Optima and Equilibria for a Model of Traffic Flow

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NSF grant: "A Theory of Complex Transportation Network Design"

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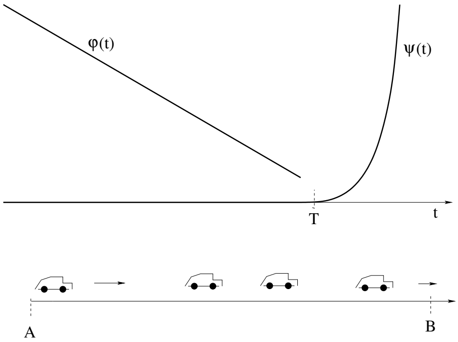
A Traffic Flow Problem

- Car drivers starting from a location A (a residential neighborhood) need to reach a destination B (a working place) at a given time T.
- There is a cost $\varphi(\tau_d)$ for departing early and a cost $\psi(\tau_a)$ for arriving late.



Elementary solution

$$L = \text{length of the road}, \qquad v = \text{speed of cars}$$

$$\tau_a = \tau_d + \frac{L}{v}$$

Optimal departure time:

$$\tau_d^{\rm opt} = \operatorname*{argmin}_t \left\{ \varphi(t) + \psi\left(t + \frac{L}{v}\right) \right\}.$$

If everyone departs exactly at the same optimal time,
a traffic jam is created and this strategy is not optimal anymore.

An optimization problem for traffic flow

Problem: choose the departure rate $\bar{u}(t)$ in order to minimize the total cost to all drivers.

$$u(t,x) \doteq \rho(t,x)\cdot v(\rho(t,x)) = \text{flux of cars}$$

$$\text{minimize:} \qquad \int \varphi(t)\cdot u(t,0)\,dt + \int \psi(t)u(t,L)\,dt$$

for a solution of

$$\begin{cases} \rho_t + [\rho\, v(\rho)]_x = 0 & x\in[0,L] \\ \rho(t,0)v(\rho(t,0)) = \bar{u}(t) \end{cases}$$

Choose the optimal departure rate $\bar{u}(t)$, subject to the constraint

$$\int \bar{u}(t)\,dt = \kappa = [\text{total number of drivers}]$$

Equivalent formulations

Boundary value problem for the density ρ:

$$\begin{aligned} \text{conservation law:} \quad & \rho_t + [\rho v(\rho)]_x = 0, \qquad (t,x) \in \mathbb{R} \times [0,L] \\ \text{control (on the boundary data):} \quad & \rho(t,0)v(\rho(t,0)) = \bar{u}(t) \end{aligned}$$

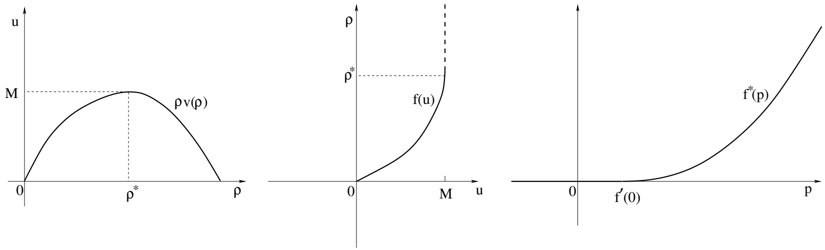
Cauchy problem for the flux u:

$$\begin{aligned} \text{conservation law:} \quad & u_x + f(u)_t = 0, \qquad u = \rho\, v(\rho), \qquad f(u) = \rho \\ \text{control (on the initial data):} \quad & u(t,0) = \bar{u}(t) \end{aligned}$$

$$\text{Cost:} \quad J(u) = \int_{-\infty}^{+\infty} \varphi(t) u(t,0)\, dt + \int_{-\infty}^{+\infty} \psi(t) u(t,L)\, dt$$

$$\text{Constraint:} \quad \int_{-\infty}^{+\infty} \bar{u}(t)\, dt = \kappa$$

The flux function and its Legendre transform



$$u \;=\; \rho\, v(\rho)\,, \qquad\qquad \rho \;=\; f(u)$$

Legendre transform: $$f^*(p) \;\doteq\; \max_u \Big\{ pu - f(u) \Big\} \qquad (1)$$

Solution to the conservation law is provided by the Lax formula

The globally optimal (Pareto) solution

$$\text{minimize:} \qquad J(u) \;=\; \int \varphi(x)\cdot u(0,x)\,dx + \int \psi(x)\,u(T,x)\,dx$$

$$\text{subject to:} \begin{cases} u_t + f(u)_x \;=\; 0 \\ u(0,x) \;=\; \bar u(x)\,, \qquad\qquad \int \bar u(x)\,dx \;=\; \kappa \end{cases}$$

(A1) The flux function $f : [0, M] \mapsto \mathbb{R}$ is continuous, increasing, and strictly convex. It is twice continuously differentiable on the open interval $]0, M[$ and satisfies

$$f(0) = 0\,, \quad \lim_{u\to M-} f'(u) = +\infty, \qquad f''(u) \geq b > 0 \quad \text{for} \quad 0 < u < M\,. \tag{2}$$

(A2) The cost functions φ, ψ satisfy $\varphi' \;<\; 0, \quad \psi, \psi' \;\geq\; 0,$

$$\lim_{x\to-\infty} \varphi(x) \;=\; +\infty\,, \qquad \lim_{x\to+\infty} \Big(\varphi(x) + \psi(x)\Big) \;=\; +\infty$$

Existence and characterization of the optimal solution

Theorem (A.B. and K. Han, 2011). Let **(A1)-(A2)** hold. Then, for any given T, κ, there exists a unique admissible initial data $\bar{u}$ minimizing the cost $J(\cdot)$. In addition,

- No shocks are present, hence $u = u(t, x)$ is continuous for $t > 0$. Moreover

$$\sup_{t\in[0,T],\, x\in\mathbb{R}} u(t,\, x) \ <\ M$$

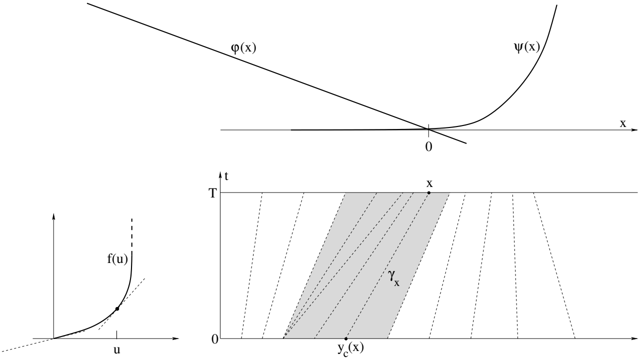
- For some constant $c = c(\kappa)$, this optimal solution admits the following characterization: For every $x \in \mathbb{R}$, let $y_c(x)$ be the unique point such that

$$\varphi(y_c(x)) + \psi(x) \ =\ c$$

  Then, the solution $u = u(t, x)$ is constant along the segment with endpoints $(0, y_c(x))$, (T, x).

  Indeed, either $f'(u) \ \equiv\ \frac{x - y_c(x)}{T}$, or $u \equiv 0$

Necessary conditions



$$\varphi(y_c(x)) + \psi(x) \ = \ c$$

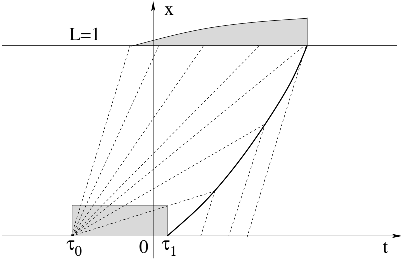
$$f'(u) \ = \ \frac{x - y_c(x)}{T} \qquad \text{on the characteristic segment} \ \ \gamma_x$$

An Example

Cost functions: $\varphi(t) = -t, \qquad \psi(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ t^2, & \text{if } t > 0 \end{cases}$

$$L = 1, \quad u = \rho(2-\rho), \quad M = 1, \quad \kappa = 3.80758$$

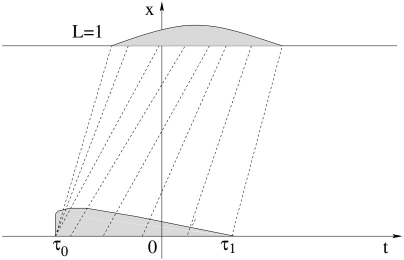
Bang-bang solution



$\tau_0 = -2.78836, \qquad \tau_1 = 1.01924$

total cost $= 5.86767$

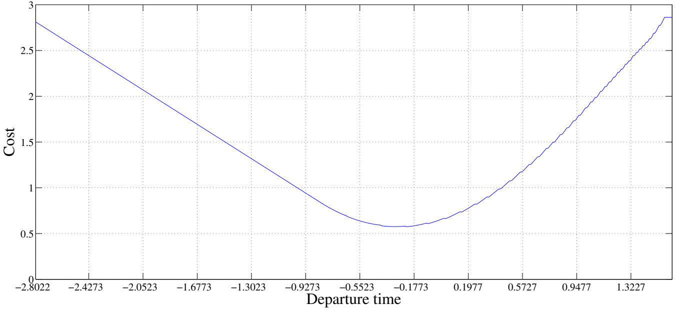
Pareto optimal solution



$\tau_0 = -2.8023, \qquad \tau_1 = 1.5976$

total cost $= 5.5714$

Does everyone pay the same cost?



Departure time vs. cost in the Pareto optimal solution

The Nash equilibrium solution

A solution $u = u(t,x)$ is a Nash equilibrium if no driver can reduce his/her own cost by choosing a different departure time.
This implies that all drivers pay the same cost.

To find a Nash equilibrium, write the conservation law $u_t + f(u)_x \;=\; 0$ in terms of a Hamilton-Jacobi equation

$$U_t + f(U_x) \;=\; 0 \qquad\qquad U(0,x) \;=\; Q(x) \tag{3}$$

$$U(t,\,x) \;\doteq\; \int_{-\infty}^{x} u(t,\,y)\,dy$$

No constraint can be imposed on the departing rate, so a queue can form at the entrance of the highway.

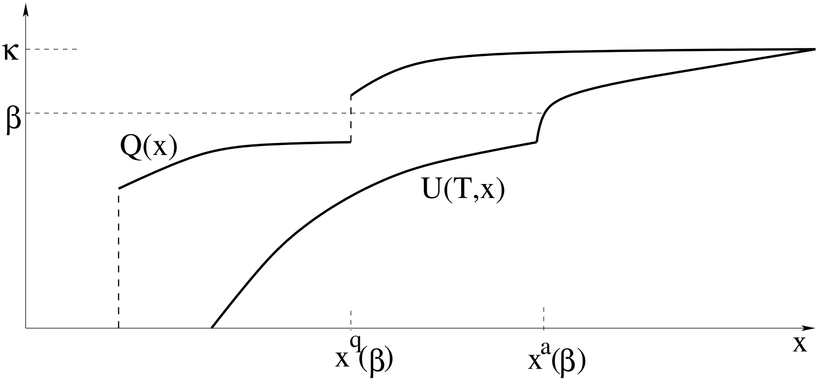
$x \mapsto Q(x) =$ number of drivers who have started their journey before time x (joining the queue, if there is any).

$$Q(-\infty) = 0, \qquad\qquad Q(+\infty) = \kappa$$

$x \mapsto U(T, x) =$ number of drivers who have reached destination within time x

$$U(T, x) = \min_{y \in \mathbb{R}} \left\{ T\, f^* \left(\frac{x-y}{T} \right) + Q(y) \right\}$$

Characterization of a Nash equilibrium



$\beta \in [0, \kappa]$ = Lagrangian variable labeling one particular driver

$x^q(\beta)$ = time when driver β joins the queue

$x^a(\beta)$ = time when driver β arrives at destination

Existence and Uniqueness of Nash equilibrium

Departure and arrival times are implicitly defined by

$$Q(x^q(\beta)-) \;\leq\; \beta \;\leq\; Q(x^q(\beta)+), \qquad\qquad U(T, x^a(\beta)) \;=\; \beta$$

$$\text{Nash equilibrium} \qquad \Longrightarrow \qquad \varphi(x^q(\beta)) + \psi(x^a(\beta)) \;\equiv\; c$$

Theorem (A.B. - K. Han, SIAM J. Applied Math., to appear).

Let the flux f and cost functions φ, ψ satisfy the assumptions (A1)-(A2). Then, for every $\kappa > 0$, the Hamilton-Jacobi equation

$$U_t + f(U_x) \;=\; 0$$

admits a unique Nash equilibrium solution with total mass κ

Sketch of the proof

1. For a given cost c, let $\mathcal{Q}_c$ be the set of all initial data $Q(\cdot)$ for which every driver has a cost $\leq c$:

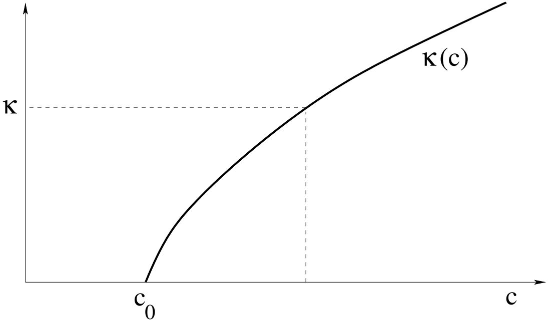
$$\varphi(x^q(\beta)) + \psi(x^a(\beta)) \ \leq \ c \qquad \text{for a.e. } \beta \in [0, \ Q(+\infty)] .$$

2. Claim: $\quad Q^*(x) \ \doteq \ \sup\Big\{ Q(x) \, ; \ \ Q \in \mathcal{Q}_c \Big\}$
is the initial data for a Nash equilibrium with common cost c.

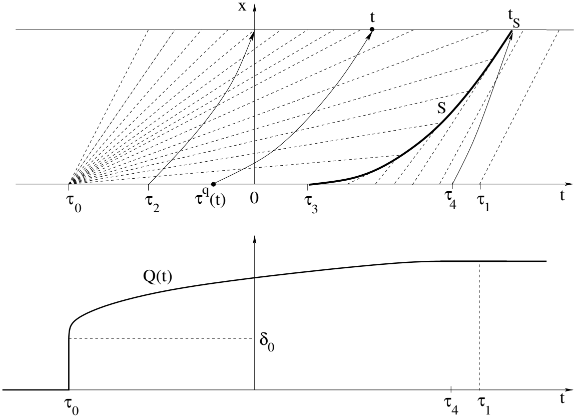
3. For a given cost c, the Nash equilibrium is unique.

4. There exists a minimum cost c_0 such that $\kappa(c) = 0$ for $c \leq c_0$.

The map $c \mapsto \kappa(c)$ is strictly increasing and continuous from $[c_0\,, +\infty[$ to $[0, +\infty[$. $\square$



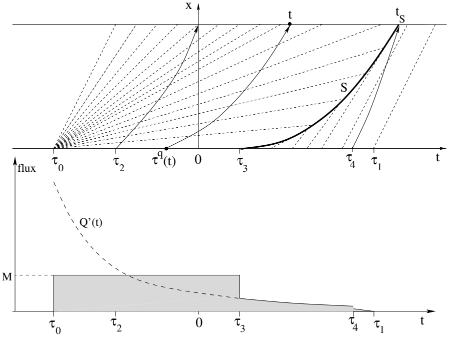
An example of Nash equilibrium



- A queue of size δ_0 forms instantly at time τ_0
- The last driver of this queue departs at τ_2, and arrives at exactly 0.
- The queue is depleted at time τ_3. A shock is formed.
- The last driver departs at τ_1.

Numerical results

$$L = 1, \quad u(\rho) = \rho(2-\rho), \quad M = 1, \quad \kappa = 3.80758, \quad c = 2.7$$



$\tau_0 = -2.7$ $\qquad \tau_2 = -0.9074$

$\tau_3 = 0.9698$ $\qquad \tau_4 = 1.52303$

$\tau_1 = 1.56525$ $\qquad t_S = 2.0550$

$\delta_0 = 1.79259$

total cost $= 10.286$

$$Q(t) = 1.7 + \sqrt{t+2.7} + 1/(4(\sqrt{t+2.7}+2.7))$$

$$Q'(t) = \Big(1 - 1/(4(\sqrt{t+2.7}+2.7)^2)\Big)/(2\sqrt{t+2.7})$$

A comparison

Total cost of the Pareto optimal solution: $J^{opt} \;=\; 5.5714$

Total cost of the Nash equilibrium solution: $J^{Nash} \;=\; 10.286$

Price of anarchy: $J^{Nash} - J^{opt} \;\approx\; 4.715$

Can one eliminate this inefficiency,
yet allowing freedom of choice to each driver ?

(goal of non-cooperative game theory: devise incentives)

Optimal pricing

Scientific American, Dec. 2010: Ten World Changing Ideas

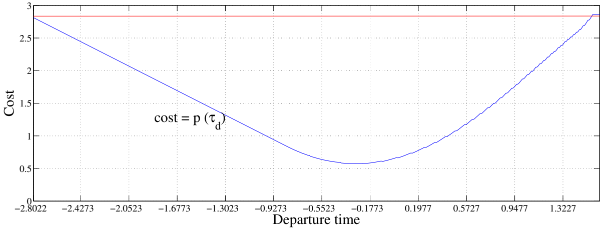
"Building more roads won't eliminate traffic. Smart pricing will."

Suppose a fee $b(t)$ is collected at a toll booth at the entrance of the highway, depending on the departure time.

$$\text{New departure cost:} \qquad \tilde{\varphi}(t) \;=\; \varphi(t) + b(t)$$

Problem: We wish to collect a total revenue R.

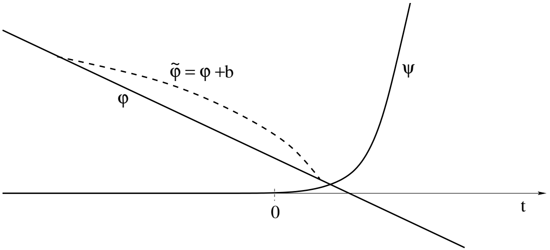
How do we choose $t \mapsto b(t) \geq 0$ so that the Nash solution with departure and arrival costs $\tilde{\varphi}, \psi$ yields the minimum total cost to each driver?



$p(t)$ = cost to a driver starting at time t, in the globally optimal solution

Optimal pricing: $b(t) = p_{max} - p(t) + C$

choosing the constant C so that [total revenue] $= R$.



Continuous dependence of the Nash solution

$\varphi_1(x),\ \varphi_2(x)$ costs for departing at time x

$\psi_1(x),\ \psi_2(x)$ costs for arriving at time x

$v_1(\rho),\ \ v_2(\rho)$ speeds of cars, when the density is $\rho \geq 0$

$Q_1(x),\ Q_2(x) =$ number of cars that have departed up to time x, in the corresponding Nash equilibrium solutions (with zero total cost to all drivers)

Theorem (A.B., C.J.Liu, and F.Yu, 2011)

Assume all cars depart and arrive within the interval $[a, b]$, and the maximum density is $\leq \rho^*$. Then

$$\|Q_1(x) - Q_2(x)\|_{\mathbf{L}^1([a,b])}$$
$$\leq\ C \cdot \left(\|\varphi_1 - \varphi_2\|_{\mathbf{L}^\infty([a,b])} + \|\psi_1 - \psi_2\|_{\mathbf{L}^\infty([a,b])} + \|v_1 - v_2\|^{1/2}_{\mathbf{L}^\infty([0,\rho^*])} \right)$$

A minimax property of Nash equilibria

For any departure distribution $Q(\cdot)$, let

$\Phi(Q) \doteq$ maximum of the total costs, among all drivers

Theorem (A.B., C.J.Liu, and F.Yu, 2011)

Among all starting distributions with κ drivers, the distribution $Q^*(\cdot)$ which yields the Nash equilibrium is a **global minimizer** of Φ.

Drivers with different costs

Assume that there are several groups of drivers, who use the same road but need to reach destination at different times.

For $i = 1, \ldots, N$, the i-th group consists of κ_i drivers, with departure and arrival costs $\varphi_i(x)$, $\psi_i(x)$.

Does there exist a unique global optima and a unique Nash equilibrium solution, in this more general situation?

Existence of a Nash equilibrium for several groups of drivers

Theorem 4 (A.B. & K. Han, 2011).

Let the flux f and cost functions φ_i, ψ_i satisfy the assumptions (A1)-(A2). Then, for every $\kappa_1, \ldots, \kappa_n > 0$, the Hamilton-Jacobi equation

$$U_t + f(U_x) \;=\; 0$$

admits a (possibly non unique) Nash equilibrium solution, where κ_i is the number of drivers of the i-th group.

Sketch of the proof

- For any given costs $c = (c_1, \ldots, c_n)$, there exists at least one Nash solution where each driver of the i-th group pays the same cost c_i.
- Consider the multifunction $c = (c_1, \ldots, c_n) \mapsto K(c)$

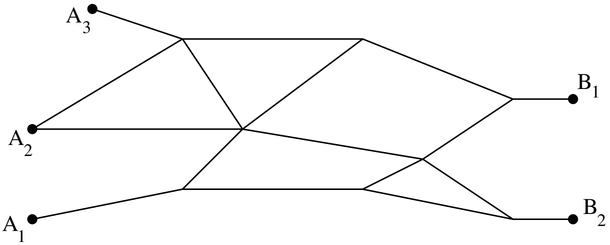
$$K(c) \doteq \Big\{(\kappa_1, \ldots, \kappa_n)\,; \quad \text{there exists a Nash solution where}$$
$$\text{each } i\text{-driver pays a total cost } c_i \text{ and the total number of } i\text{-drivers is } \kappa_i\Big\}$$

- The multifunction $c \mapsto K(c)$ is upper semicontinuous (i.e. it has closed graph), with compact, convex values.
- By a topological argument (using Cellina's approximate selection theorem), as $c = (c_1, \ldots, c_n)$ ranges over $\mathbb{R}^n$, the images $K(c)$ cover the positive cone

$$\mathbb{R}^n_+ = \Big\{(\kappa_1, \ldots, \kappa_n)\,; \quad \kappa_i \geq 0 \qquad i = 1, \ldots, n\Big\}$$

Future work: Network of roads

Extend the previous results to network of roads, including the possibility that drivers choose different routes to get to the same destination.



On each road, the flux satisfies a conservation law

+ boundary conditions at nodes

Stability of Nash equilibrium ?

To justify the practical relevance of a Nash equilibrium, we need to analyze a suitable dynamic model, and show that the rate of departures asymptotically converges to the Nash equilibrium.

Assume that drivers can change their departure time on a day-to-day basis, in order to decrease their own cost.

Introduce an additional variable θ counting the number of days on the calendar.

$$\bar{u}(x,\theta) \doteq \text{ rate of departures at time } x\text{, on day } \theta]$$

$$\Phi(x,\theta) \doteq [\text{cost to a driver starting at time } x\text{, on day } \theta]$$

Model 1: drivers gradually change their departure time, drifting toward times where the cost is smaller.
If the rate of change is proportional to the gradient of the cost, this leads to

$$\bar{u}_\theta + [\Phi_x \, \bar{u}]_x \;=\; 0$$

Model 2: drivers jump to different departure times having a lower cost.
If the rate of change is proportional to the difference between the costs, this leads to

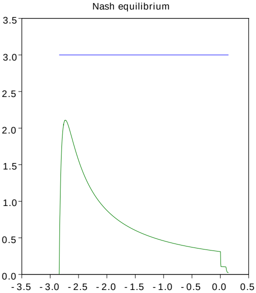
$$\bar{u}_\theta(x) \;=\; \int \bar{u}(y) \Big[\Phi(y) - \Phi(x) \Big]_+ \, dy - \int \bar{u}(x) \Big[\Phi(x) - \Phi(y) \Big]_+ \, dy$$

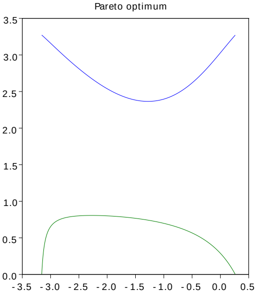
Question: as $\theta \to \infty$, does the departure rate $\overline{u}(x, \theta)$ approach the unique Nash equilibrium?

Some numerical experiments (Wen Shen)

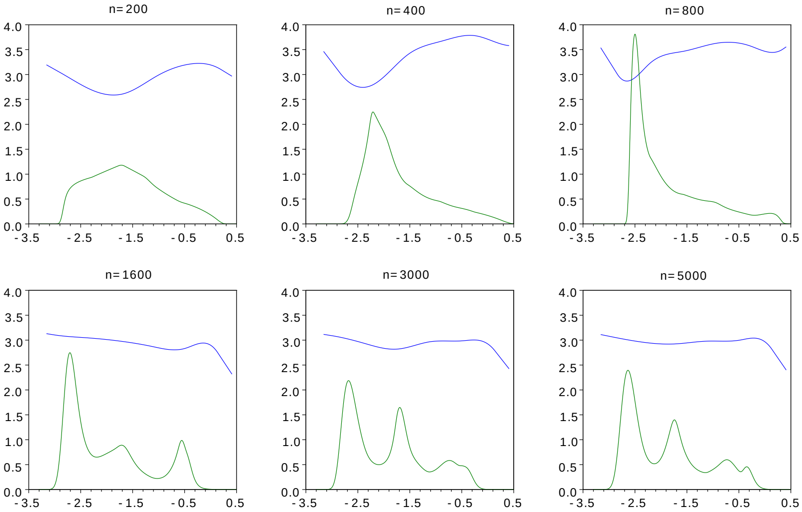
- departure and arrival costs: $\varphi(x) \;=\; -x\,, \qquad \psi(x) \;=\; e^x$
- velocity of cars: $v(\rho) \;=\; 2-\rho$ length of road $= 2$
- total number of cars $= 2.2005$
- common total cost in the Nash equilibrium $= 3$

$$\rho_t + (2\rho - \rho^2)_x \;=\; 0$$





Numerical simulation: Model 1



Numerical simulation: Model 2