otherwise.} \end{cases}$$

- Recalling (4.26), for $\xi \notin \Theta^*$ we simply set $\widetilde{\chi}(\xi, t) = \chi(\xi, t)$ for all $t \geq 0$.

- Particles $\xi \in \Theta_l$ move along the path γ up to the point P. Then the move horizontally to the left of P, stopping at a point $\widetilde{\chi}(\xi, \widetilde{T}(\xi))$ on the same vertical line as $\chi(\xi, T(\xi))$.

- Particles $\xi \in \Theta_r$ move along the path γ up to the point P. Then they move along $\widetilde{\gamma}$ until they reach the corresponding points $P_1, \ldots, P_n$. Afterwards, the remaining portions of their trajectories are exactly as before.

6. By construction we have $\mathcal{S}^{\mathbf{n}}(\widetilde{\mu}) = \mathcal{S}^{\mathbf{n}}(\mu)$. To analyze the cost of the new irrigation plan $\widetilde{\chi}$, consider the set

$$\Theta_\nu^\dagger \;\doteq\; \bigl\{\xi \in \Theta \setminus \Theta^* ; \;\; \chi(\xi, t_\nu) = \gamma(t_\nu) = P\bigr\}.$$

This is the set of particles that go through P, but do not reach P^* afterwards: either they stop along γ, or they move to some other branch bifurcating from γ before reaching P^*.

We observe that, in the case where $\Theta_\nu^\dagger = \emptyset$, one can immediately apply Lemma 4.2 and conclude

$$\mathcal{I}^\alpha(\widetilde{\mu}) \ \leq\ \mathcal{E}^\alpha(\widetilde{\chi}) \ <\ \mathcal{E}^\alpha(\chi) \ =\ \mathcal{I}^\alpha(\mu). \tag{4.33}$$

The following analysis will show that the same conclusion can still be reached, provided that P is sufficiently close to P^* and

$$\delta_\nu \ \doteq\ \mathrm{meas}(\Theta_\nu^\dagger) \tag{4.34}$$

is sufficiently small. For future use, we observe that

$$\lim_{\nu\to\infty} t_\nu \ =\ t^*, \qquad \lim_{\nu\to\infty} \delta_\nu \ =\ 0. \tag{4.35}$$

We are now ready to estimate the difference between the two irrigation costs, in the general case.

(1) For $\xi \notin \Theta^* \cup \Theta_\nu^\dagger$, we have

$$\chi(\xi,t) \ =\ \widetilde{\chi}(\xi,t), \qquad\qquad \big|\chi(\xi,t)\big|_\chi \ =\ \big|\widetilde{\chi}(\xi,t)\big|_{\widetilde{\chi}} \tag{4.36}$$

for all $t > 0$. Hence these particles do not contribute to the difference in the transportation cost.

(2) For $\xi \in \Theta_\nu^\dagger$, the first identity in (4.36) still holds for all $t > 0$. However, the second one holds only for $t \notin [t_\nu, \tau(\xi)]$, where

$$\tau(\xi) \ =\ \sup\big\{t \in [t_\nu, T(\xi)]\,;\ \ \chi(\xi,t) = \gamma(t)\big\} \ <\ t^*$$

is the time when the particle ξ either stops, or leaves the path γ. We estimate the difference

$$\begin{aligned} A \ &\doteq\ \int_{\Theta_\nu^\dagger} \int_{t_\nu}^{\tau(\xi)} \Big(\big|\widetilde{\chi}(\xi,t)\big|_{\widetilde{\chi}}^{\alpha-1} - \big|\chi(\xi,t)\big|_\chi^{\alpha-1}\Big)\,dt\,d\xi \\ &\leq\ \int_{\Theta_\nu^\dagger} \int_{t_\nu}^{\tau(\xi)} \big|\widetilde{\chi}(\xi,t)\big|_{\widetilde{\chi}}^{\alpha-1}\,dt\,d\xi \ \leq\ \big[\mathrm{meas}(\Theta_\nu^\dagger)\big]^\alpha \cdot (t^* - t_\nu). \end{aligned} \tag{4.37}$$

(3) It remains to estimate the difference in the cost for transporting particles $\xi \in \Theta^*$, namely

$$B \ \doteq\ \int_{\Theta^*} \left(\int_0^{\widetilde{T}(\xi)} \big|\widetilde{\chi}(\xi,t)\big|_{\widetilde{\chi}}^{\alpha-1}\,dt - \int_0^{T(\xi)} \big|\chi(\xi,t)\big|_\chi^{\alpha-1} dt\right) d\xi\,. \tag{4.38}$$

The estimate of B is based on the following observation. If the portion of the curve γ between P and P^* were a straight segment, and if on this segment the multiplicity were constantly equal to $\sigma + \kappa$, then we could use Lemma 4.2 and Remark 4.3. By (4.23) we could thus conclude

$$B \ \leq\ -\,c_0\,|P_1 - P^*|. \tag{4.39}$$

In the general case, recalling (4.34), the multiplicity along γ is estimated by

$$\sigma + \kappa \ \leq\ \big|\gamma(t)\big|_\chi \ \leq\ \sigma + \kappa + \delta_\nu \qquad \text{for all } t \in [t_\nu, t^*].$$

The presence of the additional particles $\xi \in \Theta_\nu^\dagger$ increase the multiplicity and hence reduce the cost of χ. The amount by which this cost is reduced can be bounded above by

$$\int_{\Theta^*}\int_{t_\nu}^{t^*}\Big[(\sigma+\kappa)^{\alpha-1}-(\sigma+\kappa+\delta_\nu)^{\alpha-1}\Big]\,dt\,d\xi \;=\; \Big[(\sigma+\kappa)^{\alpha-1}-(\sigma+\kappa+\delta_\nu)^{\alpha-1}\Big](\sigma+\kappa)\,(t^*-t_\nu). \tag{4.40}$$

On the other hand, the fact that the curve γ is not necessarily a straight line increases the cost of χ in an amount which is bounded below by

$$(\sigma+\kappa)\cdot(\sigma+\kappa+\delta_\nu)^{\alpha-1}\cdot\Big((t^*-t_\nu)-|P^*-P|\Big). \tag{4.41}$$

Combining all the previous observations, from (4.37), (4.39), (4.40), and (4.41), we conclude

$$\begin{aligned}\mathcal{E}^\alpha(\widetilde\chi)-\mathcal{E}^\alpha(\chi) \;&\le\; \delta_\nu^\alpha(t^*-t_\nu)-c_0|P_1-P^*|\\ &\quad+\Big[(\sigma+\kappa)^{\alpha-1}-(\sigma+\kappa+\delta_\nu)^{\alpha-1}\Big](\sigma+\kappa)\,(t^*-t_\nu)\\ &\quad-(\sigma+\kappa)\cdot(\sigma+\kappa+\delta_\nu)^{\alpha-1}\cdot\Big((t^*-t_\nu)-|P^*-P|\Big).\end{aligned} \tag{4.42}$$

We claim that, for some choice of $\nu\ge 1$ large enough, the right hand side of (4.42) becomes negative, thus contradicting the optimality of the measure μ. We consider two cases.

CASE 1: $(t^*-t_\nu)\le 2|\gamma(t_\nu)-P^*|$ for infinitely many integers $\nu\ge 1$. When this inequality holds, (4.42) yields

$$\begin{aligned}\mathcal{E}^\alpha(\widetilde\chi)-\mathcal{E}^\alpha(\chi) \;&\le\; 2\delta_\nu^\alpha|\gamma(t_\nu)-P^*|-c_0|P_1-P^*|\\ &\quad+\Big[(\sigma+\kappa)^{\alpha-1}-(\sigma+\kappa+\delta_\nu)^{\alpha-1}\Big]\cdot 2(\sigma+\kappa)|\gamma(t_\nu)-P^*|.\end{aligned} \tag{4.43}$$

We now observe that the limit (4.32) implies an inequality of the form

$$|\gamma(t_\nu)-P^*|\;\le\; C|P_1-P^*|,$$

for a suitable constant C and all ν sufficiently large. Therefore, as $\delta_\nu\to 0$, it is clear that the right hand side of (4.43) becomes negative. This contradicts the optimality of μ.

CASE 2: $(t^*-t_\nu)\ge 2|\gamma(t_\nu)-P^*|$ for infinitely many integers $\nu\ge 1$. When this inequality holds, (4.42) yields

$$\begin{aligned}\mathcal{E}^\alpha(\widetilde\chi)-\mathcal{E}^\alpha(\chi) \;&\le\; \delta_\nu^\alpha(t^*-t_\nu)+\Big[(\sigma+\kappa)^{\alpha-1}-(\sigma+\kappa+\delta_\nu)^{\alpha-1}\Big](\sigma+\kappa)\,(t^*-t_\nu)\\ &\quad-(\sigma+\kappa)\cdot(\sigma+\kappa+\delta_\nu)^{\alpha-1}\cdot\tfrac{1}{2}(t^*-t_\nu).\end{aligned} \tag{4.44}$$

When δ_ν is sufficiently small, the right hand side of (4.44) becomes negative. Once again, this contradicts the optimality of μ. □

5 The case $d=2$, $\alpha=0$

We give here a proof of Theorem 2.9.

1. Assume that there exists a unit vector $\mathbf{w}^* \in \mathbb{R}^2$ such that

$$K \;=\; \int_{\mathbf{n}\in S^1} \Big|\langle \mathbf{w}^*, \mathbf{n}\rangle\Big|\, \eta(\mathbf{n})\, d\mathbf{n} \;>\; c.$$

Let $\mathbf{v} = (\cos\beta, \sin\beta)$ be a unit vector perpendicular to $\mathbf{w}^*$, with $\beta \in [0, \pi]$. Let μ be the measure supported on the segment $\{r\mathbf{v}\,;\ r \in [0, \ell]\}$, with constant density λ w.r.t. 1-dimensional Lebesgue measure.

Then the payoff achieved by μ is estimated by

$$\begin{aligned} \mathcal{S}^\eta(\mu) - c\mathcal{I}^0(\mu) \;&=\; \ell \cdot \int_{S^1} \left(1 - \exp\Big\{ - \frac{\lambda}{\big|\langle \mathbf{w}^*, \mathbf{n}\rangle\big|}\Big\}\right) \Big|\langle \mathbf{w}^*, \mathbf{n}\rangle\Big|\, \eta(\mathbf{n})\, d\mathbf{n} - c\,\ell \\ &\geq\; \ell \cdot (1 - e^{-\lambda}) \int_{S^1} \Big|\langle \mathbf{w}^*, \mathbf{n}\rangle\Big|\, \eta(\mathbf{n})\, d\mathbf{n} - c\,\ell \\ &=\; \Big[(1 - e^{-\lambda})\, K - c\Big]\, \ell. \end{aligned} \tag{5.1}$$

By choosing $\lambda > 0$ large enough, the first factor on the right hand side of (5.1) is strictly positive. Hence, by increasing the length ℓ, we can render the payoff arbitrarily large.

2. Next, assume that $K \leq c$. Consider any Lipschitz curve $s \mapsto \gamma(s)$, parameterized by arc-length $s \in [0, \ell]$. Then, for any measure μ supported on γ, the total amount of sunlight from the direction $\mathbf{n}$ captured by μ satisfies the estimate

$$\mathcal{S}^{\mathbf{n}}(\mu) \;\leq\; \int_0^\ell \Big|\langle \dot\gamma(s)^\perp,\, \mathbf{n}\rangle\Big|\, ds.$$

Indeed, it is bounded by the length of the projection of γ on the line $E_{\mathbf{n}}^\perp$ perpendicular to $\mathbf{n}$. Integrating over the various sunlight directions, one obtains

$$\mathcal{S}^\eta(\mu) \;\leq\; \int_0^\ell \int_{S^1} \Big|\langle \dot\gamma(s)^\perp,\, \mathbf{n}\rangle\Big|\, \eta(\mathbf{n})\, d\mathbf{n}\, ds \;\leq\; K\,\ell.$$

More generally, $\mu = \sum_i \mu_i$ can be the sum of countably many measures supported on Lipschitz curves γ_i. In this case, since the sunlight functional is sub-additive, one has

$$\mathcal{S}^\eta(\mu) \;\leq\; \sum_i \mathcal{S}^\eta(\mu_i) \;\leq\; \sum_i K\ell_i\,.$$

Hence

$$\mathcal{S}^\eta(\mu) - c\mathcal{I}^0(\mu) \;\leq\; \sum_i K\ell_i - c\sum_i \ell_i \;\leq\; 0.$$

This concludes the proof of case (ii) in Theorem 2.9. □

6 Concluding remarks

We first clarify the role of the assumption (2.17), used in Theorem 2.8 when $0 < \alpha < 1/2$.

Consider the Gilbert problem where two masses $M_0, M_1 > 0$ need to be irrigated from the origin. Then, as shown in Fig. 12, left, the optimal bifurcation angle satisfies

$$\cos\theta \;=\; \frac{1-\lambda^{2\alpha}-(1-\lambda)^{2\alpha}}{2\lambda^\alpha(1-\lambda)^\alpha}, \qquad \lambda = \frac{M_0}{M_0+M_1}\,. \tag{6.1}$$

For a proof, see Lemma 12.2 of [1]. As a consequence, we have the implications

$$\begin{array}{rcl} \frac{1}{2}<\alpha\leq 1 & \Longrightarrow & \cos\theta\in\,]0, 2^{2\alpha-1}-1], \\ \alpha=\frac{1}{2} & \Longrightarrow & \cos\theta = 0, \\ 0<\alpha<\frac{1}{2} & \Longrightarrow & \cos\theta\in\,[2^{2\alpha-1}-1, 0[\,. \end{array}$$

Notice that $\cos\theta = 2^{2\alpha-1}-1$ when $M_0 = M_1$ and hence $\lambda = 1/2$. As a consequence, regardless of the relative sizes of M_0, M_1, we have:

- When $\alpha \geq 1/2$, a bifurcation with an angle $\theta > \pi/2$ cannot be optimal.
- When $\alpha < 1/2$, a bifurcation with an angle θ such that $\cos\theta < 2^{2\alpha-1}-1$ cannot be optimal.

This is the underlying motivation for the assumption (2.17), repeatedly used in the proofs. Notice that a similar assumption (4.3) was introduced in Lemma 4.1.

It is interesting to speculate whether the conclusion of Theorem 2.8 may still hold when $\alpha < 1/2$ while the angle $\theta_0 > 0$ is arbitrarily small. Consider any measure μ supported on the two half-lines $\Gamma_0\cup\Gamma_1$ as shown in Fig. 12, right. The necessary conditions derived for the problems (2.18)-(2.19) allow us to compute the total mass concentrated by the measure μ on each of these half-lines. Indeed, according to (2.29) and (2.27), we have

$$M_0 \;\doteq\; \mu(\Gamma_0) \;=\; \left(\frac{\sin\theta_0}{c}\right)^{1/\alpha}, \qquad M_1 \;=\; \mu(\Gamma_1) \;=\; \left(\frac{1}{c}\right)^{1/\alpha} \tag{6.2}$$

Therefore

$$\lambda \;=\; \frac{M_0}{M_0+M_1} \;=\; \frac{(\sin\theta_0)^{1/\alpha}}{1+(\sin\theta_0)^{1/\alpha}}, \qquad \sin\theta_0 \;=\; \frac{\lambda^\alpha}{(1-\lambda)^\alpha}\,. \tag{6.3}$$

In order for this configuration to be optimal, a necessary condition is

$$\cos\theta \;=\; \frac{1-\lambda^{2\alpha}-(1-\lambda)^{2\alpha}}{2\lambda^\alpha(1-\lambda)^\alpha} \;\geq\; \cos\left(\theta_0+\frac{\pi}{2}\right) \;=\; -\sin\theta_0\,. \tag{6.4}$$

Indeed, if (6.4) fails, a better configuration could be constructed as shown in Fig. 12, right. Here A and B are points along Γ_0 and Γ_1 respectively, at distance $\varepsilon > 0$ from the origin, while P is a suitable point such that the angle between PA and PB equals θ. To uniquely determine P, we again refer to the necessary conditions in Lemma 12.2 of [1]. Replacing the segments $0A$ and $0B$ with the three segments $0P$, PA, and PB, we can obtain a new configuration where the transportation cost is reduced by $\mathcal{O}(1)\cdot\varepsilon$, while keeping the same value of the sunlight functional. Therefore, our initial configuration where the measure μ is supported on $\Gamma_0\cup\Gamma_1$ would not be optimal.

The inequality (6.4) is precisely what is needed to rule out this possibility. Namely, if (6.4) holds, then the point P cannot lie inside the sector bounded by Γ_0 and Γ_1. Recalling (6.3), we can write (6.4) in the form

$$\frac{1-\lambda^{2\alpha}-(1-\lambda)^{2\alpha}}{2\lambda^\alpha(1-\lambda)^\alpha} \;\geq\; -\frac{\lambda^\alpha}{(1-\lambda)^\alpha}\,.$$

Equivalently:

$$\phi(\lambda) \;\doteq\; (1-\lambda)^{2\alpha}-\lambda^{2\alpha}-1 \;\leq\; 0. \tag{6.5}$$

We observe that $\phi(0)=0$ while $\phi(1)=-2$. In addition,

$$\phi'(\lambda) \;=\; -2\alpha[(1-\lambda)^{2\alpha-1}+\lambda^{2\alpha-1}] \;<\; 0$$

for $0<\lambda<1$. This implies that $\phi(\lambda)<0$ for $0<\lambda<1$. Therefore, the necessary conditions for optimality (6.4) are always satisfied, even when the angle θ_0 is very small.

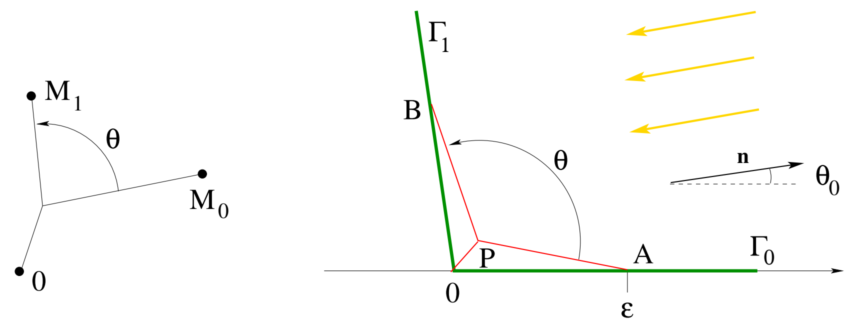

Figure 12: Changing the transport plan, when $\theta_0>0$ is very small. If in the Gilbert problem the bifurcation angle satisfies $\theta>\theta_0+\frac{\pi}{2}$, then the original configuration, where all the mass is supported along $\Gamma_0\cup\Gamma_1$, would not be optimal. The analysis at (6.5) shows that this situation never happens.

We conclude this paper by discussing possible extensions of our results.

(I) Motivated by the previous analysis, we conjecture that the conclusion of Theorem 2.8 remains valid even without the assumption (2.17). Namely, for all $0<\alpha<1/2$ and every $\theta_0\in[0,\pi/2]$, the optimal measure μ is still supported on the union of the two rays $\Gamma_0\cup\Gamma_1$. To achieve a proof, however, an additional argument will be needed. More specifically, the assumption (4.3) in Lemma 4.1 can be removed only by imposing some additional restriction on the value of $\lambda=\frac{\sigma}{\kappa+\sigma}\in[0,1]$. Our construction in Section 4, however, does not yield such information.

(II) In Theorem 2.8 it was assumed that sunlight comes from one single direction. In the case considered at (2.4), where sunlight comes with various intensity from different directions, one may conjecture that a similar result still holds true. This guess seems very reasonable if the support of the function $\eta\in\mathbf{L}^1(S^1)$ is contained within a small sector, say $[\theta_0-\delta,\theta_0+\delta]$. We remark, however, that proving such a result will require a substantially different approach. In the proof of Theorem 2.8, we repeatedly used the fact (highlighted in Remark 2.2) that, shifting part of the mass of μ along the direction $\mathbf{n}$ of sunlight, one obtains a new measure $\widetilde{\mu}$ which collects exactly the same amount of sunlight: $\mathcal{S}^{\mathbf{n}}(\widetilde{\mu})=\mathcal{S}^{\mathbf{n}}(\mu)$. This crucial property fails as soon as we replace $\mathcal{S}^{\mathbf{n}}$ with $\mathcal{S}^\eta$, allowing sunlight to come from different directions.

(III) It would be interesting to analyze the optimal branch configuration in three space dimensions. To fix the ideas, assume that sunlight comes from the direction parallel to $\mathbf{n} = (\cos\theta_0, 0, \sin\theta_0)$, and call $\mathbf{e} = (0, 0, 1)$ the unit vector in the vertical direction. Then it is easy to see that the optimal measure μ must be supported within the convex closure of the two half-planes

$$\Gamma_0 \doteq \{\mathbf{v} \in \mathbb{R}^3;\ \langle \mathbf{v}, \mathbf{n}\rangle \geq 0,\ \ \langle \mathbf{v}, \mathbf{e}\rangle = 0\}, \qquad \Gamma_1 \doteq \{\mathbf{v} \in \mathbb{R}^3;\ \langle \mathbf{v}, \mathbf{n}\rangle = 0,\ \ \langle \mathbf{v}, \mathbf{e}\rangle \geq 0\}.$$

In addition, the Hausdorff measure of Supp(μ) must be ≥ 2. Otherwise, the collected sunlight would be $\mathcal{S}^{\mathbf{n}}(\mu) = 0$. A challenging question is whether the support of μ is indeed contained in a two-dimensional surface. In this case, the optimal irrigation plan should have a structure similar to the one studied in [16].

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