Non-existence and Non-uniqueness for Multidimensional Sticky Particle Systems

Alberto Bressan(*) and Truyen Nguyen(**)

(*) Department of Mathematics, Penn State University. University Park, PA 16802, USA.
(**) Department of Mathematics, University of Akron. Akron, OH 44325-4002, USA.
e-mails: bressan@math.psu.edu , tnguyen@uakron.edu

March 15, 2014

Abstract

The paper is concerned with sticky weak solutions to the equations of pressureless gases in two or more space dimensions. Various initial data are constructed, showing that the Cauchy problem can have (i) two distinct sticky solutions, or (ii) no sticky solution, not even locally in time. In both cases the initial density is smooth with compact support, while the initial velocity field is continuous.

1 Introduction

We consider the initial value problem for the equations of pressureless gases in several space dimensions:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho v) &= 0, \\ \partial_t(\rho v) + \mathrm{div}\,\big(\rho v \otimes v\big) &= 0, \end{cases} \qquad t \in]0,T[\,, \quad x \in I\!R^n, \tag{1.1}$$

$$\rho(0,x) \;=\; \bar\rho(x), \qquad v(0,x) \;=\; \bar v(x). \tag{1.2}$$

The system (1.1) was first studied by Zeldovich [13] in the one-dimensional case to model the evolution of a sticky particle system. An example of a measure-valued solution is provided by a finite collection of particles moving with constant speed in the absence of forces. Whenever two or more particles collide, they stick to each other as a single compound particle. The mass of the new particle is equal to the total mass of the particles involved in the collision, while its velocity is determined by the conservation of momentum. The sticky particle system has been investigated extensively by many authors and is well understood in dimension $n = 1$, see [2, 3, 4, 6, 7, 8, 9, 10]. In this case it is known that, for any initial data $(\bar\rho, \bar v)$ with bounded total mass and energy, the Cauchy problem (1.1)-(1.2) has a unique global entropy-admissible weak solution (ρ, v) (see [8] and [10, Theorem 1.3]).

The present paper is concerned with the initial value problem associated with (1.1) in space dimension $n \geq 2$. We are interested in weak solutions to (1.1) obeying the sticky particle or adhesion dynamics principle, which are the most relevant from a physical point of view. For initial data containing finitely many particles, it is easy to see that a unique global solution exists, but it does not depend continuously on the initial data. In the case of countably many particles, we show that both uniqueness and existence can fail. Indeed, we construct a Cauchy problem having exactly two solutions, and a second Cauchy problem where no solution exists, not even locally in time. Both these examples can be adapted to the case of $\mathbf{L}^\infty$ initial data.

The remainder of the paper is organized as follows. In Section 2 we give precise definitions of "weak solution" and "sticky solution" for initial data containing countably many point masses and also for continuous mass distributions, following [11]. Section 3 contains an example showing the non-uniqueness of sticky solutions. In Section 4 we describe a Cauchy problem without any local solution and explain how this counterexample relates to the (erroneous) proof of global existence of sticky solutions proposed in [11]. Finally, in Section 5 we extend the analysis to initial data having smooth density and continuous velocity. Even in this case we show that local existence and uniqueness do not hold, in general.

2 Dynamics of sticky particles

We consider a system containing countably many sticky particles, moving in n-dimensional space. Let

$$\left.\begin{array}{rcl} x_i(t) & = & \text{position} \\ v_i(t) & = & \text{velocity} \\ m_i & = & \text{mass} \end{array}\right\} \quad \text{of the } i\text{-th particle at time } t.$$

In a Lagrangian formulation, the state of the system is described by countably many ODEs for the variables x_i. Let

$$x_i(0) \;=\; \bar{x}_i\,, \qquad\qquad \dot{x}_i(0+) \;=\; \bar{v}_i \tag{2.1}$$

be the initial position and the initial velocity of the i-th particle. It is natural to assume that, when particles are at a same location, they stick together traveling with a common speed determined by the conservation of momentum. At any time $t \geq 0$, the speed of the i-th particle should thus be

$$\dot{x}_i(t) \;=\; V_i(t) \;\doteq\; \frac{\sum_{j\in J_i(t)} m_j \bar{v}_j}{\sum_{j\in J_i(t)} m_j}\,, \tag{2.2}$$

where

$$J_i(t) \;\doteq\; \Big\{ j\geq 1 : \quad x_j(t) = x_i(t)\Big\}. \tag{2.3}$$

Notice that the right hand side of (2.2) is well defined provided that the total mass $M = \sum_i m_i$ and the initial energy $E \doteq \frac{1}{2}\sum_i m_i |\bar{v}_i|^2$ are finite.

Definition 1. A family of continuous maps $t \mapsto x_i(t)$ is a *weak solution* of the equations (2.2) with initial data (2.1) if, for every $i \geq 1$ and $t \geq 0$, one has

$$x_i(t) \;=\; \bar{x}_i + \int_0^t V_i(s)\,ds\,. \tag{2.4}$$

In addition, we say that the solution is *energy admissible* if the corresponding energy

$$E(t) \;\doteq\; \frac{1}{2}\sum_i m_i |\dot{x}_i(t+)|^2 \tag{2.5}$$

is a bounded, non-increasing function of time.

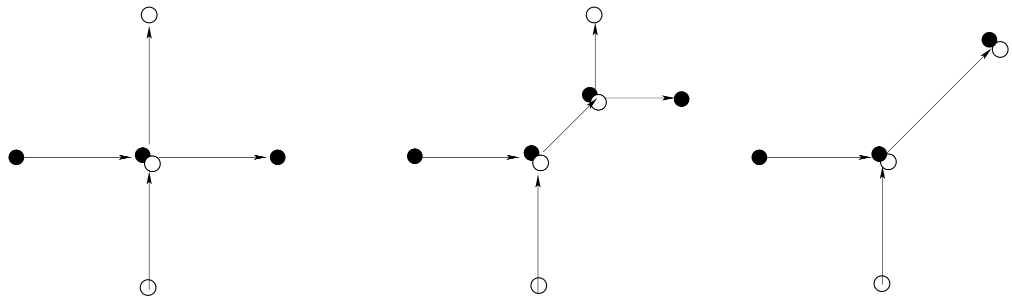

Figure 1: Left: a weak solution which is energy admissible but not sticky. Center: a weak solution which is neither sticky nor energy admissible. Right: a sticky solution.

In general, even in the case of two particles, the energy-admissible weak solution need not be unique. To achieve uniqueness (at least for finitely many particles) one more condition must be imposed.

Definition 2. We say that a weak solution $\{x_i(\cdot)\,;\quad i \geq 1\}$ is a *sticky solution* if it satisfies the additional property

(SP) *If $x_i(t_0) = x_j(t_0)$ at some time $t_0 \geq 0$, then $x_i(t) = x_j(t)$ for all $t > t_0$.*

Example 1. Given any initial data (2.1), the family of functions

$$x_i(t) \;=\; \bar{x}_i + t\bar{v}_i \tag{2.6}$$

always provides a weak, energy admissible solution. Indeed, for any i, the set of times

$$\{t > 0 : \quad x_j(t) = x_i(t) \text{ for some } j \neq i\}$$

is at most countable, therefore it has measure zero. This implies $J_i(t) = \{i\}$ for a.e. $t \geq 0$, hence (2.4) trivially holds.

Example 2. Consider two particles moving in the plane, with masses $m_1 = m_2 = 1$ and with initial positions and velocities given by

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \bar{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \bar{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.7}$$

As in Example 1, the maps

$$x_1(t) = \bar{x}_1 + t\bar{v}_1\,, \qquad x_2(t) = \bar{x}_2 + t\bar{v}_2 \tag{2.8}$$

provide an energy-admissible weak solution, which however does not satisfies the stickiness assumption **(SP)**. The unique solution that satisfies **(SP)** is given by (2.8) for $t \in [0,1]$, while

$$x_1(t) \;=\; x_2(t) \;=\; \frac{t+1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for} \quad t \geq 1\,. \tag{2.9}$$

We remark that the weak solution (2.8) depends continuously on the initial data, but the sticky solution does not. Indeed, if we slightly perturb the initial data, say by taking $\bar{x}_1 = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$ with $\varepsilon \neq 0$, then the two particles do not collide and (2.8) provides the unique sticky solution to the Cauchy problem.

We also observe that the above Cauchy problem has infinitely many weak solutions which are not energy admissible (and not sticky). Indeed, for any given time $T \geq 1$, a weak solution is defined by (2.8) for $t \in [0,1]$, by (2.9) for $t \in [1,T]$, and by

$$x_1(t) \;=\; \frac{T+1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ t-T \end{pmatrix}, \qquad x_2(t) \;=\; \frac{T+1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} t-T \\ 0 \end{pmatrix} \qquad \text{for } \; t \geq T\,. \tag{2.10}$$

Remark 1. When only finitely many particles are present, is it an easy matter to prove the global existence and uniqueness of a sticky solution to the Cauchy problem (2.1). The proof can be achieved by induction on the number N of particles. When $N = 1$ the result is trivial. Next, assume that the result is true whenever the initial number of particles is $< N$. Consider an initial data consisting of exactly N particles. Let $x_i(t) \;\doteq\; \bar{x}_i + t\bar{v}_i$ and define the first interaction time

$$\tau \;\doteq\; \inf\Big\{t > 0\,; \;\; x_i(t) = x_j(t) \quad \text{for some } i \neq j\Big\}.$$

If $\tau = +\infty$, then

$$x_i(t) \;=\; \bar{x}_i + t\bar{v}_i \qquad\qquad i = 1, \ldots, N,$$

describes the unique sticky solution. If $\tau < \infty$, then at time τ two or more colliding particles are lumped together in a single compound particle with speed determined by the conservation of momentum. This yields a new Cauchy problem, where the initial data at $t = \tau$ contains a number of particles strictly less than N. The result follows by induction.

For initial data where the density can be an arbitrary measure, a general notion of sticky weak solutions for the system (1.1) was introduced by Sever in [11]. This definition is reviewed here, and will be later used in Section 5. In the following we assume that the initial density and the initial velocity of the pressureless gas satisfy

$$\bar{\rho} \in \mathcal{P}_2(I\!R^n), \qquad\qquad \bar{v} \in \mathbf{L}^2(\bar{\rho}). \tag{2.11}$$

Here $\mathcal{P}_2(I\!R^n)$ is the set of all probability measures ρ such that

$$\int |x|^2\, d\rho(x) \;<\; \infty.$$

Let D be an open set in $I\!R^n$ with $\mathcal{L}_n(D) = 1$, where $\mathcal{L}_n$ is the Lebesgue measure on $I\!R^n$. Regarding $y \in D$ as a Lagrangian coordinate, a flow will be described by a mapping $X : [0,T] \times D \to I\!R^n$ with forward time derivatives X_t satisfying

$$\sup_{0 \leq t \leq T} \int_D |X_t(t,y)|^2\, dy < \infty.$$

In addition to the usual Lebesgue spaces $\mathbf{L}^2(D)$ and $\mathbf{L}^2\big([0,T] \times D\big)$, we shall also need the following function spaces:

- $J(X)$ is the completion of $C_c^1([0,T] \times I\!\!R^n)$ with respect to the norm

$$\|\theta\|_J \doteq \left[\int_0^T \int_D \theta(t, X(t,y))^2 \, dydt\right]^{1/2}.$$

- For each $t \in [0,T]$, the t-section $J(X,t)$ is the completion of $C_c^1(I\!\!R^n)$ with respect to the norm

$$\|\phi\|_{J,t} \doteq \left[\int_D \phi(X(t,y))^2 \, dy\right]^{1/2}.$$

- $K(X) \doteq \{\theta \circ X\,;\ \theta \in J(X)\} \subset \mathbf{L}^2([0,T] \times D)$ and $K(X,t) \doteq \{\phi \circ X(t,\cdot)\,;\ \phi \in J(X,t)\} \subset \mathbf{L}^2(D)$.

It was shown in [11, Lemma 3.1] that for every $g \in \mathbf{L}^2(D)$ there exists a unique $v \in J(X)$ such that, for every $t \in [0,T]$,

$$v(t,\cdot) \in J(X,t), \qquad \text{and} \quad \|v(t,\cdot)\|_{J,t} \le \|g\|_{\mathbf{L}^2}. \tag{2.12}$$

In addition, the following orthogonality relation holds:

$$v \circ X(t,\cdot) - g \perp K(X,t). \tag{2.13}$$

Let $W(X) : \mathbf{L}^2(D) \mapsto J(X)$ be the linear mapping defined by $W(X)g \doteq v$, where v is the unique $v \in J(X)$ satisfying (2.12)-(2.13). We can now recall the definition of sticky weak solution introduced in [11].

Definition 3. A flow mapping $X : [0,T] \times D \to I\!\!R^n$ provides a *sticky weak solution* to the Cauchy problem (1.1)–(1.2) if it satisfies the following properties.

1 - weak solution: $X_t = (W(X)X_t(0,\cdot)) \circ X$ in $\mathbf{L}^2([0,T] \times D)$.

2 - initial data: For every $\phi \in J(X,0)$ one has

$$\int_D \phi(X(0,y))\,dy = \int_{I\!\!R^n} \phi(x)\,d\bar{\rho}(x),$$
$$\int_D X_t(y,0)\phi(X(0,y))\,dy = \int_{I\!\!R^n} \bar{v}(x)\phi(x)\,d\bar{\rho}(x).$$

3 - sticky property: For every y_1, y_2, t_0 such that $X(t_0,y_1) = X(t_0,y_2)$, one has

$$X(t,y_1) = X(t,y_2) \quad \text{for all} \quad t \ge t_0.$$

If X satisfies **1 - 2** in Definition 3, then

$$v \doteq W(X)X_t(0,\cdot) \qquad \text{and} \qquad \rho(t,\cdot) \doteq X(t,\cdot)_{\#}\mathcal{L}_n \tag{2.14}$$

(i.e., the push-forward of the Lebesgue measure $\mathcal{L}_n$ on D by the map $y \mapsto X(t,y)$) provide a distributional solution to the system (1.1) of pressureless gases (see [11, Theorem 3.2] and the subsequent remark). Moreover, $v \circ X = X_t$ in $\mathbf{L}^2([0,T] \times D)$.

3 A Cauchy problem with two solutions

We construct here an initial configuration containing countably many particles in the plane, such that the Cauchy problem has two distinct sticky solutions.

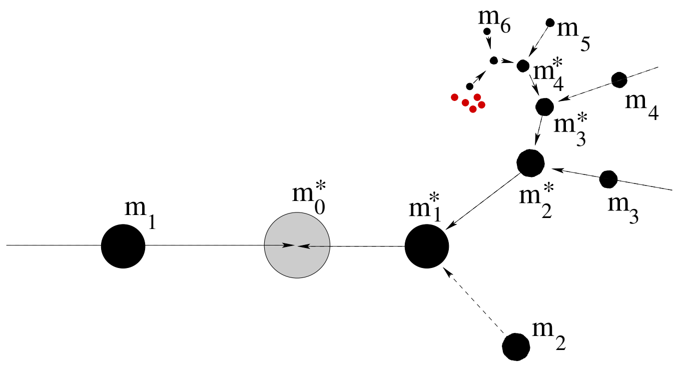

Figure 2: Countably many particles with masses $m_m = 2^{-n}$ collide, two at a time, until a single compound particle is formed.

Example 3. A solution of (2.4) will be constructed by induction, starting from the final configuration and going backward in time (see Figure 2). For notational simplicity, we denote with the same symbol a particle and its mass. We also define the times $t_n \doteq 2^{-n}$, $n = 1, 2, \ldots$ The solution is constructed by induction on the intervals $[t_{n+1},\, t_n]$.

- For $t \geq t_1 = 1/2$ there is one single particle of mass $m_0^* = 1$, located at the origin.
- For $t \in [t_2, t_1[$ there are two particles with masses $m_1 = m_1^* = 1/2$, moving toward each other with opposite speed and colliding at the origin at time $t = t_1$.
- For $t \in [t_3, t_2[$, the particle m_1^* is replaced by two equal particles with masses $m_2 = m_2^* = 1/4$, colliding at time t_2.

$\cdots$

- In general, for $t \in [t_{n+1},\, t_n[$, the particle m_{n-1}^* is replaced by two equal particles, with masses $m_n = m_n^* = 2^{-n}$, colliding at time t_n.

We now describe more precisely the inductive step of the construction.

Inductive Hypothesis (IH)$_{n-1}$: During the time interval $[t_n, t_{n-1}[$ the solution contains n point masses

$$m_1 = 2^{-1}, \quad m_2 = 2^{-2}, \quad \cdots \quad , \quad m_{n-1} = 2^{-n+1}, \quad m_{n-1}^* = 2^{-n+1},$$

located at the points

$$\begin{cases} x_j(t) = y_j + (t - t_j)v_j, & j = 1, \ldots, n-1\,, \\ x_{n-1}^*(t) = y_{n-1} + (t - t_{n-1})v_{n-1}^*\,. \end{cases} \tag{3.1}$$

Here t_j are the interaction times, y_j are the points where interaction occurs, and v_j are the constant speeds of particles before the interactions. These speeds satisfy

$$\begin{cases} |v_j| \; < \; 1\,, & j = 1, \ldots, n-1\,, \\ |v^*_{n-1}| \; < \; 1\,, & v^*_{n-1} \; \neq \; v_{n-1}\,. \end{cases} \tag{3.2}$$

Moreover, for every $t \geq 0$, the $n-1$ points $x_1(t), \ldots, x_{n-1}(t)$ in (3.1) are all distinct.

To prolong the solution backward in time, on the interval $[t_{n+1}, t_n[$, we introduce two additional particles, with mass $m_n = m^*_n = 2^{-n}$, which collide together at time $t = t_n$ generating the lumped particle m^*_{n-1}. The collision will occur at the point

$$y_n \; = \; x^*_{n-1}(t_n) \; = \; y_{n-1} + (t_n - t_{n-1})v^*_{n-1} \; = \; y_{n-1} - 2^{-n}v^*_{n-1}\,.$$

Let v_n, v^*_n be the speeds of the two interacting particles m_n, m^*_n. Conservation of momentum requires

$$\frac{v_n + v^*_n}{2} \; = \; v^*_{n-1}\,. \tag{3.3}$$

Apart from (3.3), the speeds v_n, v^*_n can be freely chosen. Recalling that $|v^*_{n-1}| < 1$, we can thus choose v_n, v^*_n so that

$$v_n \; \neq \; v^*_n\,, \qquad |v_n| \; < \; 1\,, \qquad |v^*_n| \; < \; 1\,, \tag{3.4}$$

and in such a way that the following non-intersection property holds:

(NIP) For every $t \geq 0$, the two points

$$x_n(t) \; \doteq \; y_n + (t - t_n)v_n, \qquad x^*_n(t) \; \doteq \; y_n + (t - t_n)v^*_n\,,$$

do not coincide with any of the $n-1$ points

$$x_j(t) \; \doteq \; y_j + (t - t_j)v_j, \qquad\qquad 1 \leq j < n\,.$$

With this choice, all requirements in the inductive hypothesis $(\mathrm{IH})_n$ are satisfied.

By induction on $n = 1, 2, \ldots$ we thus obtain a sticky solution to the particle equation (2.2) defined for all $t \geq 0$. The initial data consists of countably many particles with masses $m_n = 2^{-n}$, $n \geq 1$, located at the initial points

$$x_n(0) \; = \; \bar{x}_n \; = \; y_n - t_n v_n\,.$$

As time progresses, these particles collide and stick to each other. In particular during each time interval $[t_n, t_{n-1}[$ only n distinct particles are present. Notice that in this example the total mass is $\sum_{n\geq 1} m_n = 1$. Moreover, since $|v_n| < 1$ for every n, the total energy is also bounded.

We now observe that, for the same initial data, the trivial solution

$$x_n(t) \; = \; \bar{x}_n + tv_n \; = \; y_n + (t - t_n)v_n$$

is a sticky solution as well. Indeed, by the non-intersection property (NIP), the set of collision times

$$\{t \geq 0\,; \quad x_i(t) = x_j(t) \quad \text{for some} \quad i \neq j\}$$

is empty. This provides a counterexample to the uniqueness of sticky solutions, for an initial configuration containing countably many particles.

4 A Cauchy problem with no solution

In this section we shall construct an initial configuration containing countably many particles in the plane, such that the corresponding Cauchy problem has no sticky solution, even locally in time.

As a first step, consider a one-dimensional configuration consisting of countably many particles x_k, $k \geq 1$ moving along the x axis. The masses m_k of these particles, and their initial positions $\bar{x}_k$ and velocities $\bar{v}_k$ are chosen to be

$$m_k \doteq \alpha^k, \qquad \bar{x}_k = \beta^k, \qquad \bar{v}_k = 1 - \gamma^k. \tag{4.1}$$

with $0 < \alpha, \beta, \gamma < 1$. For notational convenience, we denote by

$$x_j(t) = \bar{x}_j + t\bar{v}_j \qquad\qquad j \geq 1$$

the positions of the free particles. Notice that the collection of particles $\{x_j(t)\,; \ \ j \geq k\}$ with masses m_j has barycenter located at

$$\begin{aligned} b_k(t) \ &\doteq \ \frac{\sum_{j\geq k} m_j(\bar{x}_j + t\bar{v}_j)}{\sum_{j\geq k} m_j} \ = \ \frac{\sum_{j\geq k} \alpha^j[\beta^j + t(1-\gamma^j)]}{\sum_{j\geq k}\alpha^j} \\ &= \ \frac{1-\alpha}{\alpha^k}\cdot\left[\frac{\alpha^k\beta^k}{1-\alpha\beta} + \frac{t\alpha^k}{1-\alpha} - \frac{t\alpha^k\gamma^k}{1-\alpha\gamma}\right]. \end{aligned} \tag{4.2}$$

Call t_{k-1} the time when this barycenter hits the particle x_{k-1}. Solving the equation $b_k(t) = x_{k-1}(t)$ we obtain

$$\begin{aligned} \beta^{k-1} - t\gamma^{k-1} \ &= \ (1-\alpha)\left[\frac{\beta^k}{1-\alpha\beta} - \frac{t\gamma^k}{1-\alpha\gamma}\right], \\ \frac{1-\beta}{1-\alpha\beta}\beta^{k-1} \ &= \ \frac{1-\gamma}{1-\alpha\gamma}\cdot t\gamma^{k-1}. \end{aligned}$$

Therefore,

$$t_{k-1} \ = \ \frac{1-\beta}{1-\alpha\beta}\cdot\frac{1-\alpha\gamma}{1-\gamma}\cdot\left(\frac{\beta}{\gamma}\right)^{k-1}. \tag{4.3}$$

In particular, if we choose $0 < \beta < \gamma < 1$, the sequence of times t_k will be strictly decreasing to zero.

The unique solution to the one-dimensional Cauchy problem can be explicitly described as follows (Figure 3). For $t \in [t_k,\, t_{k-1}[$ there are $k-1$ particles with masses $m_{k-1}, \ldots, m_1$, located at $x_{k-1}(t) < x_{k-2}(t) < \cdots < x_1(t)$, and one compound particle with mass $m_k^* = \sum_{j\geq k} m_j$, located at $b_k(t)$.

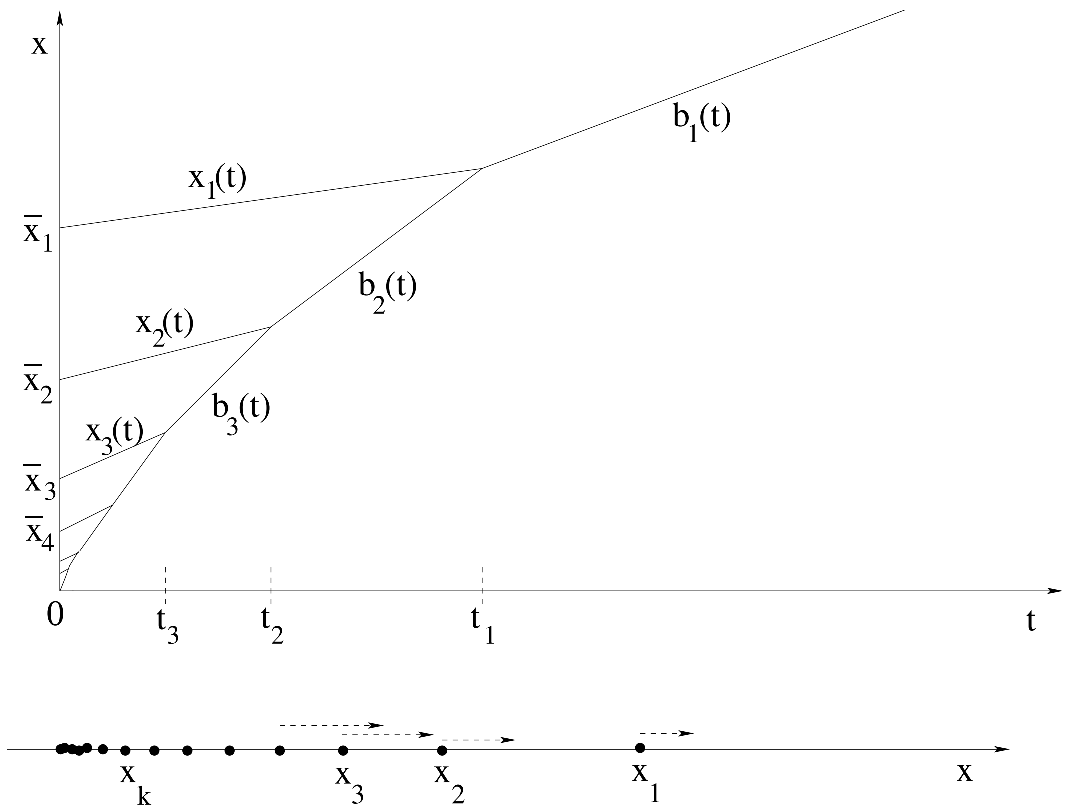

Figure 3: A sticky solution containing countably many particles, moving on the x-axis.

For future use, two lemmas will be needed.

Lemma 1. *If* $0 < \beta < \gamma < 1$ *and* $0 < \alpha < \frac{1}{1+\beta+\gamma}$, *then for every* $k > 1$ *one has*

$$x_{k+1}(t_{k-1}) \ > \ x_{k-1}(t_{k-1}) \ = \ b_k(t_{k-1}) \tag{4.4}$$

Proof. The inequality (4.4) holds provided that

$$\beta^{k+1} - \gamma^{k+1} t_{k-1} \ > \ \beta^{k-1} - \gamma^{k-1} t_{k-1}.$$

An explicit computation yields

$$t_{k-1} \ > \ \frac{1-\beta^2}{1-\gamma^2} \left(\frac{\beta}{\gamma}\right)^{k-1}.$$

By (4.3), this is equivalent to

$$\frac{1+\beta}{1+\gamma} \ < \ \frac{1-\alpha\gamma}{1-\alpha\beta}.$$

Therefore, if $0 < \beta < \gamma < 1$, the above inequality holds as soon as $0 < \alpha < \frac{1}{1+\beta+\gamma}$. □

Lemma 2. *Let $0 < \beta < \gamma < 1$ and $0 < \alpha < \frac{1}{1+\beta+\gamma}$ be as in Lemma 1. Then for every $k > 1$, there exists a time $\tau_k \in \,]t_k,\, t_{k-1}[$ such that the following holds.*

Consider any subset $S'_k \subset S_k \doteq \{j\,;\; j \geq k\}$, with $k \in S'_k \neq S_k$. Then the barycenter $b'_k(\tau_k)$ of the set $\{x_j(\tau_k)\,;\; j \in S'_k\}$ satisfies

$$b'_k(\tau_k) \;\doteq\; \frac{\sum_{j\in S'_k} m_j x_j(\tau_k)}{\sum_{j\in S'_k} m_j} \;<\; \frac{\sum_{j\in S_k} m_j x_j(\tau_k)}{\sum_{j\in S_k} m_j} \;=\; b_k(\tau_k). \tag{4.5}$$

Proof. Let $k > 1$ be given. Using Lemma 1, by continuity we can find $\tau_k \in \,]t_k,\, t_{k-1}[$ such that

$$x_{k+1}(\tau_k) \;>\; b_k(\tau_k).$$

We claim that with this choice the inequalities (4.5) hold as well. Indeed,

$$x_k(\tau_k) \;<\; b_k(\tau_k) \;<\; x_{k+1}(\tau_k) \;<\; x_{k+2}(\tau_k) \;<\; \cdots$$

Therefore, defining the set $S''_k \doteq S_k \setminus S'_k$, the corresponding barycenter satisfies

$$b''_k(\tau_k) \;\doteq\; \frac{\sum_{j\in S''_k} m_j x_j(\tau_k)}{\sum_{j\in S''_k} m_j} \;\geq\; \min_{j\in S''_k} x_j(\tau_k) \;>\; b_k(\tau_k). \tag{4.6}$$

Observe that

$$b_k(\tau_k) = \theta b'_k(\tau_k) + (1-\theta) b''_k(\tau_k)$$

for $\theta \doteq \frac{\sum_{j\in S'_k} m_j}{\sum_{j\in S_k} m_j}$. But since $0 < \theta < 1$ due to $S'_k \neq \emptyset$ and $S'_k \subsetneq S_k$, it follows from (4.6) that $b_k(\tau_k) > b'_k(\tau_k)$. □

After these preliminaries we can describe our main counterexample.

Example 4. On the plane $\mathbb{R}^2$ we shall use the canonical basis $\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$. The initial configuration consists of two countable sets of particles (Figure 4).

- A sequence of black particles moving horizontally along the x_1 axis. As in (4.1), their masses, initial positions, and initial velocities are defined as

$$m_k \;\doteq\; \alpha^k, \qquad \bar{x}_k \;=\; \beta^k \mathbf{e}_1, \qquad \bar{v}_k \;=\; (1-\gamma^k)\mathbf{e}_1\,. \tag{4.7}$$

- A sequence of white particles, moving vertically. Their masses, initial positions, and initial velocities are chosen as

$$M_k \;\doteq\; \alpha^k, \qquad \overline{X}_k \;=\; b_k(\tau_k)\mathbf{e}_1 + \tau_k \mathbf{e}_2\,, \qquad \overline{V}_k \;=\; -\,\mathbf{e}_2\,. \tag{4.8}$$

We think of the white particles as bullets, knocking the black particles away from the x_1 axis.

We claim that, with this initial configuration, no sticky solution exists. Indeed, assume that a solution exists, and call $S \subset \mathbb{N}$ be the set of all white particles that hit a target, i.e. that collide with a lumped black particle while crossing the x_1 axis. Two cases can be considered, each leading to a contradiction.

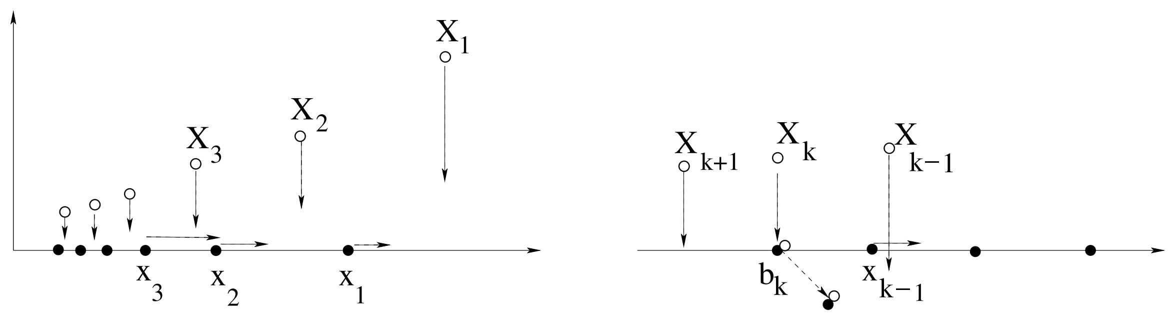

Figure 4: Left: The initial configuration of black particles x_k moving horizontally and white particles X_k moving vertically. Right: if the white particle X_k scores a hit, then no other white particle X_j with $j \neq k$ can collide with a black particle along the x_1 axis.

CASE 1: $k \in S$ for some $k \geq 1$. We claim that this is possible only if $j \notin S$ for all $j > k$. Indeed, let $S'_k \subseteq S_k = \{j \ \ j \geq k\}$ denote the set of black particles which are NOT hit by some white particle before time τ_k. If $S'_k \neq S_k$, then by Lemma 2 the barycenter of these particles satisfies $b'_k(\tau_k) < b_k(\tau_k)$. Hence

$$b'_k(\tau_k)\mathbf{e}_1 \ \neq \ b_k(\tau_k)\mathbf{e}_1 \ = \ X_k(\tau_k)$$

and the k-th white particle will not hit its target.

On the other hand, if $j \notin S$ for all $j > k$, then none of the black particles x_j with $j \geq k+1$ is hit by white bullets. At time τ_{k+1} the barycenter of this set of black particles is located at $b_{k+1}(\tau_{k+1})\mathbf{e}_1 \ = \ X_{k+1}(\tau_{k+1})$. As a result, the white particle X_{k+1} hits its target. We have thus proved the two implications

$$\begin{aligned} k \in S \qquad &\Longrightarrow \qquad j \notin S \ \text{ for all } \ j > k, \\ j \notin S \ \text{ for all } \ j > k \qquad &\Longrightarrow \qquad k+1 \in S, \end{aligned} \tag{4.9}$$

leading to a contradiction.

CASE 2: The remaining possibility is that $S = \emptyset$. But in this case the second implication in (4.9) immediately yields a contradiction.

From the above arguments, it is clear that a sticky solution does not exist, even locally in time.

Remark 2. Following [11], one can construct a family of weak solutions as follows. At each time where an interaction occurs, two particles can either stick together, or continue their separate motion without any change in the velocities. The choice (sticking to each other or not) is made in order to minimize the integral

$$J_\varepsilon \ \doteq \ \int_0^\infty e^{-t/\varepsilon} E(t)\, dt$$

where $E(t)$ is the total energy at time t. It is clear that, as two interacting particles stick together, the momentum is conserved but the energy decreases. Letting $\varepsilon \to 0$, in [11] it was claimed (but not proved) that any limit of a sequence of weak solutions which minimize J_ε should yield a solution to the sticky particle equations. This is true in the case of finitely many particles, as suggested by intuition, but false in general. In our specific example, for a given $\varepsilon > 0$ a solution which minimizes J_ε can be described as follows. The black particles traveling along the x_1 axis always stick to each other after collision. On the other hand, there is an integer $N = N(\varepsilon)$ such that:

(i) For $k > N$, at time τ_k the white particle X_k and the corresponding lumped black particle hit each other at $b_k(\tau_k)\mathbf{e}_1$, without changing their speed (i.e., without sticking).

(ii) At time τ_N, the white particle X_N hits the corresponding black particle at $b_N(\tau_N)\mathbf{e}_1$ and sticks to it, knocking it away from the x_1 axis.

(iii) For $k < N$, the remaining white particles do not hit any black particle.

As $\varepsilon \to 0$, since we are putting less and less weight on energy at later times, the single white particle that sticks to its target is $X_{N(\varepsilon)}$, with $N(\varepsilon) \to \infty$. In the limit, we obtain a weak solution where all black particles stick to each other, but all white particles hit the black particles without sticking. More precisely, for $t \in [t_k, t_{k-1}[$ this limit solution contains $k-1$ black particles with masses $m_1, \ldots, m_{k-1}$, located at

$$x_j(t) = (\bar{x}_j + t\bar{v}_j)\mathbf{e}_1 , \qquad\qquad j = 1, \ldots, k-1,$$

and one lumped black particle with mass $m_k^* = \sum_{j \geq k} m_j$, located at $b_k(t)\mathbf{e}_1$. In addition, it contains countably many white particles with masses M_i, located at

$$X_i(t) \;=\; \overline{X}_i + t\overline{V}_i \;=\; b_i(\tau_i)\mathbf{e}_1 - (t - \tau_i)\mathbf{e}_2 \,.$$

This is not a sticky solution.

Remark 3. Given a weak solution consisting of countably many particles $Y_i(t)$, on can introduce a measure of "non-stickiness" by setting

$$\Phi \;\doteq\; \sum_{(i,j)\in NS} m_i m_j \,.$$

Here m_i denotes the mass of the i-th particle, while

$$NS \;\doteq\; \{(i,j) : \quad Y_i(t_0) = Y_j(t_0) \text{ but } Y_i(t) \neq Y_j(t) \;\text{ for some times } t > t_0\}$$

describes all couples of particles that hit each other without sticking.

As $\varepsilon \to 0+$, for the sequence of weak solutions considered in Remark 2 the measure Φ_ε of non-stickiness approaches zero. However, this does not imply that the limit solution should be sticky.

5 Non-existence and non-uniqueness for continuous initial data

In this last section we extend Example 3 and Example 4 to the case of continuous initial data. Our main goal is to prove:

Theorem 1. *In any dimension $n \geq 2$ there exists an initial datum $(\bar\rho, \bar v)$, such that the system* (1.1) *does not admit any sticky weak solution in the sense of Definition 3, not even locally in time. Here $\bar\rho \in \mathcal{C}_c^\infty(I\!R^n)$ is the density of a probability measure, while $\bar v \in \mathcal{C}_c(I\!R^n)$ is a continuous initial velocity.*

Proof. The main idea is to modify the initial data in Example 4, replacing each point mass, say located at $\bar y_j$, by a smooth distribution of mass supported on a small ball $B_j = \{x : |x - \bar y_j| \leq r_j\}$. By choosing an appropriate initial velocity, after a very short time all the mass initially contained inside B_j collapses to a point mass and then continues its motion as in the previous example. However, a difficulty arises because the black particles move horizontally along the x_1 axis, while the white particles have velocity $-\mathbf{e}_2$. This would determine a discontinuity in the velocity field at the origin. To avoid this, we need to add a horizontal component to the velocities of the white particles X_k, as shown in Fig. 5. We now describe the construction in greater detail.

1. In this step we replace the black particles x_k with a smooth distribution of mass. Choose $0 < \beta < \gamma < 1$ and consider the initial points and velocities

$$\bar x_k \;=\; \beta^k \mathbf{e}_1\,, \qquad\qquad \bar v_k \;=\; (1-\gamma^k)\mathbf{e}_1 \tag{5.10}$$

as in (4.7). For each $k \geq 1$, we choose $a_k, r_k > 0$ and define the smooth function

$$\psi_k(x) \;=\; \begin{cases} a_k \exp\left\{ \dfrac{-1}{r_k^2 - |x - \bar x_k|^2}\right\} & \text{if } \ |x - \bar x_k| < r_k\,, \\[2ex] 0 & \text{otherwise.}\end{cases}$$

Notice that $\psi_k \in \mathcal{C}_c^\infty(I\!R^n)$, with support contained in the ball $B_k \doteq \{x \in I\!R^n\,;\ |x - \bar x_k| \leq r_k\}$.

As initial velocity we choose a continuous function $\bar v$ such that

$$\bar v(x) \;=\; (1-\gamma^k)\mathbf{e}_1 + \frac{\bar x_k - x}{\sqrt{r_k}} \qquad \text{if } \ |x - \bar x_k| \leq r_k\,, \quad k \geq 1. \tag{5.11}$$

Notice that, for $t \geq \sqrt{r_k}$ all the mass initially located inside the ball B_k gets concentrated at the single point

$$x_k(t) \;=\; \bar x_k + t\bar v_k \;=\; [\beta^k + t(1-\gamma^k)]\mathbf{e}_1.$$

The choice of the coefficients a_k, r_k is made in two stages.

First we choose the sequence of radii $r_k \downarrow 0$ decreasing to zero fast enough so that (i) the balls B_k, $k \geq 1$ are mutually disjoint and (ii) during the time interval $[0, \sqrt{r_k}]$, particles originating from the ball B_k do not interact with any other particles from different balls.

Afterwards, we choose the coefficients $a_k \downarrow 0$ decreasing to zero fast enough so that the function $\rho(x) \doteq \sum_{k\geq 1} \psi_k(x)$ is in $\mathcal{C}_c^\infty$. At this stage we also observe that the conclusion of Lemma 2 remains valid if the particle masses, instead of $m_k = \alpha^k$, are given by $m_k \;=\; \int \psi_k(x)\,dx$. Indeed, the only relevant assumption is that $m_k/m_{k-1} \to 0$ fast enough.

2. In this step we replace the white particles X_k with a smooth distribution of mass. For this purpose, it is worth noting that in Example 4 there is a lot of freedom in the choice of the positions and masses of the particles X_k. Indeed, the only thing that matters is the identity $X_k(\tau_k) = b_k(\tau_k)\mathbf{e}_1$.

Let the functions $b_k(\cdot)$ and the times τ_k be as in (4.5). We can then consider a sequence of particles X_k, $k \geq 1$, with initial velocity and position given respectively by

$$\overline{V}_k \;=\; \mathbf{e}_1 - \frac{\mathbf{e}_2}{k}\,, \qquad\qquad \overline{X}_k \;=\; b_k(\tau_k)\mathbf{e}_1 - \tau_k \overline{V}_k\,. \tag{5.12}$$

Observe that (5.10) and (5.12) imply

$$\lim_{k\to\infty} \bar{x}_k \;=\; \lim_{k\to\infty} \overline{X}_k \;=\; 0, \qquad\qquad \lim_{k\to\infty} \bar{v}_k \;=\; \lim_{k\to\infty} \overline{V}_k \;=\; \mathbf{e}_1\,. \tag{5.13}$$

We can now replace the countably many particles X_k with a continuous distribution of mass, as in the previous step. For each $k \geq 1$, choose $\tilde{a}_k, R_k > 0$ and define the smooth function

$$\tilde{\psi}_k(x) \;=\; \begin{cases} \tilde{a}_k \exp\left\{ \dfrac{-1}{R_k^2 - |x - \overline{X}_k|^2} \right\} & \text{if } \; |x - \overline{X}_k| \leq R_k\,, \\[3mm] 0 & \text{otherwise.} \end{cases}$$

Notice that $\tilde{\psi}_k \in \mathcal{C}_c^\infty(I\!R^n)$, with support contained inside the ball $\widetilde{B}_k \doteq \{x \in I\!R^n\,;\; |x - \overline{X}_k| \leq R_k\}$.

As initial velocity we choose a continuous function $\bar{v} : I\!R^n \mapsto I\!R^n$, with bounded support, that satisfies (5.11) together with

$$\bar{v}(x) \;=\; (1 - \gamma^k)\mathbf{e}_1 + \frac{\overline{X}_k - x}{\sqrt{R_k}} \qquad\qquad \text{if } \; |x - \overline{X}_k| \leq R_k\,, \quad k \geq 1\,. \tag{5.14}$$

Notice that, for $t \geq \sqrt{R_k}$ all the mass initially located inside the ball $\widetilde{B}_k$ gets concentrated at the single point

$$x_k(t) \;=\; \overline{X}_k + t\overline{V}_k\,.$$

The choice of the coefficients $\tilde{a}_k, R_k$ is made in two stages.

First we choose the sequence of radii $R_k \downarrow 0$ decreasing to zero fast enough so that (i) all the balls B_j, $\widetilde{B}_k$, $j, k \geq 1$ are mutually disjoint and (ii) during the time interval $[0, \sqrt{R_k}]$, particles originating from the ball $\widetilde{B}_k$ do not interact with any other particles from different balls.

Afterwards, we choose the coefficients $\tilde{a}_k \downarrow 0$ decreasing to zero fast enough so that the function $\tilde{\rho}(x) \doteq \sum_{k\geq 1} \tilde{\psi}_k(x)$ is in $\mathcal{C}_c^\infty$.

3. By the same argument introduced in Example 4, for the initial data consisting of countably many point masses $\bar{x}_k, \overline{X}_k$ with initial velocities $\bar{v}_k, \overline{V}_k$, no sticky solution exists, not even locally in time. By the above construction, the same conclusion holds for an initial distribution of mass with smooth density

$$\bar{\rho}(x) \;\doteq\; C \cdot \left(\sum_{k=1}^{\infty} \psi_k(x) + \sum_{k=1}^{\infty} \tilde{\psi}_k(x) \right)$$

and continuous initial velocity field $\bar{v}(\cdot)$ Here $C > 0$ is a normalizing constant, chosen so that $\int \bar{\rho}(x)\, dx = 1$. □

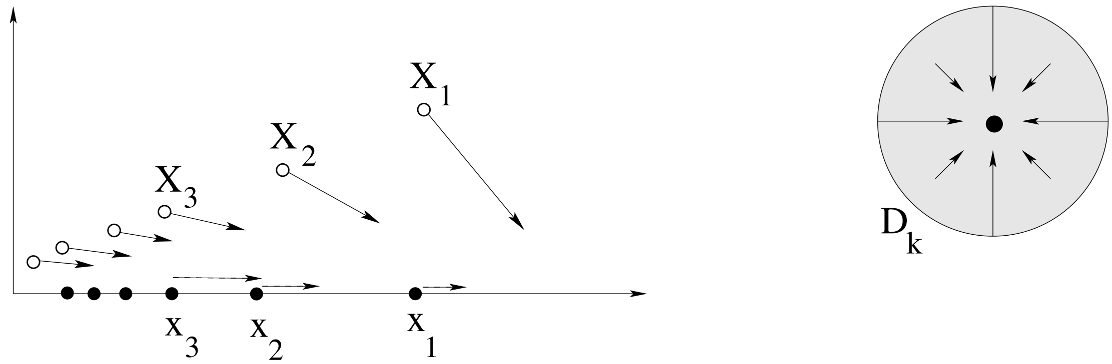

Figure 5: Modifying the initial data in Figure 4 in order to obtain a continuous distribution of initial velocities. Left: the initial speed of the particles x_k, X_k approaches the same limit as $k \to \infty$ and $\bar{x}_k, \overline{X}_k \to 0$. Right: a point mass is replaced by a continuous distribution on a ball B_k, choosing the initial velocity so that after a short time all the mass is concentrated at one single point.

In an entirely similar way, one can modify the initial data in Example 3 and obtain

Theorem 2. *In any dimension $n \geq 2$ there exists an initial datum $(\bar{\rho}, \bar{v})$ such that the system* (1.1) *admits two distinct sticky weak solutions. Here $\bar{\rho} \in \mathcal{C}_c^\infty(I\!\!R^n)$ is the density of a probability measure, while $\bar{v} \in \mathcal{C}_c(I\!\!R^n)$ is a continuous initial velocity.*

Remark 4. With a more careful construction, counterexamples to the existence and uniqueness could be achieved with an initial velocity distribution $\bar{v} \in \mathcal{C}_c^{1-\varepsilon}(I\!\!R^n)$ which is Hölder continuous, with any exponent strictly smaller than 1. However, this initial velocity field cannot be Lipschitz continuous: in order that all the mass initially inside B_k or $\widetilde{B}_k$ collapse to a point within time $t_k \to 0$, the Lipschitz constant of $\bar{v}$ in (5.11) and (5.14) must tend to infinity as $k \to \infty$.

On the other hand, if the initial velocity field $\bar{v}$ is continuous with Lipschitz constant L, then it is easy to see that the Cauchy problem (1.1)-(1.2) has a unique local solution defined for $0 \leq t < L^{-1}$. Indeed, for each $0 \leq t < L^{-1}$ the identity

$$v(t,\, x + t\bar{v}(x)) \;=\; \bar{v}(x)$$

uniquely defines a Lipschitz continuous vector field $v(t, \cdot)$. Inserting this function $v(t, \cdot)$ in (1.1), one obtains a linear transport equation for ρ, with Lipschitz velocity field and hence with a unique solution.

References

[1] A. Andrievskii, S. Gurbatov and A. Sobolevskii. Ballistic aggregation in symmetric and nonsymmetric flows. *J. Experimental Theor. Phys.* **104** (2007), 887–896.

[2] F. Bouchut and F. James. Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness. *Comm. Part. Diff. Eq.* **24** (1999), 2173–2189.

[3] Y. Brenier, W. Gangbo, G. Savaré and M. Westdickenberg. Sticky particle dynamics with interactions. *J. Math. Pures Appl.* **99** (2013), 577–617.

[4] Y. Brenier and E. Grenier. Sticky particles and scalar conservation laws. *SIAM J. Numer. Anal.* **35** (1998), 2317–2328.

[5] A. Dermoune and B. Djehiche. Global solution of the pressureless gas equation with viscosity. *Physica D* **163** (2002), no. 3-4, 184–190.

[6] W. E, Y. Rykov and Y. Sinai. Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics. *Comm. Math. Phys.* **177** (1996), 349–380.

[7] F. Huang and Z. Wang. Well Posedness for Pressureless Flow. *Comm. Math. Phys.* **222** (2001), 117–146.

[8] L. Natile and G. Savaré. A Wasserstein approach to the one-dimensional sticky particle system. *SIAM J. Math. Anal.* **41** (2009), 1340–1365.

[9] T. Nguyen and A. Tudorascu. Pressureless Euler/Euler-Poisson systems via adhesion dynamics and scalar conservation laws. *SIAM J. Math. Anal.* **40** (2008), 754–775.

[10] T. Nguyen and A. Tudorascu. One-dimensional pressureless gas systems with/without viscosity. Preprint, 2013.

[11] M. Sever. An existence theorem in the large for zero-pressure gas dynamics. *Diff. Integral Equat.* **14** (2001), 1077–1092.

[12] S. Shandarin and Y. B. Zeldovich. The large-scale structure of the universe: turbulence, intermittency, structures in a self-gravitating medium. *Rev. Modern Phys.* **61** (1989), 185–220.

[13] Y. B. Zeldovich. Gravitational instability: an approximate theory for large density perturbations. *Astro. & Astrophys.* **5** (1970), 84–89.