On Traffic Flow with Nonlocal Flux: A Relaxation Representation

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Abstract

We consider a conservation law model of traffic flow, where the velocity of each car depends on a weighted average of the traffic density ρ ahead. The averaging kernel is of exponential type: $w_\varepsilon(s) = \varepsilon^{-1}e^{-s/\varepsilon}$. By a transformation of coordinates, the problem can be reformulated as a 2×2 hyperbolic system with relaxation. Uniform BV bounds on the solution are thus obtained, independent of the scaling parameter ε. Letting $\varepsilon \to 0$, the limit yields a weak solution to the corresponding conservation law $\rho_t + (\rho v(\rho))_x = 0$. In the case where the velocity $v(\rho) = a - b\rho$ is affine, using the Hardy-Littlewood rearrangement inequality we prove that the limit is the unique entropy-admissible solution to the scalar conservation law.

1 Introduction

We consider a nonlocal PDE model for traffic flow, where the traffic density $\rho = \rho(t, x)$ satisfies a scalar conservation law with nonlocal flux

$$\rho_t + (\rho v(q))_x = 0. \tag{1.1}$$

Here $\rho \mapsto v(\rho)$ is a decreasing function, modeling the velocity of cars depending on the traffic density, while the integral

$$q(x) = \int_0^{+\infty} w(s)\, \rho(x + s)\, ds \tag{1.2}$$

computes a weighted average of the car density. On the function v and the averaging kernel w, we shall always assume

(A1) *The function $v : [0, \rho_{jam}] \mapsto \mathbb{R}_+$ is $\mathcal{C}^2$, and satisfies*

$$v(\rho_{jam}) = 0, \qquad v'(\rho) \leq -\delta_* < 0, \qquad \text{for all } \rho \in [0, \rho_{jam}]. \tag{1.3}$$

(A2) *The weight function* $w \in \mathcal{C}^1(\mathbb{R}_+)$ *satisfies*

$$w'(s) \le 0, \qquad \int_0^{+\infty} w(s)\,ds = 1. \tag{1.4}$$

In **(A1)** one can think of ρ_{jam} as the maximum possible density of cars along the road, when all cars are packed bumper-to-bumper and nobody moves. At a later stage, more specific choices for the functions w and v will be made. In particular, we shall focus on the case where $w(s) = e^{-s}$.

The conservation equation (1.1) will be solved with initial data

$$\rho(0,x) = \bar{\rho}(x) \in [0, \rho_{jam}]. \tag{1.5}$$

Given a weight function w satisfying (1.4), we also consider the rescaled weights

$$w_\varepsilon(s) \doteq \varepsilon^{-1} w(s/\varepsilon). \tag{1.6}$$

As $\varepsilon \to 0+$, the weight w_ε converges to a Dirac mass at the origin, and the nonlocal equation (1.1) formally converges to the scalar conservation law

$$\rho_t + f(\rho)_x = 0, \qquad \text{where} \qquad f(\rho) \doteq \rho v(\rho). \tag{1.7}$$

The main purpose of this paper is to analyze the convergence of solutions of the nonlocal equation (1.1) to those of (1.7).

Conservation laws with nonlocal flux have attracted much interest in recent years, because of their numerous applications and the analytical challenges they pose. Applications of nonlocal models include sedimentation [6], pedestrian flow and crowd dynamics [2, 17, 18, 19], traffic flow [7, 14], synchronization of oscillators [3], slow erosion of granular matter [4], materials with fading memory [10], some biological and industrial models [20], and many others. Due to the nonlocal flux, the equation (1.1) behaves very differently from the classical conservation law (1.7). Its analysis faces additional difficulties and requires novel techniques.

For a fixed weight function w, the well posedness of the nonlocal conservation laws was proved in [7] with a Lax-Friedrich type numerical approximation, in [26] by the method of characteristics, and in [23] using a Godunov type scheme. Traveling waves for related nonlocal models have been recently studied in [13, 31, 32, 33, 34]. See also the results for several space dimensions [1], and other related results in [21, 36].

Up to this date, however, the nonlocal to local limit for (1.1) as $\varepsilon \to 0+$ has remained a challenging question. Namely, is it true that the solutions of the Cauchy problem ρ_ε of (1.1)-(1.2), with averaging kernels w_ε in (1.6), as $\varepsilon \to 0+$ converge to the entropy admissible solutions of (1.7)? The question was already posed in [5]. For a general weight function $w(\cdot)$, whose support covers an entire neighborhood of the origin, a negative answer is provided by the counterexamples in [14]. On the other hand, the results in [14] do not apply to the physically relevant models where the velocity v is a monotone decreasing function and each driver only takes into account the density of traffic ahead (not behind) the car. Indeed, existence and uniqueness result for this more realistic model are given in [7, 12]. Furthermore, various numerical simulations [5, 7] suggest that the behavior of

ρ_ε should be stable in the limit $\varepsilon \to 0+$. See also [16] for the effect of numerical viscosity in the study of this limit. In the case of monotone initial data, a convergence result was recently proved in [25].

The main goal of the present paper is to study the limit behavior of solutions to (1.1), for the averaging kernel $w_\varepsilon(s) = \varepsilon^{-1}\exp(-s/\varepsilon)$, as $\varepsilon \to 0$. In this setting, we first show that (1.1) can be treated as a 2×2 system with relaxation, in a suitable coordinate system. This formulation allows us to obtain a uniform bound on the total variation, independent of ε. As $\varepsilon \to 0$, a standard compactness argument yields the convergence $\rho_\varepsilon \to \rho$ in $\mathbf{L}^1_{loc}$, for a weak solution ρ of (1.7). Finally, in the case of a Lighthill-Whitham speed [28, 35] of the form $v(\rho) = a - b\rho$, we prove that the limit solution ρ coincides with the unique entropy weak solution of (1.7).

The remainder of the paper is organized as follows. Section 2 contains a short proof of global existence, uniqueness, and continuous dependence on the initial data, for solutions to (1.1)-(1.2) with v, w, satisfying **(A1)-(A2)**. For Lipschitz continuous initial data, solutions are constructed locally in time, as the fixed point of a contractive transformation. By suitable a priori estimates, we then show that these Lipschitz solutions can be extended globally in time. In turn, the semigroup of Lipschitz solutions can be continuously extended (w.r.t. the $\mathbf{L}^1$ distance) to a domain containing all initial data with bounded variation.

Starting with Section 3, we restrict our attention to exponential kernels: $w_\varepsilon(s) = \varepsilon^{-1}e^{-s/\varepsilon}$. In this case, the conservation law with nonlocal flux can be reformulated as a hyperbolic system with relaxation. In Section 4, by a suitable transformation of independent and dependent coordinates, we establish a priori BV estimates which are independent of the relaxation parameter ε. We assume here that the initial density is uniformly positive. By a standard compactness argument, in Section 5 we construct the limit of a sequence of solutions with averaging kernels w_ε, as $\varepsilon \to 0$. It is then an easy matter to show that any such limit provides a weak solution to the conservation law (1.7). A much deeper issue is whether this limit coincides with the unique entropy-admissible solution. In Section 6 we prove that this is indeed true, in the special case where the velocity function is affine: $v(\rho) = a - b\rho$. This allows a detailed analysis of the convex entropy $\eta(\rho) = \rho^2$. Using the Hardy-Littlewood rearrangement inequality [24, 27], we show that the entropy production is $\leq \mathcal{O}(1)\cdot\varepsilon$. Hence, in the limit as $\varepsilon \to 0$, this entropy is dissipated.

We leave it as an open question to understand whether the same result is valid for more general velocity functions $v(\cdot)$. Say, for $v(\rho) = a - b\rho^2$. Moreover, all of our techniques heavily rely on the fact that the averaging kernel $w(\cdot)$ is exponential. It would be of much interest to understand what happens for different kind of kernels.

2 Existence of solutions

In this section we consider the Cauchy problem for (1.1)-(1.2), for a given initial datum

$$\rho(0,x) \;=\; \bar\rho(x). \tag{2.1}$$

We consider the domain

$$\mathcal{D} \;\doteq\; \Big\{\rho \in \mathbf{L}^\infty(\mathbb{R})\,; \quad \text{Tot.Var.}\{\rho\} < \infty, \quad \rho(x) \in [0, \rho_{jam}] \quad \text{for all } x \in \mathbb{R}\Big\}. \tag{2.2}$$

Theorem 2.1 *Under the assumptions* **(A1)** *and* **(A2)**, *there exists a unique semigroup* $S : [0, +\infty[\times \mathcal{D} \mapsto \mathcal{D}$, *continuous in* $\mathbf{L}^1_{loc}$, *such that each trajectory* $t \mapsto S_t\bar\rho$ *is a weak solution to the Cauchy problem (1.1)-(1.2), (2.1).*

Proof. We first construct a family of Lipschitz solutions, and show that they depend continuously on time and on the initial data, in the $\mathbf{L}^1$ distance. By an approximation argument, we then construct solutions for general BV data $\bar\rho \in \mathcal{D}$.

1. Consider the domain of Lipschitz functions

$$\begin{aligned}\mathcal{D}_L \ \doteq\ \Big\{\rho \in \mathcal{D}\,; \quad \inf_x \rho(x) > 0, \quad \sup_x \rho(x) < \rho_{jam}\,,\\ |\rho(x) - \rho(y)| \le L|x-y| \ \text{ for all } x, y \in \mathbb{R}\Big\}.\end{aligned} \tag{2.3}$$

For every initial datum $\bar\rho \in \mathcal{D}_L$, we will construct a solution $t \mapsto \rho(t,\cdot) \in \mathcal{D}_{2L}$ as the unique fixed point of a contractive transformation, on a suitably small time interval $[0, t_0]$.

Given any function $t \mapsto \rho(t,\cdot) \in \mathcal{D}_{2L}$, consider the corresponding integral averages

$$q(t,x) \ = \int_0^\infty w(s)\rho(t, x+s)\, ds\,. \tag{2.4}$$

We observe that

$$q_x(t,x) \ = \ \int_0^\infty w(s)\, \rho_x(t, x+s)\, ds.$$

Hence

$$\|q_x(t,\cdot)\|_{\mathbf{L}^\infty} \ \le\ \|\rho_x(t,\cdot)\|_{\mathbf{L}^\infty} \ \le\ 2L\,. \tag{2.5}$$

Moreover, an integration by parts yields

$$q_{xx}(t,x) = \int_0^\infty w(s)\, \rho_{xx}(t, x+s)\, ds = -w(0)\rho_x(t,x) - \int_0^\infty w'(s)\, \rho_x(t,x+s)\, ds\,,$$

therefore

$$\begin{aligned}\|q_{xx}(t,\cdot)\|_{\mathbf{L}^\infty} &\le\ w(0)\|\rho_x(t,\cdot)\|_{\mathbf{L}^\infty} + \|w'\|_{\mathbf{L}^1}\cdot \|\rho_x(t,\cdot)\|_{\mathbf{L}^\infty}\\ &=\ 2w(0)\|\rho_x(t,\cdot)\|_{\mathbf{L}^\infty} \ \le\ 4Lw(0).\end{aligned} \tag{2.6}$$

Consider the transformation $\rho \mapsto u = \Gamma(\rho)$, where u is the solution to the linear Cauchy problem

$$u_t + (v(q)u)_x \ =\ 0, \qquad u(0,x) \ =\ \bar\rho(x), \qquad t \in [0, t_0], \tag{2.7}$$

with q as in (2.4). In the next two steps we shall prove:

(i) The values $\Gamma(u)$ remain uniformly bounded in the $W^{1,\infty}$ norm.

(ii) The map $\Gamma : \mathcal{D}_{2L} \mapsto \mathcal{D}_{2L}$ is contractive w.r.t. the $\mathcal{C}^0$ norm.

By the contraction mapping theorem, a unique fixed point will thus exist, providing the solution to (2.7) on the time interval $[0, t_0]$.

2. To fix the ideas, assume

$$0 \ < \delta_0 \ \leq \ \bar{\rho}(x) \ \leq \ \rho_{jam} - \delta_0 \,, \tag{2.8}$$

for some δ_0. From the equation

$$u_t + v(q)u_x \ = \ -\, v'(q) q_x u \,, \qquad\qquad u(0,x) \ = \ \bar{\rho}, \tag{2.9}$$

integrating along characteristics and using (2.5), we obtain

$$\delta_0 - t \cdot \Big(\|v'\|_{\mathbf{L}^\infty} \, 2L \Big) \rho_{jam} \ \leq \ u(t,x) \ \leq \ \rho_{jam} - \delta_0 + t \cdot \Big(\|v'\|_{\mathbf{L}^\infty} \, 2L \Big) \rho_{jam} \,. \tag{2.10}$$

Choosing $t_0 \ < \ \delta_0 \cdot (\|v'\|_{\mathbf{L}^\infty} \, 2L)^{-1} \cdot \rho_{jam}^{-1}$, the solution u will thus remain strictly positive and smaller than ρ_{jam}, for all $t \in [0, t_0]$.

3. Differentiating the conservation law in (2.7) we obtain

$$u_{xt} + v(q) u_{xx} \ = \ -\, 2v'(q) q_x \, u_x - [v''(q) q_x^2 + v'(q) q_{xx}] u. \tag{2.11}$$

Let $Z(t)$ be the solution to the ODE

$$\dot{Z} \ = \ aZ + b, \qquad\qquad Z(0) \ = \ L \,,$$

where

$$a \ \doteq \ 2\|v'\|_{\mathbf{L}^\infty} \cdot 2L \,, \qquad b \ \doteq \ \Big[4L^2 \|v''\|_{\mathbf{L}^\infty} + 4Lw(0) \|v'\|_{\mathbf{L}^\infty} \Big] \cdot \rho_{jam} \,.$$

Since

$$\|u_x(0,\cdot)\|_{\mathbf{L}^\infty} \ = \ \|\bar{\rho}_x\|_{\mathbf{L}^\infty} \ \leq \ L \,,$$

in view of (2.11) and the bounds (2.5)–(2.6), a comparison argument yields

$$\|u_x(t,\cdot)\|_{\mathbf{L}^\infty} \ \leq \ Z(t). \tag{2.12}$$

In particular, for $t \in [0, t_0]$ with t_0 sufficiently small, we have

$$\|u_x(t,\cdot)\|_{\mathbf{L}^\infty} \ \leq \ 2L \,. \tag{2.13}$$

4. Using the identity

$$q_x(t,x) \ = \ -\, w(0) \rho(t,x) - \int_0^\infty w'(s) \rho(t, x+s) \, ds$$

and recalling that $w'(s) \leq 0$, one obtains the bound

$$\|q_x(t,\cdot)\|_{\mathbf{L}^\infty} \ \leq \ 2w(0) \|\rho(t,\cdot)\|_{\mathbf{L}^\infty}. \tag{2.14}$$

Next, consider two functions $t \mapsto \rho_1(t,\cdot)$, $t \mapsto \rho_2(t,\cdot)$, both taking values inside $\mathcal{D}_{2L}$. Then, for all $t \in [0, t_0]$, the corresponding weighted averages q_1, q_2 satisfy

$$\|q_1(t,\cdot) - q_2(t,\cdot)\|_{W^{1,\infty}} \ \leq \ (1 + 2w(0)) \cdot \sup_{\tau \in [0, t_0]} \|\rho_1(\tau,\cdot) - \rho_2(\tau,\cdot)\|_{\mathbf{L}^\infty} \,. \tag{2.15}$$

By choosing $t_0 > 0$ small enough, we claim that the corresponding solutions u_1, u_2 of (2.7) satisfy

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{\mathbf{L}^\infty} \;\leq\; \frac{1}{2} \sup_{\tau\in[0,t]} \|\rho_1(\tau,\cdot) - \rho_2(\tau,\cdot)\|_{\mathbf{L}^\infty} \qquad \text{for all } t \in [0,t_0]\,. \tag{2.16}$$

Indeed, consider a point (τ, y). Call $t \mapsto x_i(t)$, $i = 1, 2$, the corresponding characteristics. These solve the equations

$$\dot{x}_i \;=\; v(q_i(t, x_i(t))), \qquad\qquad x_i(\tau) = y. \tag{2.17}$$

Hence, moving backward in time, we have

$$\begin{aligned}
&-\frac{d}{dt}|x_1(t) - x_2(t)| \\
&\quad\leq\; \Big|v(q_1(t,x_1(t))) - v(q_1(t,x_2(t)))\Big| + \Big|v(q_1(t,x_2(t))) - v(q_2(t,x_2(t)))\Big| \\
&\quad\leq\; \|v'\|_{\mathbf{L}^\infty}\|q_{1,x}\|_{\mathbf{L}^\infty} \cdot |x_1(t) - x_2(t)| + \|v'\|_{\mathbf{L}^\infty}\|q_1 - q_2\|_{\mathbf{L}^\infty}\,.
\end{aligned}$$

By (2.5), the quantity $\|q_{1,x}(t,\cdot)\|_{\mathbf{L}^\infty}$ remains uniformly bounded. The distance $Z(t) \doteq |x_1(t) - x_2(t)|$ between the two characteristics thus satisfies a differential inequality of the form

$$-\frac{d}{dt}Z(t) \;\leq\; a_* Z(t) + b_*\|q_1(t,\cdot) - q_2(t,\cdot)\|_{\mathbf{L}^\infty}, \qquad Z(\tau) = 0,$$

for some constants a_*, b_*. This implies

$$|x_1(t) - x_2(t)| \;\leq\; \int_t^\tau e^{(t-s)a_*} \cdot b_*\|q_1(s,\cdot) - q_2(s,\cdot)\|_{\mathbf{L}^\infty}\, ds\,. \tag{2.18}$$

The values $u_i(\tau, y)$, $i = 1, 2$, can now be obtained by integrating along characteristics. Indeed,

$$\frac{d}{dt}u_i(t, x_i(t)) \;=\; v'(q_i(t,x_i(t))) \cdot q_{i,x}(t,x_i(t)) \cdot u_i(t,x_i(t)), \qquad u_i(0,x_i(0)) \;=\; \bar{\rho}(x_i(0)).$$

Thanks to the a priori bounds (2.6) on $\|q_{i,xx}(t,\cdot)\|_{\mathbf{L}^\infty}$, using (2.18) for any $\epsilon > 0$ we can choose $t_0 > 0$ such that

$$|u_1(\tau,y) - u_2(\tau,y)| \;\leq\; \epsilon \cdot \sup_{t\in[0,\tau]} \|q_1(t,\cdot) - q_2(t,\cdot)\|_{\mathbf{L}^\infty}\,,$$

for all $\tau \in [0, t_0]$ and $y \in \mathbb{R}$. In view of (2.15), this implies (2.16).

5. By the contraction mapping principle, there exists a unique function $t \mapsto \rho(t,\cdot)$ such that $\rho(t,\cdot) = u(t,\cdot)$ for all $t \in [0,t_0]$. This fixed point of the transformation Γ provides the unique solution to the Cauchy problem (1.1)-(1.2) with initial data (2.1).

6. In this step we show that this solution can be extended to all times $t > 0$. This requires (i) a priori upper and lower bounds of the form

$$0 \;<\; \delta_0 \;\leq\; \rho(t,x) \;\leq\; \rho_{jam} - \delta_0\,, \tag{2.19}$$

independent of time, and (ii) a priori estimates on the Lipschitz constant, which should remain uniformly bounded on bounded intervals of time.

To establish an upper bound on the solution $\rho(t,\cdot)$, $t \in [0, t_0]$, we analyze its behavior along a characteristic. Fix $\epsilon > 0$. Consider any point (τ, ξ) such that

$$\rho(\tau,\xi) \;\geq\; \sup_{x\in\mathbb{R}} \rho(\tau, x) - \epsilon.$$

At the point (τ, ξ) one has

$$\begin{aligned}
\rho_t + v(q)\rho_x &= -\rho v'(q) q_x \;=\; -\,\rho(\tau,\xi)v'(q(\tau,\xi))\cdot \frac{\partial}{\partial\xi}\left[\int_\xi^{+\infty} \rho(\tau,y)\,w(y-\xi)\,dy\right]\\
&= -\rho(\tau,\xi)v'(q(\tau,\xi))\cdot\left[-\rho(\tau,\xi)\,w(0) - \int_\xi^{+\infty} \rho(\tau,y)\,w'(y-\xi)\,dy\right]\\
&= -\rho(\tau,\xi)v'(q(\tau,\xi))\cdot \int_\xi^{+\infty} \big[\rho(\tau,\xi)-\rho(\tau,y)\big]\,w'(y-\xi)\,dy\\
&\leq \rho_{jam}\cdot \max_{0\leq q\leq \rho_{jam}} |v'(q)|\cdot w(0)\cdot\epsilon \;\doteq\; C_0\,\epsilon.
\end{aligned} \tag{2.20}$$

The above implies

$$\frac{d}{dt}\left(\sup_x \rho(t,x)\right) \;\leq\; C_0 \epsilon,$$

as long as $0 < \rho(t,y) < \rho_{jam}$ for all $y \in \mathbb{R}$.

Since $\bar\rho$ satisfies (2.8) and $\epsilon > 0$ is arbitrary, this establishes the upper bound in (2.19). The lower bound is proved in an entirely similar way.

Next, from the analysis in step **3** it follows

$$\|\rho_x(t,\cdot)\|_{\mathbf{L}^\infty} \;\leq\; Z(t), \tag{2.21}$$

which immediately yields the a priori bound on the Lipschitz constant.

By induction, we can thus construct a unique solution $\rho = \rho(t,x)$ on a sequence of time intervals $[0,t_0]$, $[t_0,t_1]$, $[t_1,t_2]$, $\ldots$, where the length of each interval $[t_k\, t_{k+1}]$ depends only on (i) the constant δ_0 in (2.19), and (ii) the Lipschitz constant of $\rho(t_k,\cdot)$. Thanks to (2.21), this Lipschitz constant remains $\leq Z(t_k)$. This implies $t_k \to +\infty$ as $k \to \infty$, hence the solution can be extended to all times $t > 0$.

We remark that, by a further differentiation of the basic equation (1.1), one can prove that, if $\bar\rho \in C^k$, then every derivatives up to order k remains uniformly bounded on bounded intervals of time.

7. To complete the proof, it remains to show that the semigroup of solutions can be extended by continuity to all initial data $\bar\rho \in \mathcal{D}$.

Toward this goal, we first prove that the total variation of the solution $\rho(t,\cdot)$ remains uniformly bounded on bounded time intervals. Indeed, from

$$\rho_{xt} + (v(q)\rho_x)_x \;=\; -\,(v'(q)q_x\rho)_x\,,$$

it follows

$$\begin{aligned}
\frac{d}{dt}\|\rho_x\|_{\mathbf{L}^1} &\le \|(v'(q)q_x\rho)_x\|_{\mathbf{L}^1}\\
&\le \|v'\|_{\mathbf{L}^\infty}\|q_x\|_{\mathbf{L}^\infty}\|\rho_x\|_{\mathbf{L}^1}+\|v'\|_{\mathbf{L}^\infty}\|q_{xx}\|_{\mathbf{L}^1}\|\rho\|_{\mathbf{L}^\infty}+\|v''\|_{\mathbf{L}^\infty}\|q_x\|_{\mathbf{L}^\infty}\|q_x\|_{\mathbf{L}^1}\|\rho\|_{\mathbf{L}^\infty}\\
&\le C\|\rho_x\|_{\mathbf{L}^1}\,. \qquad (2.22)
\end{aligned}$$

Above we used the estimates

$$\|q_x\|_{\mathbf{L}^1}\le\|\rho_x\|_{\mathbf{L}^1},\qquad \|q_{xx}\|_{\mathbf{L}^1}\le 2w(0)\cdot\|\rho_x\|_{\mathbf{L}^1}. \qquad (2.23)$$

Note that in (2.22) the constant C depends on the velocity function $v:[0,\rho_{jam}]\mapsto\mathbb{R}_+$ and the averaging kernel w, but it does not depend on the Lipschitz constant $\|\rho_x\|_{\mathbf{L}^\infty}$ of the solution. According to (2.22), the total variation of the solution grows at most at an exponential rate. In particular, it remains bounded on bounded intervals of time.

8. Thanks to the a priori bounds (2.22) on the total variation and (2.12) on the Lipschitz constant, the solution can be extended to an arbitrarily large time interval $[0,T]$. This already defines a family of trajectories $t\mapsto S_t\bar\rho$ defined for every $L>0$, every $\bar\rho\in\mathcal{D}_L$, and $t\ge 0$.

In order to extend the semigroup S by continuity to the entire domain $\mathcal{D}$, we need to prove that for every $t>0$ the map $\bar\rho\mapsto S_t\bar\rho$ is Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance.

Indeed, consider a family of smooth solutions, say $\rho^\theta(t,\cdot)$, $\theta>0$. Define the first order perturbations

$$\zeta^\theta(t,\cdot)\;=\;\lim_{h\to0}\frac{\rho^{\theta+h}(t,\cdot)-\rho^\theta(t,\cdot)}{h}\,,\qquad Q^\theta(t,\cdot)\;=\;\lim_{h\to0}\frac{q^{\theta+h}(t,\cdot)-q^\theta(t,\cdot)}{h}\,.$$

Notice that

$$Q^\theta(t,x)\;=\;\int_0^{+\infty}w(s)\,\zeta^\theta(t,x+s)\,ds.$$

Then ζ^θ satisfies the linearized equation

$$\zeta_t+(v(q)\zeta)_x+\big(v'(q)Q\rho\big)_x\;=\;0, \qquad (2.24)$$

where for simplicity we dropped the upper indices. Using the estimates

$$\|Q(t,\cdot)\|_{\mathbf{L}^1}\le\|\zeta(t,\cdot)\|_{\mathbf{L}^1},\qquad \|Q_x(t,\cdot)\|_{\mathbf{L}^1}\le 2w(0)\cdot\|\zeta(t,\cdot)\|_{\mathbf{L}^1}\,, \qquad (2.25)$$

$$\|q_x(t,\cdot)\|_{\mathbf{L}^\infty}\le 2w(0)\cdot\rho_{jam},\qquad \|Q(t,\cdot)\|_{\mathbf{L}^\infty}\le w(0)\cdot\|\zeta(t,\cdot)\|_{\mathbf{L}^1}\,, \qquad (2.26)$$

we compute

$$\begin{aligned}
\frac{d}{dt}\|\zeta(t,\cdot)\|_{\mathbf{L}^1} &\le \|(v'(q)Q\rho)_x\|_{\mathbf{L}^1}\\
&\le \|v''\|_{\mathbf{L}^\infty}\|q_x\|_{\mathbf{L}^\infty}\|Q\|_{\mathbf{L}^1}\|\rho\|_{\mathbf{L}^\infty}+\|v'\|_{\mathbf{L}^\infty}\|Q_x\|_{\mathbf{L}^1}\|\rho\|_{\mathbf{L}^\infty}+\|v'\|_{\mathbf{L}^\infty}\|Q\|_{\mathbf{L}^\infty}\|\rho_x\|_{\mathbf{L}^1}\\
&\le C(t)\cdot\|\zeta(t,\cdot)\|_{\mathbf{L}^1}\,. \qquad (2.27)
\end{aligned}$$

Here $C(t)$ depends on time because the total variation $\|\rho_x(t,\cdot)\|_{\mathbf{L}^1}$ may grow at an exponential rate. On the other hand, it is important to observe that $C(t)$ does not depend on the Lipschitz constant of the solutions. From (2.27) we deduce

$$\|\zeta(t,\cdot)\|_{\mathbf{L}^1}\;\le\;\exp\left\{\int_0^t C(\tau)\,d\tau\right\}\|\zeta(0,\cdot)\|_{\mathbf{L}^1}\,. \qquad (2.28)$$

For any two Lipschitz solutions ρ^0, ρ^1 of (1.1)-(1.2), we now construct a 1-parameter family of solutions $\rho^\theta(t,\cdot)$ with initial data

$$\rho^\theta(0,\cdot) \;=\; \theta\rho^1(0,\cdot)+(1-\theta)\rho^0(0,\cdot).$$

Using (2.28) one obtains

$$\begin{aligned}\|\rho^1(t,\cdot)-\rho^0(t,\cdot)\|_{\mathbf{L}^1} \;&\leq\; \int_0^1 \|\zeta^\theta(t,\cdot)\|_{\mathbf{L}^1}\,d\theta \;\leq \int_0^1 \exp\left\{\int_0^t C(\tau)\,d\tau\right\}\cdot\|\zeta^\theta(0,\cdot)\|_{\mathbf{L}^1}\,d\theta\\ &\leq\; \exp\left\{\int_0^t C(\tau)\,d\tau\right\}\cdot\|\rho^1(0,\cdot)-\rho^0(0,\cdot)\|_{\mathbf{L}^1}\,. \qquad (2.29)\end{aligned}$$

This establishes Lipschitz continuity of the semigroup w.r.t. the initial data. Notice that this Lipschitz constant may well depend on time. Since every initial datum $\bar\rho\in\mathcal{D}$ can be approximated in the $\mathbf{L}^1$ distance by a sequence of Lipschitz continuous functions $\bar\rho_n\in\mathcal{D}_{L_n}$ (possibly with $L_n\to+\infty$), by continuity we obtain a unique semigroup defined on the entire domain $\mathcal{D}$. □

Remark 2.1 By the argument in step **6** of the above proof, if the initial condition satisfies

$$0\;\leq\;\bar a\;\leq\;\bar\rho(x)\;\leq\;\bar b\;\leq\;\rho_{jam}\qquad\text{for all } x\in\mathbb{R},$$

then the solution satisfies

$$\bar a\;\leq\;\rho(t,x)\;\leq\;\bar b\qquad\text{for all } t\geq 0,\; x\in\mathbb{R}.$$

3 A hyperbolic system with relaxation

From now on, we focus on the case where $w(s)=e^{-s}$, so that the rescaled kernels are

$$w_\varepsilon(s)=\varepsilon^{-1}e^{-s/\varepsilon}.$$

This yields

$$\frac{\partial}{\partial x}\left[\int_x^{+\infty}\rho(t,s)\,\frac{1}{\varepsilon}\,e^{-(s-x)/\varepsilon}\,ds\right]\;=\;-\,\frac{1}{\varepsilon}\,\rho(t,x)+\frac{1}{\varepsilon}\int_x^{+\infty}\rho(t,s)\,\frac{1}{\varepsilon}\,e^{-(s-x)/\varepsilon}\,ds. \qquad (3.1)$$

Therefore, the averaged density q satisfies the ODE

$$q_x\;=\;\varepsilon^{-1}q-\varepsilon^{-1}\rho\,.$$

The conservation law with nonlocal flux (1.1)-(1.2) can thus be written as

$$\begin{cases}\rho_t+(\rho v(q))_x \;=\; 0,\\ q_x \;=\; \varepsilon^{-1}(q-\rho)\,.\end{cases} \qquad (3.2)$$

To make further progress, we choose a constant $K>v(0)$ and consider new independent coordinates (τ,y) defined by

$$\tau\;=\;t-\frac{x}{K},\qquad\qquad y\;=\;x\,. \qquad (3.3)$$

For future use, we derive the relations between the partial derivative operators in these two sets of coordinates:

$$\partial_\tau \;=\; \partial_t\,, \qquad \partial_y \;=\; \partial_x + K^{-1}\partial_t\,, \qquad \partial_x \;=\; \partial_y - K^{-1}\partial_\tau\,. \tag{3.4}$$

A direct computation yields

$$\rho_t = \rho_\tau, \qquad (\rho v(q))_x = -K^{-1}(\rho v(q))_\tau + (\rho v(q))_y, \qquad q_x = -K^{-1}q_\tau + q_y.$$

In these new coordinates, the equations (3.2) take the form

$$\begin{cases} (K\rho - \rho v(q))_\tau + (K\rho v(q))_y &=& 0, \\ \qquad q_\tau - Kq_y &=& \dfrac{K}{\varepsilon}(\rho - q). \end{cases} \tag{3.5}$$

One can easily verify that the above system of balance laws is strictly hyperbolic, with two distinct characteristic speeds

$$\lambda_1 = -K, \qquad \lambda_2 = \frac{Kv(q)}{K - v(q)}\,. \tag{3.6}$$

We observe that $\lambda_1 < 0 < \lambda_2$, provided that K is sufficiently large such that $K > v(0)$. Moreover, both characteristic families are linearly degenerate.

In the zero relaxation limit, letting $\varepsilon \to 0+$ one formally obtains $q \to \rho$. Hence (3.5) formally converges to the scalar conservation law

$$(K\rho - \rho v(\rho))_\tau + (K\rho v(\rho))_y \;=\; 0. \tag{3.7}$$

Recalling the function f defined in (1.7), one obtains

$$(K\rho - f(\rho))_\tau + (Kf(\rho))_y \;=\; 0. \tag{3.8}$$

Note that (3.8) is equivalent to the conservation law (1.7) in the original (t, x) coordinates.

The characteristic speed for (3.8) is

$$\lambda^* \;=\; \frac{Kf'(\rho)}{K - f'(\rho)} \;=\; \frac{K^2}{K - f'(\rho)} - K. \tag{3.9}$$

Since $K > v(0) \geq v(\rho) > f'(\rho)$, we clearly have $\lambda^* > -K = \lambda_1$. Furthermore, since $f'(\rho) < v(\rho)$, we conclude that $\lambda^* < \lambda_2$. The sub-characteristic condition

$$\lambda_1 \;<\; \lambda^* \;<\; \lambda_2 \tag{3.10}$$

is thus satisfied. This is a crucial condition for stability of the relaxation system, see [29]. For other related general references on zero relaxation limit, we refer to [9, 11].

From (3.5) it follows

$$\begin{aligned} (K - v(q))\rho_\tau + Kv(q)\rho_y &= \rho\big[v(q)_\tau - Kv(q)_y\big] \\ &= \rho v'(q)(q_\tau - Kq_y) \;=\; \rho v'(q)\cdot\frac{K}{\varepsilon}(\rho - q). \end{aligned}$$

We can thus write (3.5) in diagonal form:

$$
\begin{cases}
\rho_\tau + \dfrac{Kv(q)}{K-v(q)}\rho_y & = \dfrac{K}{\varepsilon}\cdot(\rho-q)\cdot\dfrac{\rho v'(q)}{K-v(q)}, \\
q_\tau - Kq_y & = \dfrac{K}{\varepsilon}\cdot(\rho-q).
\end{cases}
\tag{3.11}
$$

To further analyze (3.11), it is convenient to introduce the new dependent variables

$$
u \;=\; \ln\rho, \qquad z \;=\; \ln(K-v(q)), \tag{3.12}
$$

so that

$$
\rho \;=\; e^u, \qquad v(q) \;=\; K-e^z. \tag{3.13}
$$

Using these new variables, (3.11) becomes

$$
\begin{cases}
u_\tau + K(Ke^{-z}-1)u_y & = \dfrac{K}{\varepsilon}\Lambda(u,z), \\
z_\tau - Kz_y & = -\dfrac{K}{\varepsilon}\Lambda(u,z),
\end{cases}
\tag{3.14}
$$

where the source term Λ is given by

$$
\Lambda(u,z) \;=\; (\rho(u)-q(z))\frac{v'(q(z))}{K-v(q(z))}\,. \tag{3.15}
$$

Introducing the monotone function

$$
g(u) \;\doteq\; \ln(K-v(e^u)), \qquad \text{where} \quad g'(u) \;=\; \frac{-v'(e^u)e^u}{K-v(e^u)} \;>\; 0, \tag{3.16}
$$

one checks that

$$
\Lambda(u,g(u)) \;=\; 0 \qquad \text{for all } u. \tag{3.17}
$$

Letting $\varepsilon \to 0$, we expect that $z \to g(u)$ hence the system (3.14) formally converges to the scalar conservation law

$$
(u+g(u))_\tau + K(Ke^{-g(u)}-1)u_y - Kg(u)_y = 0. \tag{3.18}
$$

Using the identities

$$
u+g(u) \;=\; \ln(e^u(K-v(e^u))), \quad e^{-g(u)} \;=\; \frac{1}{K-v(e^u)}, \quad Ke^{-g(u)}-1 \;=\; \frac{v(e^u)}{K-v(e^u)}\,,
$$

we get

$$
\frac{(e^u(K-v(e^u)))_\tau}{e^u(K-v(e^u))} + \frac{K(e^u v(e^u))_y}{e^u(K-v(e^u))} \;=\; 0.
$$

Writing $\rho = e^u$, we obtain once again the conservation law (3.7).

4 A priori BV bounds

In order to prove a rigorous convergence result, we need an a priori BV bound on the solution to the system (3.14), independent of the relaxation parameter ε. We always assume that the velocity v satisfies the assumptions **(A1)**.

Differentiating (3.14) w.r.t. y one obtains

$$\begin{cases} u_{y\tau} + [K(Ke^{-z} - 1)u_y]_y &= \dfrac{K}{\varepsilon}\,[\Lambda_u u_y + \Lambda_z z_y], \\ z_{y\tau} - K z_{yy} &= -\dfrac{K}{\varepsilon}\,[\Lambda_u u_y + \Lambda_z z_y]. \end{cases} \tag{4.1}$$

A kinetic interpretation of the above system is shown in Figure 1.

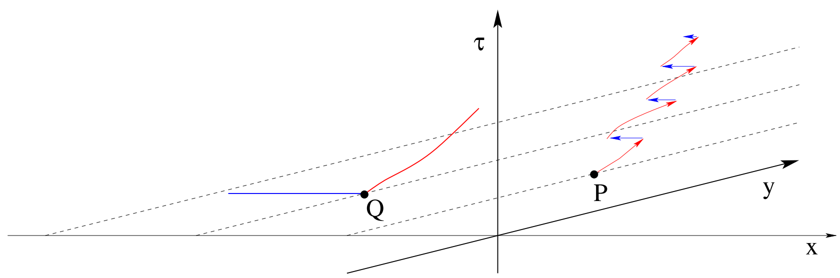

Figure 1: The new system of coordinates (τ, y) defined at (3.3), is illustrated here together with the original coordinates (t, x). The two characteristics through a point Q have speeds $\lambda_1 < 0 < \lambda_2$, as in (3.6). With reference to the system (4.1), one can think of z_y as the density of backward-moving particles, with speed $\lambda_1 = -K$, while u_y is the density of forward-moving particles, with speed $\lambda_2 > 0$. Backward particles are transformed into forward particles at rate $K\Lambda_z/\varepsilon$, while forward particles turn into backward ones with rate $-K\Lambda_u/\varepsilon$. The total number of particles does not increase; actually, it decreases when positive and negative particles of the same type cancel out.

We observe that

$$\begin{aligned} &\frac{d}{d\tau}\int |u_y(\tau, y)|\,dy + \frac{d}{d\tau}\int |z_y(\tau, y)|\,dy \\ &\quad= \frac{K}{\varepsilon}\int \Big\{\mathrm{sign}(u_y)[\Lambda_u u_y + \Lambda_z z_y] - \mathrm{sign}(z_y)[\Lambda_u u_y + \Lambda_z z_y]\Big\}\,dy \\ &\quad\leq \frac{K}{\varepsilon}\int \Big\{\Lambda_u |u_y| + |\Lambda_z|\cdot|z_y| + |\Lambda_u|\cdot|u_y| - \Lambda_z |z_y|\Big\}\,dy. \end{aligned}$$

Therefore, if

$$\Lambda_u \;\leq\; 0, \qquad\qquad \Lambda_z \;\geq\; 0, \tag{4.2}$$

then the map

$$\tau \;\mapsto\; \|u_y(\tau,\cdot)\|_{\mathbf{L}^1} + \|z_y(\tau,\cdot)\|_{\mathbf{L}^1}$$

will be non-increasing. By (3.15), a direct computation yields

$$\Lambda_u \;=\; e^u \frac{v'(q(z))}{K - v(q(z))} \;<\; 0.$$

It remains to verify that $\Lambda_z \geq 0$. Since $\frac{\partial q}{\partial z} > 0$, it suffices to show that $\Lambda_q \geq 0$. We compute

$$\begin{aligned} \Lambda_q &= (\rho - q)\frac{v''(q)(K - v(q)) + (v'(q))^2}{(K - v(q))^2} - \frac{v'(q)}{K - v(q)} \\ &= \frac{1}{K - v(q)} \left[(\rho - q)\Big(v''(q) + \frac{(v'(q))^2}{K - v(q)} \Big) - v'(q) \right]. \end{aligned} \tag{4.3}$$

Since $v'(q) < 0$, the above inequality will hold provided that

$$|\rho - q| \cdot \left(|v''(q)| + \frac{(v'(q))^2}{K - v(q)} \right) \leq \big|v'(q)\big| \qquad \text{for all } q. \tag{4.4}$$

Notice that, by choosing K sufficiently large, the factor $\frac{(v')^2}{K-v}$ can be rendered as small as we like. Hence we can always achieve the inequality (4.4) provided that:

- Either $|\rho - q|$ remains small. This is certainly the case if the oscillation of the initial datum is small.
- Or else, $|v''|$ is small compared with $|v'|$.

As a consequence of the above analysis, we have:

Lemma 4.1 *Let (u, z) be a Lipschitz solution to the relaxation system (4.1). Assume that $\rho(\tau, y) = e^{u(\tau,y)} \in [\rho_1, \rho_2]$ for all (τ, y), and moreover*

$$\min_{q \in [\rho_1, \rho_2]} |v'(q)| \geq (\rho_2 - \rho_1) \cdot \left(\|v''\|_{\mathbf{L}^\infty} + \frac{\|v'\|^2_{\mathbf{L}^\infty}}{K - \|v\|_{\mathbf{L}^\infty}} \right). \tag{4.5}$$

Then the total variation function

$$\tau \mapsto \|u_y(\tau, \cdot)\|_{\mathbf{L}^1(\mathbb{R})} + \|z_y(\tau, \cdot)\|_{\mathbf{L}^1(\mathbb{R})} \tag{4.6}$$

is non-increasing.

We observe that, in the case where v is affine, say

$$v(\rho) = a_o - b_o \rho \tag{4.7}$$

for some $a_o, b_o > 0$, by (1.3) we can always choose K large enough so that

$$\rho_{jam} \cdot \frac{\|v'\|^2_{\mathbf{L}^\infty}}{K - \|v\|_{\mathbf{L}^\infty}} \leq \min_{0 \leq q \leq \rho_{jam}} |v'(q)|. \tag{4.8}$$

Hence (4.5) is satisfied.

Our main goal is to obtain uniform BV bounds for solutions to the nonlocal conservation law (1.1)-(1.2). This will be achieved by working in the (τ, y) coordinate system.

Theorem 4.1 *Consider the Cauchy problem for (1.1)-(1.2), with kernel $w(s) = \varepsilon^{-1}e^{-s/\varepsilon}$. Assume that the velocity function v satisfies*

$$\min_{\rho\in[0,\rho_{jam}]} |v'(q)| \;>\; \rho_{jam}\cdot\|v''\|_{\mathbf{L}^\infty([0,\rho_{jam}])}\,. \tag{4.9}$$

Moreover, assume that the initial density $\bar\rho$ has bounded variation and is uniformly positive. Namely,

$$0 \;<\; \rho_{\min} \;\leq\; \bar\rho(x) \;\leq \rho_{\max} \leq\; \rho_{jam} \qquad\qquad \textit{for all } x\in\mathbb{R}. \tag{4.10}$$

Then the total variation remains uniformly bounded in time:

$$\text{Tot.Var.}\{\rho(t,\cdot)\} \;\leq\; \frac{\rho_{\max}}{\rho_{\min}}\cdot \text{Tot.Var.}\{\bar\rho\} \qquad\qquad \textit{for all } t\geq 0. \tag{4.11}$$

Proof. 1. Assume first that ρ is Lipschitz continuous. By (4.1) it follows

$$\text{div}\begin{pmatrix} u_y\\ K(Ke^{-z}-1)u_y\end{pmatrix} + \text{div}\begin{pmatrix} z_y\\ -Kz_y\end{pmatrix} \;=\; 0. \tag{4.12}$$

Thanks to (4.9), we can choose a constant K large enough so that (4.2) holds. In this case we also have

$$\text{div}\begin{pmatrix} |u_y|\\ K(Ke^{-z}-1)|u_y|\end{pmatrix} + \text{div}\begin{pmatrix} |z_y|\\ -K|z_y|\end{pmatrix} \;\leq\; 0. \tag{4.13}$$

In terms of the original (t,x) coordinates, by (3.4) the inequality (4.13) takes the form

$$\begin{aligned}
&\partial_t\left(\Big|u_x+\frac{u_t}{K}\Big| + \Big|z_x+\frac{z_t}{K}\Big|\right) + \left(\frac{1}{K}\partial_t+\partial_x\right)\left(K(Ke^{-z}-1)\Big|u_x+\frac{u_t}{K}\Big| - K\Big|z_x+\frac{z_t}{K}\Big|\right)\\
&\quad=\;\partial_t\left(Ke^{-z}\Big|u_x+\frac{u_t}{K}\Big|\right) + \partial_x\left(K(Ke^{-z}-1)\Big|u_x+\frac{u_t}{K}\Big| - K\Big|z_x+\frac{z_t}{K}\Big|\right) \;\leq\; 0.
\end{aligned} \tag{4.14}$$

2. Integrating (4.14) over any time interval $[0,T]$, we obtain

$$\begin{aligned}
&\int \frac{K}{K-v(q(T,x))}\Big|u_x(T,x)+\frac{u_t(T,x)}{K}\Big|\,dx\\
&\qquad\leq\;\int \frac{K}{K-v(q(0,x))}\Big|u_x(0,x)+\frac{u_t(0,x)}{K}\Big|\,dx\,.
\end{aligned} \tag{4.15}$$

Since we are choosing $K > v(0)\geq v(q(t,x))$ for all t,x, the above denominators remain uniformly positive and bounded. This implies

$$\int \Big|u_x(T,x)+\frac{u_t(T,x)}{K}\Big|\,dx \;\leq\; C_K\cdot\int\Big|u_x(0,x)+\frac{u_t(0,x)}{K}\Big|\,dx\,, \tag{4.16}$$

with $C_K \doteq \frac{K}{K-v(0)}$.

Repeating the same argument, with K replaced by γK where $\gamma>1$, we obtain

$$\int \Big|u_x(T,x)+\frac{u_t(T,x)}{\gamma K}\Big|\,dx \;\leq\; C_{\gamma K}\cdot\int\Big|u_x(0,x)+\frac{u_t(0,x)}{\gamma K}\Big|\,dx\,, \tag{4.17}$$

where the constant is now $C_{\gamma K} = \frac{\gamma K}{\gamma K - v(0)}$.

3. Next, we observe that, for any two numbers α, β and any number $\gamma > 1$ one has

$$\alpha \;=\; \frac{\gamma}{\gamma-1}\left(\alpha + \frac{\beta}{\gamma}\right) - \frac{1}{\gamma-1}(\alpha+\beta),$$

so

$$|\alpha| \;\leq\; \frac{\gamma}{\gamma-1}\left|\alpha + \frac{\beta}{\gamma}\right| + \frac{1}{\gamma-1}|\alpha+\beta|.$$

Applying the above inequality with $\alpha = u_x$, $\beta = K^{-1}u_t$, from (4.16)-(4.17) one obtains

$$\begin{aligned}\int \big|u_x(T,x)\big|\,dx \;\;&\leq\;\; \frac{\gamma C_{\gamma K}}{\gamma-1}\int \left|u_x(0,x) + \frac{u_t(0,x)}{\gamma K}\right| dx \\ &\qquad + \frac{C_K}{\gamma-1}\int \left|u_x(0,x) + \frac{u_t(0,x)}{K}\right| dx\,. \end{aligned} \tag{4.18}$$

4. By the assumption (4.10) and Remark 2.1 it follows that

$$0 \;<\; \rho_{\min} \;\leq\; \rho(t,x) \;\leq\; \rho_{\max} \qquad \text{for all} \;\; t \geq 0,\;\; x \in \mathbb{R}.$$

By the change of variables (3.12)-(3.13), one has

$$|u_x| \;=\; \frac{|\rho_x|}{\rho} \;\leq\; \frac{|\rho_x|}{\rho_{\min}}\,, \qquad |u_t| \;=\; \frac{|\rho_t|}{\rho} \;\leq\; \frac{|\rho_t|}{\rho_{\min}}\,, \qquad |\rho_x| \;\leq\; \rho_{\max}|u_x|\,. \tag{4.19}$$

Combining (4.19) with (4.18) we conclude

$$\begin{aligned}\rho_{\max}^{-1}\int \big|\rho_x(T,x)\big|\,dx \;&\leq\; \int \big|u_x(T,x)\big|\,dx \\ \leq\;& \frac{\gamma C_{\gamma K}}{\gamma-1}\int \left|u_x(0,x) + \frac{u_t(0,x)}{\gamma K}\right| dx \;+\; \frac{C_K}{\gamma-1}\int \left|u_x(0,x) + \frac{u_t(0,x)}{K}\right| dx \\ \leq\;& \frac{\gamma C_{\gamma K}}{(\gamma-1)\rho_{\min}}\int \left[|\rho_x(0,x)| + \frac{|\rho_t(0,x)|}{\gamma K}\right] dx \\ &\qquad + \frac{C_K}{(\gamma-1)\rho_{\min}}\int \left[|\rho_x(0,x)| + \frac{|\rho_t(0,x)|}{K}\right] dx. \end{aligned} \tag{4.20}$$

We observe that

$$\begin{aligned}\int |\rho_t(0,x)|\,dx \;\;&\leq\;\; \int |\bar\rho_x(x)v(q(0,x))| + \big|\bar\rho(x)v'(q(0,x))q_x(0,x)\big|\,dx \\ &\leq\;\; \|v\|_{\mathbf{L}^\infty}\cdot\|\bar\rho_x\|_{\mathbf{L}^1} + \rho_{\max}\cdot\|v'\|_{\mathbf{L}^\infty}\cdot\|q_x(0,\cdot)\|_{\mathbf{L}^1} \;\leq\; C_0\|\bar\rho_x\|_{\mathbf{L}^1}, \end{aligned}$$

where $C_0 \doteq \|v\|_{\mathbf{L}^\infty} + \rho_{\max}\cdot\|v'\|_{\mathbf{L}^\infty}$ is a bounded constant. Recalling the values of the constants $C_K, C_{\gamma K}$, from (4.20) we obtain

$$\begin{aligned}&\int \big|\rho_x(T,x)\big|\,dx \\ &\quad\leq\; \frac{\rho_{\max}}{\rho_{\min}}\cdot\frac{1}{\gamma-1}\left(\frac{\gamma^2 K}{\gamma K - v(0)}\Big(1+\frac{C_0}{\gamma K}\Big) + \frac{K}{K-v(0)}\Big(1+\frac{C_0}{K}\Big)\right)\cdot\|\bar\rho_x\|_{\mathbf{L}^1}\,. \end{aligned} \tag{4.21}$$

Since the constant K can be chosen arbitrarily large, letting $K \to +\infty$ in (4.21) we obtain

$$\|\rho_x(T,\cdot)\|_{\mathbf{L}^1} \;\leq\; \frac{\rho_{\max}}{\rho_{\min}} \cdot \frac{\gamma+1}{\gamma-1} \cdot \|\bar{\rho}_x\|_{\mathbf{L}^1} \,.$$

We note that as $K \to \infty$, (4.5) reduces to (4.9). Again, since $\gamma > 1$ can be chosen arbitrarily large, letting $\gamma \to \infty$ we obtain

$$\|\rho_x(T,\cdot)\|_{\mathbf{L}^1} \;\leq\; \frac{\rho_{\max}}{\rho_{\min}} \cdot \|\bar{\rho}_x\|_{\mathbf{L}^1} \,. \tag{4.22}$$

For any Lipschitz solution, this provides an a priori bound on the total variation, which does not depend on time or on the relaxation parameter ε. By an approximation argument we conclude that (4.11) holds, for every uniformly positive initial condition $\bar{\rho}$ with bounded variation. □

5 Existence of a limit solution

Relying on the a priori bound on the total variation, proved in Theorem 4.1, we now show the existence of a limit $\rho = \lim_{\varepsilon\to 0+} \rho_\varepsilon$, which provides a weak solution to the conservation law (1.7).

Theorem 5.1 *Let $\bar{\rho} : \mathbb{R} \mapsto [\rho_{\min}, \rho_{\max}]$ be a uniformly positive initial datum, with bounded variation. Call ρ_ε the corresponding solutions to (1.1)-(1.2), with averaging kernel $w_\varepsilon(s) = \varepsilon^{-1}e^{-s/\varepsilon}$. Then, by possibly extracting a subsequence $\varepsilon_n \to 0$, one obtains the convergence $\rho_{\varepsilon_n} \to \rho$ in $\mathbf{L}^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$. The limit function ρ provides a weak solution to the conservation law (1.7).*

Proof. By Theorem 4.1, all solutions $\rho_\varepsilon(t,\cdot)$ have uniformly bounded total variation. The same is thus true for the weighted averages $q_\varepsilon(t,\cdot)$, where

$$q_\varepsilon(t,x) \;=\; \int_0^{+\infty} \varepsilon^{-1}e^{-s/\varepsilon}\rho_\varepsilon(t,x+s)\,ds\,. \tag{5.1}$$

By (1.1), this implies that the map $t \mapsto \rho_\varepsilon(t,\cdot)$ is uniformly Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance.

By a compactness argument based on Helly's theorem (see for example Theorem 2.4 in [8]), we can select a sequence $\varepsilon_n \downarrow 0$ such that

$$\rho_{\varepsilon_n} \;\to\; \rho \qquad \text{in} \;\; \mathbf{L}^1_{loc}(\mathbb{R}_+ \times \mathbb{R}), \tag{5.2}$$

$$\rho_{\varepsilon_n}(t,\cdot) \;\to\; \rho(t,\cdot) \qquad \text{in} \;\; \mathbf{L}^1_{loc}(\mathbb{R}), \qquad \text{for a.e. } t \geq 0. \tag{5.3}$$

By (5.1), it now follows

$$\begin{aligned}
\big\|q_\varepsilon(t,\cdot) - \rho_\varepsilon(t,\cdot)\big\|_{\mathbf{L}^1} \;&=\; \iint_{x<y} \varepsilon^{-1}e^{(x-y)/\varepsilon}\big|\rho_\varepsilon(t,y) - \rho_\varepsilon(t,x)\big|\,dy\,dx \\
&\leq\; \iiint_{x<s<y} \varepsilon^{-1}e^{(x-y)/\varepsilon}\big|\rho_{\varepsilon,x}(t,s)\big|\,ds\,dy\,dx \\
&=\; \int_{-\infty}^{+\infty}\left(\int_0^{+\infty}\int_0^{+\infty} \varepsilon^{-1}e^{-\sigma/\varepsilon}e^{-\xi/\varepsilon}\,d\xi\,d\sigma\right)\big|\rho_{\varepsilon,x}(t,s)\big|\,ds \\
&=\; \varepsilon \cdot \text{Tot.Var.}\{\rho_\varepsilon(t,\cdot)\},
\end{aligned}$$

where the variables $\sigma = y - s$, $\xi = s - x$ were used. Therefore, as $\varepsilon_n \to 0$, we have the convergence $q_{\varepsilon_n} \to \rho$ in $\mathbf{L}^1_{loc}$. By (1.1), this implies that the limit function $\rho = \rho(t, x)$ is a weak solution to the scalar conservation law (1.7). $\square$

6 Entropy admissibility of the limit solution

In the previous section we proved that, as $\varepsilon \to 0$, any limit in $\mathbf{L}^1_{loc}$ of solutions u_ε to (1.1), (1.5) with $\bar\rho \in BV$ and q_ε given by (5.1) is a weak solution to the conservation law (1.7). A key question is whether this limit is the unique entropy admissible solution. The following analysis shows that this is indeed the case when the velocity function is affine, namely

$$v(\rho) \;=\; a - b\rho\,. \tag{6.1}$$

Theorem 6.1 *Let the velocity function v be affine. Consider any uniformly positive initial datum $\bar\rho \in BV$. Then as $\varepsilon \to 0$, the corresponding solutions ρ_ε to (1.1), (5.1), (1.5) converge to the unique entropy admissible solution of (1.7).*

Proof. For simplicity, we consider the case where $v(\rho) = 1 - \rho$. The general case (6.1) is entirely similar. According to [22, 30], to prove uniqueness it suffices to prove that the limit solution dissipates one single strictly convex entropy. We thus consider the entropy and entropy flux pair

$$\eta(\rho) \;=\; \frac{\rho^2}{2}\,, \qquad\qquad \psi(\rho) \;=\; \frac{\rho^2}{2} - \frac{2\rho^3}{3}\,. \tag{6.2}$$

When $v(\rho) = 1 - \rho$, the equation (1.1) can be written as

$$\rho_t + (\rho(1-\rho))_x \;=\; (\rho(1-\rho) - \rho(1-q))_x \;=\; (\rho(q-\rho))_x\,.$$

Multiplying both sides by $\eta'(\rho) = \rho$, we obtain

$$\eta(\rho)_t + \psi(\rho)_x \;=\; \rho(\rho(q-\rho))_x \;=\; (\rho^2(q-\rho))_x - (q-\rho)\rho\rho_x\,. \tag{6.3}$$

Given a test function $\varphi \in \mathcal{C}^1_c(\mathbb{R})$, $\varphi \geq 0$, we thus need to estimate the quantity

$$J \;=\; J_1 - J_2\,,$$

where

$$J_1 \;\doteq\; \int (\rho^2(q-\rho))_x \varphi\, dx \;=\; -\int \rho^2(q-\rho)\varphi_x\, dx\,, \tag{6.4}$$

$$\begin{aligned} J_2 \;&\doteq\; \int \big(q(x) - \rho(x)\big)\cdot \rho(x)\rho_x(x)\varphi(x)\, dx \\ &=\; \int \left(\int_x^{+\infty} \frac{1}{\varepsilon} e^{(x-y)/\varepsilon} \left(\int_x^y \rho_x(s)\, ds\right) dy\right) \rho(x)\,\rho_x(x)\,\varphi(x)\, dx\,. \end{aligned} \tag{6.5}$$

Our ultimate goal is to show that

$$J \;\leq\; \mathcal{O}(1)\cdot\varepsilon.$$

Since we have

$$|J_1| \;\leq\; \|\rho\|^2_{\mathbf{L}^\infty} \cdot \|\varphi_x\|_{\mathbf{L}^\infty} \cdot \int |q(x) - \rho(x)|\, dx \;=\; \mathcal{O}(1) \cdot \varepsilon,$$

it remains to show that

$$J_2 \;\geq\; \mathcal{O}(1) \cdot \varepsilon. \tag{6.6}$$

A key tool to achieve this estimate is

Lemma 6.1 (Hardy-Littlewood inequality). *For any two functions $g_1, g_2 \geq 0$ vanishing at infinity, one has*

$$\int g_1(x)\, g_2(x)\, dx \;\leq\; \int g_1^*(x) g_2^*(x)\, dx, \tag{6.7}$$

where g_1^*, g_2^* are the symmetric decreasing rearrangements of g_1, g_2, respectively.

For a proof, see [24] or [27].

Starting from (6.5) we compute

$$\begin{aligned}
J_2 \;&=\; \iiint_{x<s<y} \frac{1}{\varepsilon} e^{(x-y)/\varepsilon} \rho_x(s)\rho(x)\rho_x(x)\,\varphi(x)\, dy\, ds\, dx \\
&=\; \iint_{x<s} e^{(x-s)/\varepsilon} \rho_x(s)\, \rho(x)\rho_x(x)\,\varphi(x)\, dx\, ds \\
&=\; \int \left(\int_x^{+\infty} e^{-s/\varepsilon} \rho_x(s)\, ds \right) e^{x/\varepsilon} \rho(x)\rho_x(x)\,\varphi(x)\, dx \\
&=\; -\int \rho^2(x)\rho_x(x)\,\varphi(x)\, dx + \frac{1}{\varepsilon} \iint_{x<s} e^{-s/\varepsilon} \rho(s)\, e^{x/\varepsilon} \rho(x)\rho_x(x)\,\varphi(x)\, dx\, ds \\
&=\; \int \frac{\rho^3(x)}{3} \varphi_x(x)\, dx + \frac{1}{\varepsilon} \int e^{-s/\varepsilon} \rho(s) \;\left(\int_{-\infty}^s \left(\frac{\rho^2(x)}{2}\right)_x e^{x/\varepsilon} \varphi(x)\, dx \right) ds \\
&\doteq\; A + B - C - D,
\end{aligned}$$

where

$$\begin{aligned}
A \;&\doteq\; \int \frac{\rho^3(x)}{3} \varphi_x(x)\, dx\,, \\
B \;&\doteq\; \frac{1}{\varepsilon} \int \rho(s)\, \frac{\rho^2(s)}{2}\, \varphi(s)\, ds\,, \\
C \;&\doteq\; \frac{1}{\varepsilon^2} \int \int_{-\infty}^s e^{(x-s)/\varepsilon} \frac{\rho^2(x)}{2}\, \rho(s)\, \varphi(x)\, dx\, ds\,, \\
D \;&\doteq\; \frac{1}{\varepsilon} \int \int_{-\infty}^s e^{(x-s)/\varepsilon} \frac{\rho^2(x)}{2}\, \rho(s)\, \varphi_x(x)\, dx\, ds\,.
\end{aligned}$$

To achieve some cancellations, using a Taylor expansion of the term C we obtain

$$C \doteq C_1 + C_2 + C_3\,,$$

where

$$
\begin{aligned}
C_1 &\doteq \frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{\rho^2(x)}{2}\,\rho(s)\,\varphi(s)\,dx\,ds\,,\\
C_2 &\doteq \frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{\rho^2(x)}{2}\,\rho(s)\,(x-s)\varphi_x(x)\,dx\,ds\,,\\
C_3 &\doteq \frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{\rho^2(x)}{2}\,\rho(s)\,\frac{(x-s)^2}{2}\varphi_{xx}(\zeta)\,dx\,ds\,.
\end{aligned}
\tag{6.8}
$$

In the integral for C_3, it is understood that for each x, s one must choose a suitable $\zeta = \zeta(x,s) \in [x,s]$.

We now compare the integrals B and C_1. Without loss of generality one can assume $\varphi = \phi^3$ for some $\phi \in \mathcal{C}^2_c$, $\phi \geq 0$. For any $\sigma \geq 0$, we now apply the Hardy-Littlewood inequality with

$$
g_1(x) \;=\; \rho^2(x)\phi^2(x), \qquad g_2(x) \;=\; \rho(x+\sigma)\,\phi(x+\sigma),
$$

and obtain

$$
\int \frac{\rho^2(x)}{2}\,\rho(x)\,\varphi(x)\,dx \;\geq\; \int \frac{\rho^2(x)}{2}\,\phi^2(x)\cdot\rho(x+\sigma)\,\phi(x+\sigma)\,dx\,. \tag{6.9}
$$

Indeed, the level sets of the two functions $\rho^2\phi^2$ and $\rho\phi$ are the same. By (6.7), the integral on the right hand side of (6.9) is maximum (and coincides with $\int g_1^* g_2^*\,dx$) when $\sigma = 0$.

Performing the change of variable $s = x+\sigma$, a further integration w.r.t. s yields

$$
\begin{aligned}
B &= \frac{1}{\varepsilon}\int \frac{\rho^2(x)}{2}\,\rho(x)\,\varphi(x)\,dx \;\geq\; \frac{1}{\varepsilon}\int \frac{\rho^2(x)}{2}\,\phi^2(x)\cdot\rho(s)\,\phi(s)\,dx\\
&\geq \frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{\rho^2(x)}{2}\,\phi^2(x)\cdot\rho(s)\,\phi(s)\,dx\,ds \;\doteq\; B_1 - B_2\,,
\end{aligned}
\tag{6.10}
$$

where

$$
\begin{aligned}
B_1 &\doteq \frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{\rho^2(x)}{2}\,\rho(s)\,\phi^3(s)\,dx\,ds \;=\; C_1\,, \qquad (6.11)\\
B_2 &\doteq \frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{\rho^2(x)}{2}\,\phi(x)\rho(s)\,[\phi^2(s)-\phi^2(x)]\,dx\,ds.
\end{aligned}
$$

To compute the last integral for B_2 we use the Taylor expansion

$$
\phi^2(s) - \phi^2(x) \;=\; 2\phi(x)\phi_x(x)\cdot(s-x) + [2\phi_x^2(\zeta) + 2\phi_{xx}(\zeta)]\cdot\frac{(s-x)^2}{2}\,,
$$

where $\zeta = \zeta(x,s) \in [x,s]$. This yields

$$
\begin{aligned}
B_2 &= \frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}(s-x)\cdot\frac{\rho^2(x)}{2}\,\rho(s)\,\phi^2(x)\,2\phi_x(x)dx\,ds\\
&\quad+\frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{(s-x)^2}{2}\cdot\frac{\rho^2(x)}{2}\,\rho(s)\,\phi(x)\,[2\phi_x^2(\zeta)+2\phi_{xx}(\zeta)]\,dx\,ds\\
&= B_{21} + B_{22} + B_{23}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
B_{21} &\doteq \frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}(s-x)\cdot\rho^3(x)\,\phi^2(x)\phi_x(x)dx\,ds\,,\\
B_{22} &\doteq \frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}(s-x)\cdot\rho^2(x)\,\left(\int_x^s \rho_x(\sigma)\,d\sigma\right)\,\phi^2(x)\phi_x(x)dx\,ds\,,\\
B_{23} &\doteq \frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{(s-x)^2}{2}\cdot\rho^2(x)\,\rho(s)\,\phi(x)\,[\phi_x^2(\zeta)+\phi_{xx}(\zeta)]\,dx\,ds\,.
\end{aligned}
$$

The term B_{21} is computed by

$$
B_{21} \;=\; \int \rho^3(x)\,\frac{\varphi_x(x)}{3}\,dx \;=\; A\,. \tag{6.12}
$$

Concerning B_{22}, using σ, x, and $\xi = s-x$ as variables of integration, we obtain

$$
\begin{aligned}
|B_{22}| &\leq \|\rho\|^2_{\mathbf{L}^\infty}\cdot\frac{1}{3}\|\varphi_x\|_{\mathbf{L}^\infty}\cdot\frac{1}{\varepsilon^2}\iiint_{x<\sigma<s} e^{(x-s)/\varepsilon}(s-x)|\rho_x(\sigma)|\,dx\,d\sigma\,ds\\
&= \|\rho\|^2_{\mathbf{L}^\infty}\cdot\frac{1}{3}\|\varphi_x\|_{\mathbf{L}^\infty}\cdot\frac{1}{\varepsilon^2}\int\int_0^{+\infty} e^{-\xi/\varepsilon}\xi\left(\int_{\sigma-\xi}^{\sigma} dx\right)d\xi\,|\rho_x(\sigma)|\,d\sigma\\
&= \|\rho\|^2_{\mathbf{L}^\infty}\cdot\frac{1}{3}\|\varphi_x\|_{\mathbf{L}^\infty}\cdot\int\left(\int_0^{+\infty}\frac{e^{-\xi/\varepsilon}}{\varepsilon^2}\xi^2\,d\xi\right)|\rho_x(\sigma)|\,d\sigma\\
&= \|\rho\|^2_{\mathbf{L}^\infty}\cdot\frac{1}{3}\|\varphi_x\|_{\mathbf{L}^\infty}\cdot\|\rho_x\|_{\mathbf{L}^1}\cdot 2\varepsilon\,.
\end{aligned} \tag{6.13}
$$

The term B_{23} can be estimated by

$$
\begin{aligned}
|B_{23}| &\leq \|\rho\|^2_{\mathbf{L}^\infty}\|\phi\|_{\mathbf{L}^\infty}\Big(\|\phi_x\|^2_{\mathbf{L}^\infty}+\|\phi_{xx}\|_{\mathbf{L}^\infty}\Big)\int|\rho(s)|\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{(x-s)^2}{2\varepsilon^2}\,dx\,ds\\
&= \|\rho\|^2_{\mathbf{L}^\infty}\|\phi\|_{\mathbf{L}^\infty}\Big(\|\phi_x\|^2_{\mathbf{L}^\infty}+\|\phi_{xx}\|_{\mathbf{L}^\infty}\Big)\cdot\|\rho\|_{\mathbf{L}^1}\int_0^{+\infty} e^{-\sigma/\varepsilon}\frac{\sigma^2}{2\varepsilon^2}\,d\sigma\\
&= \|\rho\|^2_{\mathbf{L}^\infty}\|\phi\|_{\mathbf{L}^\infty}\Big(\|\phi_x\|^2_{\mathbf{L}^\infty}+\|\phi_{xx}\|_{\mathbf{L}^\infty}\Big)\cdot\|\rho\|_{\mathbf{L}^1}\cdot\varepsilon\,.
\end{aligned} \tag{6.14}
$$

An entirely similar argument shows that the integral defining C_3 at (6.8) also approaches zero as $\varepsilon \to 0$. Indeed,

$$
\begin{aligned}
|C_3| &= \frac{1}{\varepsilon^2}\left|\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{\rho^2(x)}{2}\,\rho(s)\,\frac{(x-s)^2}{2}\varphi_{xx}(\zeta)\,dx\,ds\right|\\
&\leq \|\varphi_{xx}\|_{\mathbf{L}^\infty}\cdot\|\rho\|^2_{\mathbf{L}^\infty}\cdot\frac{1}{2\varepsilon^2}\int|\rho(s)|\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{(x-s)^2}{2}\,dx\,ds\\
&\leq \|\varphi_{xx}\|_{\mathbf{L}^\infty}\cdot\|\rho^2\|_{\mathbf{L}^\infty}\cdot\|\rho\|_{\mathbf{L}^1}\cdot\frac{1}{2\varepsilon^2}\int_0^{+\infty} e^{-\sigma/\varepsilon}\frac{\sigma^2}{2}\,d\sigma\\
&= \|\varphi_{xx}\|_{\mathbf{L}^\infty}\cdot\|\rho\|^2_{\mathbf{L}^\infty}\cdot\|\rho\|_{\mathbf{L}^1}\cdot\frac{\varepsilon}{2}\,.
\end{aligned} \tag{6.15}
$$

Finally, we estimate the sum of the remaining two terms:

$$
\begin{aligned}
D + C_2 &= \frac{1}{\varepsilon}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{\rho^2(x)}{2}\,\rho(s)\,\varphi_x(x)\,dx\,ds \\
&\qquad -\frac{1}{\varepsilon^2}\int\int_{-\infty}^{s} e^{(x-s)/\varepsilon}\frac{\rho^2(x)}{2}\,\rho(s)\,(s-x)\varphi_x(x)\,dx\,ds \\
&= \int \frac{\rho^2(x)}{2}\varphi_x(x)\left(\int_x^{+\infty} e^{(x-s)/\varepsilon}\Big(\frac{1}{\varepsilon}-\frac{s-x}{\varepsilon^2}\Big)\rho(s)\,ds\right)\,dx\,.
\end{aligned}
$$

Using the identity

$$
\int_x^{+\infty} e^{(x-s)/\varepsilon}\Big(\frac{1}{\varepsilon}-\frac{s-x}{\varepsilon^2}\Big)\,ds = 0\,,
$$

we compute

$$
\begin{aligned}
D + C_2 &= \int \frac{\rho^2(x)}{2}\varphi_x(x)\left(\int_x^{+\infty} e^{(x-s)/\varepsilon}\Big(\frac{1}{\varepsilon}-\frac{s-x}{\varepsilon^2}\Big)[\rho(s)-\rho(x)]\,ds\right)\,dx \\
&= \int \frac{\rho^2(x)}{2}\varphi_x(x)\int_x^{+\infty} e^{(x-s)/\varepsilon}\Big(\frac{1}{\varepsilon}-\frac{s-x}{\varepsilon^2}\Big)\int_x^{s}\rho_x(\sigma)\,d\sigma\,ds\,dx \\
&= \int \frac{\rho^2(x)}{2}\varphi_x(x)\int_x^{+\infty}\rho_x(\sigma)\int_\sigma^{+\infty} e^{(x-s)/\varepsilon}\Big(\frac{1}{\varepsilon}-\frac{s-x}{\varepsilon^2}\Big)\,ds\,d\sigma\,dx \\
&= \int \frac{\rho^2(x)}{2}\varphi_x(x)\int_x^{+\infty}\rho_x(\sigma)\,e^{(x-\sigma)/\varepsilon}\,\frac{x-\sigma}{\varepsilon}\,d\sigma\,dx \\
&= \int \rho_x(\sigma)\int_{-\infty}^{\sigma}\frac{\rho^2(x)}{2}\varphi_x(x)e^{(x-\sigma)/\varepsilon}\,\frac{x-\sigma}{\varepsilon}\,dx\,d\sigma\,.
\end{aligned}
$$

As a consequence, we obtain the following estimate

$$
\begin{aligned}
|D + C_2| &\le \|\rho_x\|_{\mathbf{L}^1}\cdot\left\|\frac{\rho^2}{2}\right\|_{\mathbf{L}^\infty}\cdot\|\varphi_x\|_{\mathbf{L}^\infty}\cdot\int_{-\infty}^{\sigma} e^{(x-\sigma)/\varepsilon}\,\frac{\sigma-x}{\varepsilon}\,dx \\
&= \|\rho_x\|_{\mathbf{L}^1}\cdot\left\|\frac{\rho^2}{2}\right\|_{\mathbf{L}^\infty}\cdot\|\varphi_x\|_{\mathbf{L}^\infty}\cdot\varepsilon\,. \qquad (6.16)
\end{aligned}
$$

Summarizing all the above estimates (6.8)-(6.16), we have

$$
\begin{aligned}
J_2 &= A + B - C - D \\
&\ge A + B_1 - (B_{21}+B_{22}+B_{23}) - (C_1+C_2+C_3) - D \qquad (6.17) \\
&= (A - B_{21}) + (B_1 - C_1) - (D + C_2) - B_{22} - B_{23} - C_3 \qquad (6.18) \\
&= \mathcal{O}(1)\cdot\varepsilon.
\end{aligned}
$$

Indeed, on the line (6.18) the first two terms are zero, while the remaining four terms have size $\mathcal{O}(1)\cdot\varepsilon$. Letting $\varepsilon \to 0$ we thus obtain the desired entropy inequality.

We remark that the inequality on the line (6.17), accounting for possible entropy dissipation, is due to the relation $B \ge B_1 - B_2$ in (6.10). This follows from the Hardy-Littlewood rearrangement inequality. □

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