Uniqueness Questions for Hyperbolic Conservation Laws

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Continuous dependence on initial data

\[ u_t + f(u)_x = 0 \]

Given two solutions \( u, v \), estimate the difference \( \|u(t, \cdot) - v(t, \cdot)\|_{L^1} \)

Standard approach: set \( w = u - v \), show that

\[ \frac{d}{dt} \|w(t)\| \leq C \|w(t)\| \]

Gronwall’s lemma \( \implies \|w(t)\| \leq e^{Ct} \|w(0)\| \)

Works for Lipschitz solutions, not in the presence of shocks
For two solutions $u, v$ of a hyperbolic system containing shocks, the $L^1$ distance can increase rapidly during short time intervals.
Dependence of approximate solutions on initial data

- **Glimm approximations**: discontinuous dependence
- **Front tracking**: continuous dependence by shifting wave-fronts

\[ \|u - v\|_{L^1} = \sum_{\alpha} |\sigma_{\alpha}| \cdot |\xi_{\alpha}| = \sum_{\alpha} \text{[jump size]} \times \text{[shift]} \]
\[ \| u - v \| = \sum_{\alpha} |\sigma_\alpha| \cdot |\xi_\alpha| \]

How does this distance change across wave front interactions?
CASE 1: interacting fronts of different families $i \neq j$

\[ |\xi_i - \xi'| = \mathcal{O}(1) \cdot |\sigma''| (|\xi'| + |\xi''|) \]

\[ |\xi_k| = \mathcal{O}(1) \cdot (|\xi'| + |\xi''|), \quad |\sigma_k| = \mathcal{O}(1) \cdot |\sigma'\sigma''| \]
CASE 2: interacting fronts of the same $i$-family

$$|\sigma_j| = O(1) \cdot |\sigma'\sigma''|(|\sigma'| + |\sigma''|), \quad |\xi_j| = O(1) \cdot \frac{|\xi'| + |\xi''|}{|\sigma'| + |\sigma''|}$$
Weighted distance: \[ \sum_{\alpha} W_\alpha \cdot |\sigma_\alpha| \cdot |\xi_\alpha| \]

\[ W_\alpha = [\text{total strength of all fronts that are approaching the jump at } x_\alpha] \]

+ \[ C_1 \cdot [\text{interaction potential}] \]

If the total variation is small enough, the weighted distance decreases at every interaction time

\[ \implies \text{uniqueness and Lipschitz continuous dependence in } L^1 \]
Uniqueness for discontinuous ODEs (A.B., Proc. A.M.S., 1988)

\[ \dot{x} = g(t, x) \quad x(0) = x_0 \quad (CP) \]

Assume: BV + transversality:

- \( g(t, x) \in [a, b] \)
- \( g \) has (locally) bounded variation along any path \( t \mapsto x(t) \) such that \( a - \varepsilon \leq \dot{x}(t) \leq b + \varepsilon \)

Then the solution of \((CP)\) is unique and depends Lipschitz continuously on the initial datum \( x_0 \)
For large BV initial data, finite time blow up is possible

\[
\lim_{t \to T^-} \| u(t, \cdot) \|_{L^\infty} = \infty \quad \text{(H.K. Jenssen, SIAM J.Math.Anal. 2000)}
\]

\[
\begin{align*}
  u_t + (uv + w)_x &= 0 \\
  v_t + (v^2/16)_x &= 0 \\
  w_t + (u - uv^2 - vw)_x &= 0
\end{align*}
\]
Unbounded variation can produce non-uniqueness


Strictly hyperbolic $3 \times 3$ system, all fields linearly degenerate

\[
\begin{align*}
    u_t + u_x &= 0 \\
    v_t - v_x &= 0 \\
    w_t + [\phi(u, v) \cdot w]_x &= 0
\end{align*}
\]

Choose initial data $(\bar{u}, \bar{v})$ with unbounded variation, and a smooth function $\phi : \mathbb{R}^2 \mapsto [-1/3, 1/3]$ such that the O.D.E. for characteristics

\[
\dot{x} = g(t, x) \equiv \phi(u(t, x), v(t, x))
\]

has multiple (forward and backward) solutions
\begin{equation*}
\begin{cases}
  u_t + u_x = 0 \\
  v_t - v_x = 0 \\
  w_t + \left[ \frac{1-uv}{6} w \right]_x = 0
\end{cases}
\end{equation*}

\begin{equation*}
u(0, x) = v(0, x) = \varphi(x) = \begin{cases}
  1 & \text{if } 2^{-2n-1} < |x| < 2^{-2n} \\
  -1 & \text{if } 2^{-2n} < |x| < 2^{-2n+1}
\end{cases}
\end{equation*}
An ODE for characteristics with multiple solutions
Well posedness for $L^\infty$ data

A.B. & Paola Goatin, 2000

$$u_t + f(u)_x = 0, \quad u(0) = \bar{u} \in L^\infty$$

- system is Temple class
- all characteristic fields are genuinely nonlinear

Then the Cauchy problem has a unique solution, continuously depending on the initial data.

Temple class $\implies$ wave fronts cross each other, without producing new waves

genuine nonlinearity $\implies$ total variation decays as $t^{-1}$
Uniqueness for $2 \times 2$ systems, small $L^\infty$ data?

\[ u_t + f(u)_x = 0, \quad u(0) = \bar{u} \in L^\infty \]

**Theorem.** (J. Glimm & P. Lax, *A.M.S. Memoir* 1970)

Consider a $2 \times 2$ strictly hyperbolic, genuinely nonlinear system. If $\|\bar{u}\|_{L^\infty}$ is sufficiently small, then the Cauchy problem admits global entropy admissible weak solution.

NOTE: the total variation still decays as $t^{-1}$, but now the interactions can produce additional waves

Is the entropy admissible weak solution unique?

Does it depend continuously on the initial data in $L^1$?
Uniqueness for variational wave equation

\[ u_{tt} - c(u)(c(u)u_x)_x = 0, \quad u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1 \]


Assume \(0 < c_0 \leq c(u) \leq M\).

Then, for any initial data \(u_0 \in H^1(\mathbb{R})\) and \(u_1 \in L^2(\mathbb{R})\), the Cauchy problem admits a conservative solution, defined for all times \(t \in \mathbb{R}\).

**Theorem (uniqueness).**


The conservative solution is unique.

Proof is based on the uniqueness of solutions to the ODE for characteristics, using BV + transversality.
Multi-dimensional, radially symmetric solutions?

Multi-dimensional, variational wave equation:

\[ u_{tt} - c(u) \text{div}(c(u) \nabla u) = 0 \]

Total energy:

\[ E \doteq \frac{1}{2} \int \left( u_t^2 + c^2(u)|\nabla u|^2 \right) dx \]

Can one prove similar existence and uniqueness results, for radially symmetric, conservative solutions?
Hyperbolic conservation laws in several space dimensions

(a personal guess:)

- In one space dimension, the Cauchy problem is well posed as long as the total variation remains bounded.
- In two or more space dimensions, the Cauchy problem is hopelessly ill posed.
Some examples of non-uniqueness

**Example 1:**
\[ \dot{x} = |x|^{1/2} \quad x(0) = \bar{x} \]

A solution can be selected, depending continuously on the initial datum: namely, the unique strictly increasing solution.

**Example 2:**
\[ \dot{x} = x^{1/3} \quad x(0) = \bar{x} \]

There is no way to select a solution depending continuously on the initial datum.
Hopelessly ill posed problems

\[ \frac{d}{dt} u = \Phi(u) \quad u(0) = \bar{u} \in X \]  

(\text{CP})

**Definition.**

The Cauchy problem (CP) is **totally ill posed** on the space \( X \) if there exists a dense subset \( D^u \subset X \) such that the following holds.

- For each initial datum \( \bar{u} \in D^u \), (CP) has a unique solution \( t \mapsto u(t) = S_t \bar{u} \)

- For any open set \( V \subset X \) and \( \tau > 0 \), the map \( (t, \bar{u}) \mapsto S_t \bar{u} \) does NOT admit any continuous extension to \( [0, \tau] \times V \)
The incompressible Euler equation in $\mathbb{R}^2$

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p \\ \text{balance of momentum} \\
\text{div } u \equiv 0 \\ \text{incompressibility condition} \end{cases}$$

$$u = \text{fluid velocity} \quad p = \text{pressure}$$

$$\omega = \text{curl } u = (-u_{1,x_2} + u_{2,x_1}) = \text{vorticity}$$

vorticity equation: $$\omega_t + u \cdot \nabla \omega = 0$$

$u$ is recovered from $\omega$ by the Biot-Savart formula

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^2} \omega(y) \, dy$$
Smooth solutions of the incompressible Euler equation are unique and conserve the energy.

What is the minimum regularity that guarantees that energy is conserved?

What is the minimum regularity that guarantees that solutions are unique?
Regularity thresholds

1 - For conservation of energy (Onsager)

- \( u \in C^\alpha \) for \( \alpha > 1/3 \) \( \implies \) energy is conserved


- \( u \notin C^{1/3} \) \( \implies \) energy may not be conserved

C. DeLellis, L. Szekelyhidi, T. Buckmaster, P. Isett, 2009 – 2017

- To prove non-uniqueness, the Baire category approach is an overkill
- One may still hope that, by imposing further admissibility conditions, a “good solution” can be selected, depending continuously on initial data
2 - For uniqueness of solutions to the Cauchy problem

- \( \text{curl} \ u \in L^\infty(\mathbb{R}^2) \implies \text{solution is unique} \)


- \( \text{curl} \ u \in L^p(\mathbb{R}^2) \setminus L^\infty(\mathbb{R}^2) \implies \text{multiple solutions can occur} ?? \)
Self-similar solutions, with scaling $\frac{1}{2} < \mu < 1$

\[
\begin{align*}
    u(t, x) &= t^{\mu-1} U \left( \frac{x}{t^{\mu}} \right) \quad \text{(velocity)} \\
    \omega(t, x) &= t^{-1} \Omega \left( \frac{x}{t^{\mu}} \right) \quad \text{(vorticity)} \\
    \psi(t, x) &= t^{2\mu-1} \Psi \left( \frac{x}{t^{\mu}} \right) \quad \text{(stream function)}
\end{align*}
\]

\[
\begin{align*}
    \left( \nabla \perp \psi - \mu y \right) \cdot \nabla \Omega &= \Omega \\
    \Delta \psi &= \Omega
\end{align*}
\]

velocity: $U = \nabla \perp \psi$
Constructing two solutions with the same initial data

initial vorticity: \( \bar{\omega}(x) \approx r^{-1/\mu} \cdot \phi(\theta) \)

(support of the initial vorticity)
Plots of the vorticity at time $t = 1$

setting $\bar{\omega}(x) = \varepsilon^{-1/\mu}$ for $|x| < \varepsilon$
setting $\tilde{\omega}(x) = 0$ for $|x| < \varepsilon$
Plots with vorticity $\bar{\omega} \in L^\infty(\mathbb{R}^2)$

setting $\bar{\omega}(x) = 1$ for $|x| < \varepsilon$

setting $\bar{\omega}(x) = 0$ for $|x| < \varepsilon$
Extensions to 2-D hyperbolic systems?

Can we construct Cauchy problems with multiple solutions, also for *slightly compressible, inviscid fluid flow*?
References - existence of small BV solutions


- Definition of genuinely nonlinear, linearly degenerate field
- Solution of the Riemann problem for $n \times n$ systems


- Global existence of weak solutions for small BV initial data
- Approximation scheme by solving Riemann problems
- Restarting procedure by random sampling
- Estimates on the total variation, introducing a wave interaction potential
- Cancellation of errors, by a probabilistic limit theorem
Continuous dependence of entropy weak solutions


- construction of a semigroup of weak solutions by a homotopy method, for $2 \times 2$ systems
- estimates on strength and shifts of wave-fronts, across interactions


- Introduction of a new Lyapunov functional, equivalent to the $L^1$ distance between front tracking solutions
- For $n \times n$ systems, limits of front tracking approximations are unique, and depend Lipschitz continuously on initial data
Uniqueness criteria


- Introduced conditions which imply that weak solution coincides with a semigroup trajectory
- These conditions are satisfied by solutions generated by the Glimm scheme


- Entropy admissibility $+$ tame oscillation condition $\implies$ uniqueness


- Entropy admissibility $+$ tame variation condition $\implies$ uniqueness
Uniqueness of solutions with large data


- For a wide class of $n \times n$ systems, if the initial data is a small BV perturbation of a Riemann solution, the total variation remains bounded.
- Solutions depend continuously on initial data.


- Assuming: (i) system is Temple class, (ii) all fields are genuinely nonlinear, one obtains uniqueness and $L^1$ continuous dependence of solutions, possibly with unbounded variation.
Threshold between uniqueness and non-uniqueness


- Results and counterexamples on the uniqueness for ODEs with discontinuous right hand side, extending Caratheodory’s theorem


- Examples of non-uniqueness for hyperbolic conservation laws with $L^\infty$ initial data
Existence for $L^\infty$ initial data


- For genuinely nonlinear $2 \times 2$ systems, if the initial data $\bar{u}$ has sufficiently small oscillation:
  $$\| \bar{u} - u_0 \|_{L^\infty(\mathbb{R})} < \delta$$
  then a solution exists, globally in time.

- The total variation is locally bounded, for all times $t > 0$.


- Provides a simpler proof of a slightly more general result.
Variational wave equations


- Introduces a change of variables that reduce the equation to a semilinear system with smooth coefficients
- Proves global existence of conservative solutions


- Proves uniqueness, looking at the equations for characteristics (a Hölder continuous ODE, with bounded directional variation)