Growth Models for Tree Stems and Vines

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Abstract

The paper introduces a PDE model for the growth of a tree stem or a vine. The equations describe the elongation due to cell growth, and the response to gravity and to external obstacles. An additional term accounts for the tendency of a vine to curl around branches of other plants.

When obstacles are present, the model takes the form of a differential inclusion with state constraints. At each time t, a cone of admissible reactions is determined by the minimization of an elastic deformation energy. The main theorem shows that local solutions exist and can be prolonged globally in time, except when a specific "breakdown configuration" is reached. Approximate solutions are constructed by an operator-splitting technique. Some numerical simulations are provided at the end of the paper.

1 Introduction

We consider a simple mathematical model describing how the stem of a plant grows, and how it reacts to external constraints, such as branches of other plants. At each time t the stem is described by a curve $\gamma(t, \cdot)$ in 3-dimensional space. The model takes into account the linear elongation due to cell growth and the upward bending as a response to gravity. In the case of vines, an additional term accounts for the tendency to curl around branches of other plants.

From a theoretical perspective, the main challenge comes from the presence of external obstacles, resulting in a number of unilateral constraints. Ultimately, this yields a differential inclusion on a closed subset of $H^2([0,T]; \mathbb{R}^3)$. We remark that most of the literature on differential inclusions with constraints is concerned with the case where the cone of admissible reactions produced by the (possibly moving) obstacle is perpendicular to its boundary [4, 5, 7, 8]. In Moreau's "sweeping process", this assumption plays an essential role in the proof of existence and continuous dependence of solutions. In our model, at a time when part of the stem touches the obstacle, the evolution is governed by the minimization of an instantaneous elastic deformation energy, subject to the external constraints. As a consequence, the cone of admissible velocities determined by the obstacle's reaction can be very different from the normal cone. In certain "breakdown configurations", as shown in Fig. 4, this cone of admissible reactions actually happens to be tangent.

Our main result, Theorem 1 in Section 3, establishes the local existence of solutions to the growth model with obstacles. These solutions can be extended globally in time, provided that a specific "breakdown configuration" is never reached. As already mentioned, since the cone of admissible reactions is not a normal cone, the uniqueness and continuous dependence of solutions is a difficult problem that requires a substantially different approach from [4, 5, 7, 8]. A detailed analysis of this issue will appear in the forthcoming paper [3].

The remainder of this paper is organized as follows. Section 2 introduces the basic model and derives an evolution equation satisfied by the growing curve. If obstacles are present, this takes the form of a differential inclusion in the space $H^2([0,T]\,;\,\mathbb{R}^3)$. This is supplemented by unilateral constraints, requiring that at all times the curve $\gamma(t,\cdot)$ remains outside a given set. In Section 3 we give a definition of solution and state the main existence theorem. Namely, solutions exist locally in time and can be prolonged up to the first time when a "breakdown configuration" is reached. A precise definition of these "bad" configurations is given at (3.9)-(3.10) and illustrated in Fig. 3. In essence, this happens when the tip of the stem touches the obstacle perpendicularly, and all the portions of the stem that do not touch the obstacle are straight segments.

The existence of solutions is proved in Sections 4 and 5, constructing a sequence of approximations by an operator-splitting technique. Each time step involves:

- a regular evolution operator, modeling the linear growth and the bending in response to gravity (possibly including also the curling of vines around branches of other plants),
- a singular operator, accounting for the obstacle reaction.

Much of the analytical work is carried out in Section 4, where we introduce a "push-out" operator and derive some key a priori estimates. Section 5 completes the proof of the main theorem. This is based on a compactness argument, which yields a convergent subsequence of approximate solutions.

In Section 6 we briefly describe how our results can be extended to more general models, including the case where the elastic energies associated with twisting and bending of the stem come with different coefficients. Finally, Section 7 presents some numerical simulations, in the case of one or two obstacles, in two space dimensions. The code used for these simulations can be downloaded at [10].

2 The basic model

We assume that new cells are generated at the tip of the stem, then they grow in size. At time $t \geq 0$, the length of the cells born during the time interval $[s,\, s+ds]$ is measured by

$$d\ell \;=\; (1-e^{-\alpha(t-s)})\,ds\,, \tag{2.1}$$

for some constant $\alpha > 0$. The total length of the stem is thus

$$L(t) \;=\; \int_0^t (1-e^{-\alpha(t-s)})\,ds \;=\; t - \frac{1-e^{-\alpha t}}{\alpha}\,. \tag{2.2}$$

At any given time t, the stem will be described by a curve $s \mapsto P(t,s)$ in 3-dimensional space. For $s \in [0,t]$, the point $P(t,s)$ describes the position at time t of the cell born at time s. In addition, we denote by $\mathbf{k}(t,s)$ the unit tangent vector to the stem at the point $P(t,s)$, so that

$$\mathbf{k}(t,s) \;=\; \frac{P_s(t,s)}{|P_s(t,s)|}\,, \qquad\qquad P_s(t,s) \;\doteq\; \frac{\partial}{\partial s}P(t,s)\,. \tag{2.3}$$

The above implies

$$P(t,s) \;=\; \int_0^s (1-e^{-\alpha(t-\sigma)})\mathbf{k}(t,\sigma)\,d\sigma\,. \tag{2.4}$$

We shall always assume that the curvature vanishes at the tip of the stem, so that

$$\frac{\partial}{\partial s}\mathbf{k}(t,s)\Big|_{s=t} \;=\; 0. \tag{2.5}$$

Our description of the growing stem takes into account:

(i) the upward bending, as a response to gravity,

(ii) an additional bending, in case of a vine clinging to branches of other plants,

(iii) the reaction produced by obstacles,

(iv) the linear elongation.

For a detailed description of plant growth from a biological point of view we refer to [6].

Without loss of generality, one can assume that $P(t,0) = 0 \in \mathbb{R}^3$. Most of our analysis will be concerned with the limit case where $\alpha \to +\infty$, so that $d\ell = d\sigma$ and (2.4) simplifies to

$$P(t,s) \;=\; \int_0^s \mathbf{k}(t,\sigma)\,d\sigma\,. \tag{2.6}$$

As shown in Section 6, all results can be extended to the case $0 < \alpha < \infty$, with only minor changes in the proofs.

2.1 Response to gravity.

To model the response to gravity, we assume that, if a portion of the stem is not vertical, a local change in the curvature will be produced, affecting the position of the upper section of the stem.

More precisely, let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard orthonormal basis in $\mathbb{R}^3$, with $\mathbf{e}_3$ oriented in the upward direction. At every point $P(t,\sigma)$, $\sigma \in [0,t]$, consider the cross product

$$\omega(t,\sigma) \;\doteq\; \mathbf{k}(t,\sigma)\times\mathbf{e}_3.$$

The change in the position of points on the stem, in response to gravity, is described by (see Fig. 1)

$$\frac{\partial}{\partial t}P(t,s) \;=\; \int_0^s \kappa\, e^{-\beta(t-\sigma)}\big(\mathbf{k}(t,\sigma)\times\mathbf{e}_3\big)\times\big(P(t,s)-P(t,\sigma)\big)\,d\sigma \;\doteq\; F_1(t,s)\,. \tag{2.7}$$

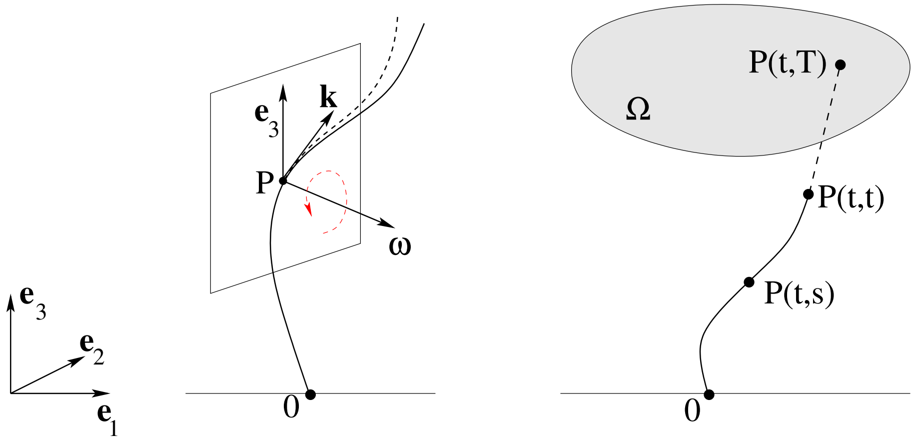


Figure 1: Left: at any point $P = P(t, \sigma)$ along the stem, if the tangent vector $\mathbf{k}$ is not vertical, consider the plane spanned by $\mathbf{k}$ and $\mathbf{e}_3$. Then the change in curvature of the stem at P produces a slight rotation of all points $P(t,s)$ with $s \in [\sigma, t]$. The angular velocity is given by the vector $\omega(\sigma)$. Right: At a given time t, the curve $P(t, \cdot)$ is parameterized by $s \in [0, t]$. It is convenient to prolong this curve by adding a segment of length $T - t$ at its tip (dotted line, possibly entering inside the obstacle). This yields an evolution equation on a fixed functional space $H^2([0, T]; \mathbb{R}^3)$.

Differentiating w.r.t. s one obtains

$$\frac{\partial}{\partial t}\mathbf{k}(t,s) \;=\; \left(\int_0^s \kappa\, e^{-\beta(t-\sigma)} \left(\mathbf{k}(t,\sigma) \times \mathbf{e}_3\right) d\sigma\right) \times \mathbf{k}(t,s) \;\doteq\; G_1(t,s)\,. \tag{2.8}$$

Notice that in the above integrands:

- $\kappa > 0$ is a constant, measuring the strength of the response, while $e^{-\beta(t-s)}$ is a **stiffness factor**. It accounts for the fact that older parts of the stem are more rigid and hence they bend more slowly.
- $\omega(t, \sigma) = \kappa\, e^{-\beta(t-\sigma)}\mathbf{k}(t,\sigma) \times \mathbf{e}_3$ is an angular velocity, determined by the response to gravity at the point $P(t, \sigma)$. This affects the upper portion of the stem, i.e. all points $P(t,s)$ with $s \in [\sigma, t]$.

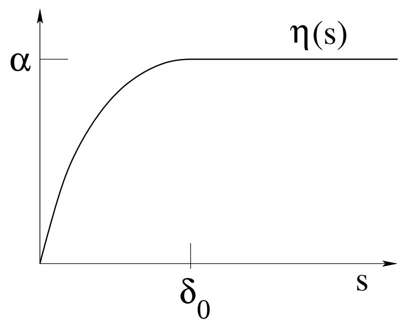
2.2 Clinging to obstacles.

Some plants, rather than growing in the vertical direction, prefer to curl around branches of other plants. To model this behavior, we assume that the stem can feel the presence of an obstacle within a distance δ_0. This triggers a local change of the curvature, in the appropriate direction.

More precisely, let $\Omega \subset \mathbb{R}^3$ be a bounded open set, whose closure $\overline{\Omega}$ does not contain the origin and whose boundary $\partial\Omega$ is a surface with $\mathcal{C}^3$ regularity.

Let $\eta : \mathbb{R} \mapsto [0, 1]$ be a smooth function such that

$$\eta(0) = 0, \qquad \eta(s) = \eta_0 \quad \text{for} \quad x \geq \delta_0, \qquad \eta' \geq 0, \quad \eta'' \leq 0\,, \tag{2.9}$$



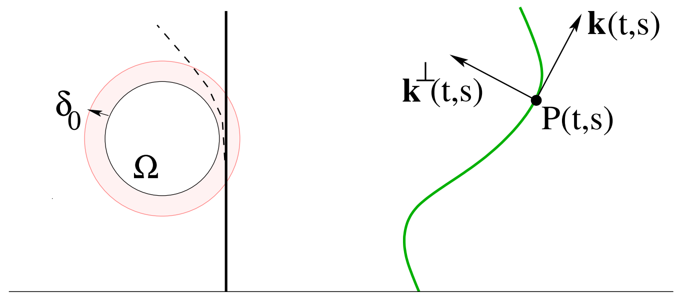


Figure 2: Left: the function η in (2.9). Center: points on the stem at a distance $\leq \delta_0$ from Ω feel the presence of the obstacle and produce an increase of curvature in the appropriate direction.

for some constants $\eta_0, \delta_0 > 0$. For $x \in \mathbb{R}^3 \setminus \Omega$, we then set

$$\psi(x) \ \doteq \ \eta\big(d(x,\Omega)\big).$$

The bending of the stem around the obstacle Ω can now be described by

$$\frac{\partial}{\partial t}P(t,s) \ = \ \int_0^s e^{-\beta(t-\sigma)}\Big(\nabla\psi(P(t,\sigma)) \times \mathbf{k}(t,\sigma)\Big) \times \big(P(t,s) - P(t,\sigma)\big)\, d\sigma \ \doteq \ F_2(t,s)\,. \tag{2.10}$$

As before, a differentiation w.r.t. s yields

$$\frac{\partial}{\partial t}\mathbf{k}(t,s) \ = \ \left(\int_0^s e^{-\beta(t-\sigma)}\Big(\nabla\psi(P(t,\sigma)) \times \mathbf{k}(t,\sigma)\Big)\, d\sigma\right) \times \mathbf{k}(t,s) \ \doteq \ G_2(t,s). \tag{2.11}$$

2.3 Unilateral constraints.

Finally, we seek to model a family of admissible reactions produced by an obstacle Ω, which guarantee that the stem will never penetrate inside Ω.

As a preliminary, consider a stem which partly lies inside the open region Ω. Call

$$\Phi(x) \ \doteq \ \left\{ \begin{array}{ll} d(x,\ \partial\Omega) & \text{if } \ x \notin \Omega, \\[2mm] -d(x,\ \partial\Omega) & \text{if } \ x \in \Omega, \end{array}\right. \tag{2.12}$$

the signed distance of a point $x \in \mathbb{R}^3$ to the boundary $\partial\Omega$. Let $s' \in [0,t]$ be fixed. If $\Phi(P(t,s')) < 0$, consider the problem of bending the stem, so that the point $P(t,s')$ is pushed out of the obstacle. Calling $\omega(\sigma) \in \mathbb{R}^3$ a rotation vector at the point $P(t,\sigma)$, in first approximation the displacement of the point $P(t,s')$ on the stem is computed by

$$\widetilde{P}(t,s') - P(t,s') \ = \ \int_0^{s'} \omega(\sigma) \times \big(P(t,s') - P(t,\sigma)\big)\, d\sigma\,. \tag{2.13}$$

We seek a function $\omega(\cdot)$ which minimizes the elastic deformation energy

$$\mathcal{E} \ \doteq \ \frac{1}{2}\int_0^{s'} e^{\beta(t-\sigma)}|\omega(\sigma)|^2\, d\sigma\,, \tag{2.14}$$

subject to

$$\nabla\Phi(P(t,s'))\cdot\big(\widetilde{P}(t,s')-P(t,s')\big)+\Phi(P(t,s')) \;=\; 0. \tag{2.15}$$

Here the factor $e^{\beta(t-\sigma)}$ accounts for the fact that older sections of the stem are stiffer, and offer more resistance to bending. Inserting (2.13) in (2.15) we obtain

$$\nabla\Phi(P(t,s'))\cdot\left(\int_0^{s'}\omega(\sigma)\times\big(P(t,s')-P(t,\sigma)\big)\,d\sigma\right)+\Phi(P(t,s')) \;=\; 0. \tag{2.16}$$

To derive necessary conditions for optimality, consider a family of perturbations

$$\omega_\varepsilon(\sigma) \;=\; \omega(\sigma)+\varepsilon\,\tilde\omega(\sigma)\,.$$

Differentiating w.r.t. ε, at $\varepsilon=0$ we obtain

$$\int_0^{s'}e^{\beta(t-\sigma)}\omega(\sigma)\cdot\tilde\omega(\sigma)\,d\sigma+\lambda\,\nabla\Phi(P(t,s'))\cdot\left(\int_0^{s'}\tilde\omega(\sigma)\times\big(P(t,s')-P(t,\sigma)\big)\,d\sigma\right) \;=\; 0, \tag{2.17}$$

where the constant λ is a suitable Lagrange multiplier.

Using the property of the mixed product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$, one obtains

$$\int_0^{s'}e^{\beta(t-\sigma)}\omega(\sigma)\cdot\tilde\omega(\sigma)\,d\sigma+\lambda\left(\int_0^{s'}\tilde\omega(\sigma)\cdot\Big(\nabla\Phi(P(t,s'))\times\big(P(t,s')-P(t,\sigma)\big)\Big)\,d\sigma\right) \;=\; 0. \tag{2.18}$$

Since (2.18) must hold for all perturbations $\tilde\omega(\cdot)$, this implies

$$e^{\beta(t-\sigma)}\omega(\sigma) \;=\; \lambda\,\nabla\Phi(P(t,s'))\times\big(P(t,s')-P(t,\sigma)\big), \tag{2.19}$$

for some constant $\lambda\in\mathbb{R}$ and all $\sigma\in[0,s']$. Imposing the boundary condition (2.15), we conclude

$$\omega(\sigma) \;=\; \lambda\,e^{-\beta(t-\sigma)}\nabla\Phi(P(t,s'))\times\big(P(t,s')-P(t,\sigma)\big), \tag{2.20}$$

where the constant λ is determined by the identity (2.15). Namely

$$\begin{aligned}\nabla\Phi(P(t,s'))\cdot&\left(\int_0^{s'}\lambda e^{-\beta(t-\sigma)}\Big(\nabla\Phi(P(t,s'))\times\big(P(t,s')-P(t,\sigma)\big)\Big)\times\big(P(t,s')-P(t,\sigma)\big)\,d\sigma\right)\\&+\Phi(P(t,s')) \;=\; 0.\end{aligned} \tag{2.21}$$

Using the vector identities

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} \;=\; (\mathbf{c}\cdot\mathbf{a})\mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\mathbf{a}\,, \tag{2.22}$$

$$\mathbf{b}\cdot\Big((\mathbf{b}\times\mathbf{c})\times\mathbf{c}\Big) \;=\; (\mathbf{b}\cdot\mathbf{c})^2-|\mathbf{b}|^2|\mathbf{c}|^2, \tag{2.23}$$

and recalling that $\Phi(P(t,s'))<0$, from (2.21) we obtain

$$\begin{aligned}\lambda \;=\; \Bigg[\int_0^{s'}e^{-\beta(t-\sigma)}\Big\{&\Big|\nabla\Phi(P(t,s'))\Big|^2\Big|P(t,s')-P(t,\sigma)\Big|^2\\&-\Big(\nabla\Phi(P(t,s'))\cdot\big(P(t,s')-P(t,\sigma)\big)\Big)^2\Big\}\,d\sigma\Bigg]^{-1}\Phi((P(t,s')) \;\leq\; 0.\end{aligned} \tag{2.24}$$

Notice that the integral in (2.24) vanishes only if the vector $\nabla\Phi(P(t,s'))$ is parallel to all vectors $P(t,s') - P(t,\sigma)$. Examples of these "bad" configurations are shown in Figure 3, right.

Next, assume that at a given time t the stem lies entirely outside the obstacle, but part of it touches the boundary. Call

$$\chi(t) \doteq \Big\{s \in [0,t]\,;\quad P(t,s) \in \partial\Omega\Big\} \tag{2.25}$$

the set where the stem touches the obstacle. For $s \in \chi(t)$, let $\mathbf{n}(t,s)$ be the unit outer normal to the boundary $\partial\Omega$ at the point $P(t,s)$.

Motivated by the previous analysis, we define the **cone of admissible velocities** produced by the obstacle reaction to be the set of velocity fields

$$\Gamma(t) \doteq \Bigg\{\mathbf{v} : [0,t] \mapsto \mathbb{R}^3\,;\quad \text{there exists a positive measure } \mu \text{ supported on } \chi(t) \text{ such that}$$

$$\mathbf{v}(s) = -\int_0^s e^{-\beta(t-\sigma)}\left(\int_{[\sigma,t]}\Big(\mathbf{n}(t,s') \times \big(P(t,s') - P(t,\sigma)\big)\Big)d\mu(s')\right) \times \big(P(t,s) - P(t,\sigma)\big)\,d\sigma\Bigg\}. \tag{2.26}$$

Note that, for every $s' \in \chi(t)$, the reaction produced by the obstacle can yield a deformation of the stem given by

$$P_t(t,s) = \int_0^s \omega(\sigma) \times (P(t,s) - P(t,\sigma))\,d\sigma,$$

where, by (2.20),

$$\omega(\sigma) = \lambda(s') \cdot e^{-\beta(t-\sigma)}\mathbf{n}(t,s') \times \big(P(t,s') - P(t,\sigma)\big),$$

for some $\lambda(s') \leq 0$. Integrating over all points $s' \in \chi(t)$, with arbitrary choices of the factor $\lambda(s') \leq 0$, we obtain (2.26).

Remark 1. One may adopt a more accurate model, where the bending and twisting of the stem are penalized in different ways. More precisely, at each point $P(t,\sigma)$ one may split the rotation vector into a component parallel to $\mathbf{k}(t,\sigma)$ (twisting) and a component perpendicular to $\mathbf{k}(t,\sigma)$ (bending):

$$\omega(\sigma) = \omega^{twist}(\sigma) + \omega^{bend}(\sigma),$$

where

$$\omega^{twist}(\sigma) = \Big(\mathbf{k}(t,\sigma) \cdot \omega(\sigma)\Big)\,\mathbf{k}(t,\sigma), \qquad \omega^{bend}(\sigma) = \omega(\sigma) - \omega^{twist}(\sigma).$$

The energy functional $\mathcal{E}$ in (2.14) can then be replaced by

$$\mathcal{E} \doteq \frac{1}{2}\int_0^{s'} e^{\beta(t-\sigma)}\Big(c_1\,\big|\omega^{twist}(\sigma)\big|^2 + c_2\big|\omega^{bend}(\sigma)\big|^2\Big)\,d\sigma\,, \tag{2.27}$$

for suitable constants c_1, c_2. When $c_1 = c_2$, this is equivalent to (2.14).

2.4 Summary of the equations.

Taking into account all terms (i)–(iii), the evolution of the stem in the presence of obstacles can be described by

$$P_t(t,s) \;=\; F_1(t,s) + F_2(t,s) + \mathbf{v}(t,s). \tag{2.28}$$

Here F_1, F_2 are the integral terms defined at (2.7), (2.10), while $\mathbf{v}(t,\cdot) \in \Gamma(t)$ is an admissible reaction, in the cone defined at (2.26). The equation (2.28) needs to be solved on a domain of the form

$$\mathcal{D} \;\doteq\; \big\{(t,s)\,;\;\; t \geq t_0\,,\;\; s\in[0,t]\big\}, \tag{2.29}$$

with initial and boundary conditions

$$P(t_0,s) \;=\; \overline{P}(s), \qquad\qquad s\in[0,t_0], \tag{2.30}$$

$$P_{ss}(t,s)\Big|_{s=t} \;=\; 0, \qquad\qquad t > t_0\,, \tag{2.31}$$

and the constraint

$$P(t,s) \;\notin\; \Omega \qquad \text{for all} \quad (t,s)\in\mathcal{D}. \tag{2.32}$$

Recalling (2.6), one obtains an equivalent evolution equation for the unit tangent vector $\mathbf{k}$, namely

$$\mathbf{k}_t(t,s) \;=\; G_1(t,s) + G_2(t,s) + \mathbf{h}(t,s). \tag{2.33}$$

Here G_1, G_2 are the integral terms defined at (2.8), (2.11), respectively. Moreover, $\mathbf{h}(t,\cdot)$ is any element of the cone

$$\Gamma'(t) \;\doteq\; \Bigg\{\mathbf{h} : [0,t]\mapsto\mathbb{R}^3\,;\;\; \text{there exists a positive measure } \mu \text{ supported on } \chi(t) \text{ such that}$$
$$\mathbf{h}(s) \;=\; -\int_0^s \left(\int_{[\sigma,t]} e^{-\beta(t-\sigma)}\mathbf{n}(t,s')\times\big(P(t,s')-P(t,\sigma)\big)\,d\mu(s')\right) d\sigma\times\mathbf{k}(t,s)\Bigg\}. \tag{2.34}$$

The equation (2.33) should be solved on the domain $\mathcal{D}$ in (2.29), with initial and boundary conditions

$$\mathbf{k}(t_0,s) \;=\; \overline{\mathbf{k}}(s), \qquad\qquad s\in[0,t_0]\,, \tag{2.35}$$

$$\mathbf{k}_s(t,s)\Big|_{s=t} \;=\; 0 \qquad\qquad t>t_0\,. \tag{2.36}$$

2.5 The two-dimensional case

In the planar case $n = 2$, the evolution equation for the growing stem takes a simpler form. For any vector $\mathbf{v} = (v_1, v_2)$, let $\mathbf{v}^\perp = (-v_2, v_1)$ be the perpendicular vector obtained by a counterclockwise rotation of $\pi/2$. Setting $\mathbf{k} = (k_1, k_2)$, the equations (2.33) can be written as

$$\begin{aligned}\mathbf{k}_t(t,s) &= \left(\int_0^s \kappa\, e^{-\beta(t-\sigma)} k_1(t,\sigma)\, d\sigma\right) \mathbf{k}^\perp(t,s)\\ &\quad -\left(\int_0^s e^{-\beta(t-\sigma)} \Big(\nabla\psi(P(t,\sigma))\cdot \mathbf{k}^\perp(t,\sigma)\Big)\, d\sigma\right) \mathbf{k}^\perp(t,s)\\ &\quad -\left(\int_0^s \left(\int_{[\sigma,t]} e^{-\beta(t-\sigma)} \Big(\mathbf{n}(t,s')\cdot \big(P(t,s') - P(t,\sigma)\big)^\perp\Big) d\mu(s')\right) d\sigma\right) \mathbf{k}^\perp(t,s).\end{aligned} \tag{2.37}$$

Here μ is any positive measure supported on the contact set $\chi(t)$ in (2.25). As before, for $s' \in \chi(t)$ we denote by $\mathbf{n}(t,s')$ the unit outer normal to Ω at the boundary point $P(t,s') \in \partial\Omega$.

3 Statement of the main results

At each time t, the position of the stem is described by a map $P(t,\cdot)$ from $[0,t]$ into $\mathbb{R}^3$. Of course, the domain of this map grows with time. It is convenient to reformulate our model as an evolution problem on a functional space independent of t. For this purpose, we fix $T > t_0$ and consider the Hilbert-Sobolev space $H^2([0,T];\mathbb{R}^3)$. Any function $P(t,\cdot) \in H^2([0,t];\mathbb{R}^3)$ will be canonically extended to $H^2([0,T];\mathbb{R}^3)$ by setting (see Fig. 1, right)

$$P(t,s) \doteq P(t,t) + (s-t)P_s(t,t) \qquad \text{for} \quad s \in [t,T]. \tag{3.1}$$

Notice that the above extension is well defined because $P(t,\cdot)$ and $P_s(t,\cdot)$ are continuous functions. In all of the following analysis, we shall study functions defined on a domain of the form

$$\mathcal{D}_T \doteq \big\{(t,s);\ \ 0 \le s \le t,\ \ t \in [t_0,T]\big\}, \tag{3.2}$$

and extended to the rectangle $[t_0,T]\times[0,T]$ as in (3.1). In particular, the partial derivative $P_s(t,s)$ will be constant for $s \in [t,T]$. This will already account for the boundary condition (2.31).

Adopting the notation $a \wedge b \doteq \min\{a,b\}$, we thus consider an evolution problem on the space $H^2([0,T];\mathbb{R}^3)$, having the more general form

$$P_t(t,s) = \int_0^{s\wedge t} \Psi\Big(t,\sigma,P(t,\sigma),P_s(t,\sigma)\Big) \times \Big(P(t,s) - P(t,\sigma)\Big) d\sigma + \mathbf{v}(t,s). \tag{3.3}$$

Here $s \in [0,T]$, $\Psi : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3 \mapsto \mathbb{R}^3$ is a smooth function, and $\mathbf{v}(t,\cdot)$ is an admissible velocity field produced by the constraint reaction. More precisely, given the configuration $P(t,\cdot)$ of the stem at time t, the cone of admissible velocities is defined as

$$\begin{aligned}\Gamma(t) \doteq \Bigg\{&\mathbf{w} : [0,T] \mapsto \mathbb{R}^3;\ \text{ there exists a positive measure } \mu, \text{ supported on}\\ &\qquad \text{the coincidence set } \chi(t) \text{ in (2.25), such that for every } s \in [0,T] \text{ one has}\\ &\mathbf{w}(s) = -\int_0^s e^{-\beta(t-\sigma)} \left(\int_{[\sigma,t]} \Big(\mathbf{n}(t,s') \times \big(P(t,s') - P(t,\sigma)\big)\Big) d\mu(s')\right) \times \big(P(t,s) - P(t,\sigma)\big)\, d\sigma\Bigg\}.\end{aligned} \tag{3.4}$$

Remark 2. In view of (2.7) and (2.10), we can write the evolution equation (2.28) in the form (3.3) by taking

$$\Psi(t,\sigma,P,\mathbf{k}) \doteq e^{-\beta(t-\sigma)}\Big(\kappa(\mathbf{k}\times\mathbf{e}_3)+(\nabla\psi(P)\times\mathbf{k})\Big). \tag{3.5}$$

Before stating our main existence theorem, we introduce a precise definition of solution.

Definition 1. *We say that a function $P = P(t,s)$, defined for $(t,s)\in[t_0,T]\times[0,T]$ is a solution to the equation (3.3)-(3.4) with initial and boundary conditions (2.30)–(2.32) if the following holds.*

(i) The map $t\mapsto P(t,\cdot)$ is Lipschitz continuous from $[t_0,T]$ into $H^2([0,T];\mathbb{R}^3)$.

(ii) For every t,s one has

$$\begin{aligned} P(t,s) &= P(t_0,s)+\int_{t_0}^{t}\int_0^{s\wedge t}\Psi\Big(\tau,\sigma,P(\tau,\sigma),P_s(\tau,\sigma)\Big)\times\Big(P(\tau,s)-P(\tau,\sigma)\Big)d\sigma\,d\tau \\ &\qquad +\int_0^t \mathbf{v}(\tau,s)\,d\tau\,, \end{aligned} \tag{3.6}$$

where each $\mathbf{v}(\tau,\cdot)$ is an element of the cone $\Gamma(\tau)$ defined as in (3.4).

(iii) The initial conditions hold:

$$P(t_0,s) = \begin{cases} \overline{P}(s) & \text{if } s\in[0,t_0],\\ \overline{P}(t_0)+(s-t_0)\overline{P}'(t_0) & \text{if } s\in[t_0,T]. \end{cases} \tag{3.7}$$

(iv) The pointwise constraints hold:

$$P(t,s)\notin\Omega \qquad \text{for all } t\in[t_0,T],\quad s\in[0,t]. \tag{3.8}$$

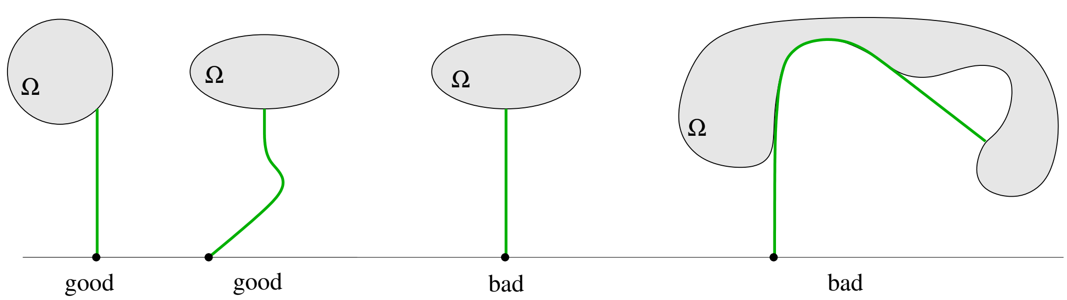


Figure 3: For the two initial configurations on the left, the constrained growth equation (2.28) admits a unique solution. The two configurations on the right satisfy the condition **(B)**. In such cases, the Cauchy problem is ill posed.

Given an initial data $P(t_0,s)=\overline{P}(s)$, our main result states local existence of solutions, locally in time, as long as the following breakdown configuration is not attained (see Fig. 3).

(B) *The tip of the stem touches the obstacle perpendicularly, namely*

$$\overline{P}(t_0) \;\in\; \partial\Omega\,, \qquad\qquad \overline{P}_s(t_0) \;=\; -\,\mathbf{n}(\overline{P}(t_0)). \tag{3.9}$$

Moreover,

$$\overline{P}_{ss}(s) \;=0 \qquad \textit{for all } s\in\;]0,t[\;\textit{ such that } \overline{P}(s)\notin\partial\Omega\,. \tag{3.10}$$

Here $\mathbf{n}(P)$ denotes the unit outer normal to Ω at a boundary point $P\in\partial\Omega$.

Our main result shows that the equations of growth with obstacle admit a solution (in the sense of Definition 1). Moreover, this solution can be prolonged in time until a breakdown configuration is reached, as described in **(B)**.

Theorem 1 (existence of solutions). *Let Ψ in (3.3) be a $\mathcal{C}^2$ function, and let $\Omega\subset\mathbb{R}^3$ be a bounded open set with $\mathcal{C}^2$ boundary. At time t_0, consider the initial data (2.30), where the curve $s\mapsto\overline{P}(s)$ is in $H^2([0,t_0];\mathbb{R}^3)$ and satisfies*

$$\overline{P}(0)=0\notin\partial\Omega, \qquad\qquad |\overline{P}\,'(s)|\equiv 1, \qquad \overline{P}(s)\notin\Omega \qquad \textit{for all}\;\; s\in[0,t_0]. \tag{3.11}$$

Moreover, assume that the condition **(B)** *does NOT hold.*

Then there exists $T>t_0$ such that the equations (3.3)-(3.4) with initial and boundary conditions (2.30)–(2.32) admit at least one solution for $t\in[t_0,T]$.

Either (i) the solution is globally defined for all times $t\geq t_0$, or (ii) the solution can be extended to a maximal time interval $[0,T]$, where $P(T,\cdot)$ satisfies all conditions in **(B)**.

Remark 3. From an abstract point of view, our evolution problem has the form

$$\frac{d}{dt}u(t) \;\in\; \Psi(u(t))+\Gamma(u(t)), \tag{3.12}$$

where $u(t)=P(t,\cdot)\in\mathcal{A}\subset H^2([0,T];\mathbb{R}^3)$. Here $\mathcal{A}$ is the set of admissible configurations, satisfying the constraint (3.8), while $\Gamma(u)$ is the cone of admissible velocities, defined as in (3.4).

If $\Gamma(u)$ were the (inward pointing) normal cone to $\mathcal{A}$ at u, then for any two solutions u_1,u_2 one could expect an estimate of the type

$$\frac{d}{dt}\|u_1(t)-u_2(t)\|_{H^2}\;\leq\; C\cdot\|u_1(t)-u_2(t)\|_{H^2}\,. \tag{3.13}$$

By Gronwall's lemma, this would imply the uniqueness and continuous dependence of solutions on the initial data [4, 5, 7, 8].

Unfortunately, in the present setting the cone $\Gamma(u)$ determined by the constraint reaction is not at all perpendicular to the boundary of the admissible set (see Fig. 4). For this reason, the well-posedness of the Cauchy problem for the growing stem with obstacles is a delicate issue, which will be separately addressed in the forthcoming paper [3].

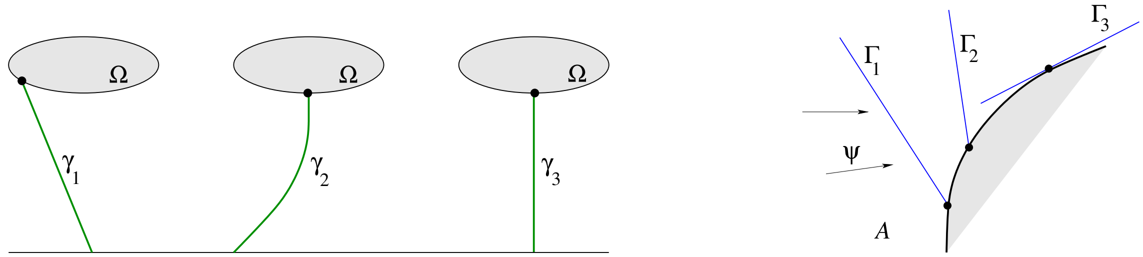


Figure 4: Right: the abstract evolution equation (3.12). In general, the cone Γ of constraint reactions is not perpendicular to the boundary of the set $\mathcal{A}$ of admissible configurations. For a stem γ_3 which satisfies all conditions in **(B)**, the corresponding cone Γ_3 is tangent to $\partial\mathcal{A}$.

4 A push-out operator

Consider a curve $\gamma_0 \in H^2([0,t_0];\mathbb{R}^3)$, parameterized by arc-length. More precisely, assume that

$$\gamma_0(0) \;=\; 0\in\mathbb{R}^3, \qquad |\gamma_0'(s)| \;=\; 1, \qquad \gamma_0(s)\notin\Omega \qquad \text{for all} \quad s\in[0,t_0]. \tag{4.1}$$

Moreover, assume that not all of the following conditions hold:

$$\gamma_0(t_0)\in\partial\Omega, \qquad\qquad \gamma_0'(t_0) = -\mathbf{n}(\gamma_0(t_0)), \tag{4.2}$$

$$\gamma_0''(s) \;=\; 0 \qquad \text{for all} \quad s\in[0,t_0] \;\; \text{such that} \;\; \gamma_0(s)\notin\partial\Omega. \tag{4.3}$$

Given $T > t_0$ we can extend γ_0 to a map $[0,T]\mapsto\mathbb{R}^3$ by setting

$$\gamma_0(s) \;\doteq\; \gamma_0(t_0)+\gamma_0'(t_0)\,(s-t_0) \qquad\qquad \text{for all} \quad s\in[t_0,T]. \tag{4.4}$$

For a fixed radius $\rho>0$ and $T>t_0$, consider the tube $\mathcal{V}_\rho\subset H^2([0,T];\,\mathbb{R}^3)$ around γ_0, defined by

$$\begin{aligned}\mathcal{V}_\rho \;\doteq\; \Big\{&\gamma\in H^2([0,T];\,\mathbb{R}^3),\quad \gamma(0)=0,\quad \gamma'(0)=\gamma_0'(0),\\ &\qquad |\gamma'(s)|=1 \quad \text{for all} \;\; s\in[0,t], \qquad \int_0^t |\gamma''(s)-\gamma_0''(s)|^2\,ds\leq\rho\Big\}.\end{aligned} \tag{4.5}$$

Given a curve $\gamma\in H^2([0,T];\,\mathbb{R}^3)$ and a function $\omega\in\mathbf{L}^2([0,T];\,\mathbb{R}^3)$, we define the rotated curve γ_ω as

$$\gamma_\omega(s) \;\doteq\; \gamma(s)+\int_0^s \omega(\sigma)\times\big(\gamma(s)-\gamma(\sigma)\big)\,d\sigma. \tag{4.6}$$

Notice that, in general, the map $s\mapsto\gamma_\omega(s)$ is not an arc-length parameterization of γ_ω. However, for $\|\omega\|_{\mathbf{L}^2}$ small, we have

$$\begin{aligned}\big|\gamma_\omega'(s)\big|^2 \;&=\; \Big\langle \gamma'(s)+\int_0^s\omega(\sigma)\,d\sigma\times\gamma'(s)\;,\;\gamma'(s)+\int_0^s\omega(\sigma)\,d\sigma\times\gamma'(s)\Big\rangle\\ &=\; 1+\left|\int_0^s\omega(\sigma)\,d\sigma\times\gamma'(s)\right|^2 \;=\; 1+\mathcal{O}(1)\cdot\|\omega\|_{\mathbf{L}^2}^2\,.\end{aligned} \tag{4.7}$$

Next, consider a curve $\gamma \in \mathcal{V}_\rho$ and a subinterval $[0,t] \subseteq [0,T]$. If $\gamma(s) \in \Omega$ for some $s \in [0,t]$, we wish to push γ outside Ω, but bending the curve as little as possible. This leads to the constrained optimization problem

$$\text{minimize:} \qquad J(\omega) \;\doteq\; \int_0^T e^{\beta(t-s)}|\omega(s)|^2\,ds, \tag{4.8}$$

$$\text{subject to:} \qquad \gamma_\omega(s) \notin \Omega \qquad \text{for all} \;\; s \in [0,t]. \tag{4.9}$$

Notice that here we allow $\gamma_\omega(s) \in \Omega$ for $t < s \le T$. Before proving the existence of a minimizer, we prove that there exists at least one angular velocity ω that pushes every point of the curve γ away from the obstacle. This will be achieved by the first two lemmas. In the following, $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^3$, while $\Phi(x)$ denotes the signed distance of point x to $\partial\Omega$, as in (2.12).

Lemma 1. *Let γ_0 be as in (4.1). Then there exist $T > t_0$, $\rho > 0$, and a constant C_0 such that the following holds.*

Extend γ_0 as in (4.4) and let $\gamma \in \mathcal{V}_\rho$ as in (4.5). Assume $\mathbf{v} \in H^2([0,T];\mathbb{R}^3)$, with

$$\mathbf{v}(0) = \mathbf{v}'(0) = 0, \qquad \big\langle \mathbf{v}'(s),\, \gamma'(s)\big\rangle \;=\; 0 \qquad \textit{for all} \;\; s \in [0,T]. \tag{4.10}$$

Then there exists a unique angular velocity field $\omega \in \mathbf{L}^2([0,T];\mathbb{R}^3)$ such that

$$\|\omega\|_{\mathbf{L}^2} \;\le\; C_0\,\|\mathbf{v}\|_{H^2}\,, \tag{4.11}$$

$$\big\langle \omega(s),\, \gamma'(s)\big\rangle \;=\; 0 \qquad \textit{for all} \;\; s \in [0,T], \tag{4.12}$$

$$\mathbf{v}(s) \;=\; \int_0^s \omega(\sigma) \times (\gamma(s) - \gamma(\sigma))\,d\sigma \qquad \textit{for all} \;\; s \in [0,T]. \tag{4.13}$$

Proof. 1. By the initial conditions in (4.10), differentiating (4.13) we see that $\omega(\cdot)$ satisfies (4.13) if and only if

$$\mathbf{v}'(s) \;=\; \left(\int_0^s \omega(\sigma)\,d\sigma\right) \times \gamma'(s) \qquad \text{for all} \;\; s \in [0,T]. \tag{4.14}$$

2. Next, consider a family of orthonormal frames $\{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)\}$, with $\mathbf{e}_1(s) = \gamma'(s)$ for all $s \in [0,T]$. We shall determine two scalar functions $\omega_2, \omega_3 : [0,T] \mapsto \mathbb{R}$ such that the vector function

$$\omega(s) \;=\; \omega_2(s)\mathbf{e}_2(s) + \omega_3(s)\mathbf{e}_3(s)$$

satisfies (4.13).

Using the orthogonality assumption in (4.10), we obtain two scalar functions z_2, z_3 such that

$$\mathbf{v}'(s) \;=\; z_2(s)\mathbf{e}_2(s) + z_3(s)\mathbf{e}_3(s) \;=\; \int_0^s \Big(\omega_2(\sigma)\mathbf{e}_2(\sigma) + \omega_3(\sigma)\mathbf{e}_3(\sigma)\Big)d\sigma \times \mathbf{e}_1(s). \tag{4.15}$$

Projecting along $\mathbf{e}_2(s)$ we obtain

$$z_2(s) \;=\; \int_0^s \Big\langle \mathbf{e}_2(\sigma) \times \mathbf{e}_1(s),\, \mathbf{e}_2(s)\Big\rangle \omega_2(\sigma)\,d\sigma + \int_0^s \Big\langle \mathbf{e}_3(\sigma) \times \mathbf{e}_1(s),\, \mathbf{e}_2(s)\Big\rangle \omega_3(\sigma)\,d\sigma.$$

By a property of the triple product, this is equivalent to

$$z_2(s) \;=\; \int_0^s \Big[\langle \mathbf{e}_2(\sigma),\, \mathbf{e}_3(s)\rangle\, \omega_2(\sigma) + \langle \mathbf{e}_3(\sigma),\, \mathbf{e}_3(s)\rangle\, \omega_3(\sigma)\Big]\, d\sigma. \tag{4.16}$$

Similarly, projecting along $\mathbf{e}_3(s)$ we obtain

$$z_3(s) \;=\; -\int_0^s \Big[\langle \mathbf{e}_2(\sigma),\, \mathbf{e}_2(s)\rangle\, \omega_2(\sigma) + \langle \mathbf{e}_3(\sigma),\, \mathbf{e}_2(s)\rangle\, \omega_3(\sigma)\Big]\, d\sigma. \tag{4.17}$$

Observing that all quantities $\mathbf{v}, z_i, \omega_i, \mathbf{e}_i$ in (4.15) are functions in H^1, we can differentiate one more time and obtain the linear system of Volterra integral equations

$$\left\{ \begin{array}{rcl} \omega_3(s) &=& z_2'(s) - \displaystyle\int_0^s \Big[\langle \mathbf{e}_2(\sigma),\, \mathbf{e}_3'(s)\rangle\, \omega_2(\sigma) + \langle \mathbf{e}_3(\sigma),\, \mathbf{e}_3'(s)\rangle\, \omega_3(\sigma)\Big]\, d\sigma\,, \\ \omega_2(s) &=& -\, z_3'(s) - \displaystyle\int_0^s \Big[\langle \mathbf{e}_2(\sigma),\, \mathbf{e}_2'(s)\rangle\, \omega_2(\sigma) + \langle \mathbf{e}_3(\sigma),\, \mathbf{e}_2'(s)\rangle\, \omega_3(\sigma)\Big]\, d\sigma\,. \end{array} \right. \tag{4.18}$$

3. The unique solution to the system (4.18) can be obtained by a standard fixed point argument. Adopting vector notation, set $U = \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix}$, $Z = \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}$. Then (4.18) can be written as

$$U(s) \;=\; Z'(s) + \int_0^s B(s,\sigma) U(\sigma)\, d\sigma \;\doteq\; \mathcal{P}[U](s), \tag{4.19}$$

where the matrix $B(s,\sigma)$ has norm

$$|B(s,\sigma)| \;\leq\; 2|\mathbf{e}_2'(s)| + 2|\mathbf{e}_3'(s)| \;\doteq\; b(s).$$

We claim that the operator $\mathcal{P}$ defined at (4.19) is a strict contraction on the space $\mathbf{L}^1([0,T];\, \mathbb{R}^2)$ with equivalent norm

$$\|U\| \;\doteq\; \int_0^t \exp\left\{-2\int_0^s b(\sigma)\, d\sigma\right\} |U(s)|\, ds.$$

Indeed, for any $U_1, U_2 \in \mathbf{L}^1$ an integration by parts yields

$$\begin{aligned} \|\mathcal{P}[U_1] - \mathcal{P}[U_2]\| \;&\leq\; \int_0^t \exp\left\{-2\int_0^s b(\sigma)\, d\sigma\right\} b(s) \left(\int_0^s |U_1(\sigma) - U_2(\sigma)|\, d\sigma\right) ds \\ &=\; \int_0^t \frac{1}{2} \exp\left\{-2\int_0^s b(\sigma)\, d\sigma\right\} |U_1(s) - U_2(s)| ds \\ &\qquad -\frac{1}{2} \exp\left\{-2\int_0^t b(\sigma)\, d\sigma\right\} \left(\int_0^t |U_1(\sigma) - U_2(\sigma)|\, d\sigma\right) ds \\ &\leq\; \frac{1}{2} \int_0^t \exp\left\{-2\int_0^s b(\sigma)\, d\sigma\right\} |U_1(s) - U_2(s)| ds \\ &=\; \frac{1}{2} \|U_1 - U_2\|\,. \end{aligned}$$

By the contraction mapping principle, the equation (4.19) has a unique solution in $\mathbf{L}^1([0,T];\, \mathbb{R}^2)$. In addition, we have

$$\|U\|_{\mathbf{L}^1} \;\leq\; C_1 \|Z'\|_{\mathbf{L}^1} \;\leq\; C_2 \|Z'\|_{\mathbf{L}^2}\,, \tag{4.20}$$

for some constants C_1, C_2 depending on t and on the function $b(\cdot)$. In turn, this implies

$$\begin{aligned}\|U\|^2_{\mathbf{L}^2} &\leq 2\|Z'\|^2_{\mathbf{L}^2} + 2\int_0^t b^2(s)\left(\int_0^s U(\sigma)\,d\sigma\right)^2 ds \\ &\leq 2\|Z'\|^2_{\mathbf{L}^2} + 2\|b\|^2_{\mathbf{L}^2}\|U\|^2_{\mathbf{L}^1} \\ &\leq \left(2+2\|b\|^2_{\mathbf{L}^2}C_2^2\right)\|Z'\|^2_{\mathbf{L}^2}\,.\end{aligned} \tag{4.21}$$

4. Returning to the original variables, we see that $\|Z'\|_{\mathbf{L}^2} = \mathcal{O}(1)\cdot\|\mathbf{v}\|_{H^2}$. On the other hand, $\|b\|_{\mathbf{L}^2} = \mathcal{O}(1)\cdot\|\gamma\|_{H^2}$ is uniformly bounded as γ ranges in $\mathcal{V}_\rho$. This completes the proof of (4.11). □

Remark 4. In the above proof, the uniqueness of the function $\omega(\cdot)$ was achieved by imposing the orthogonality condition (4.12). Without this restriction, infinitely many solutions are possible. For example, if $\gamma'(s) = \mathbf{e}_0$ for all $s \in [0,t]$ and $\omega(\cdot)$ is a solution, then $\tilde\omega(s) = \omega(s) + \phi(s)\mathbf{e}_0$ is another solution, for any scalar function ϕ.

The next lemma shows that, if a stem is not close to a bad configuration described in **(B)**, then it can be pushed away from the obstacle by a small rotation.

Lemma 2. *Assume that the initial curve $\gamma_0 : [0,t_0] \mapsto \mathbb{R}^3\setminus\Omega$ satisfies (4.1), but it does NOT satisfy simultaneously all conditions in (4.2)-(4.3). Then there exist $T > t_0$, $\rho, \delta > 0$, and a constant C_0 such that the following holds.*

Assume $t \in [t_0, T]$ and consider any curve $\gamma \in \mathcal{V}_\rho$, as in (4.5). Then there exists $\omega : [0,t] \mapsto \mathbb{R}^3$, with

$$\|\omega\|_{\mathbf{L}^2} \leq C_0\,, \tag{4.22}$$

such that, for every $s \in [0,t]$ such that $\big|\Phi(\gamma(s))\big| \leq \delta$, one has

$$\left\langle \int_0^s \omega(\sigma)\times\big(\gamma(s)-\gamma(\sigma)\big)\,d\sigma\,,\ \nabla\Phi(\gamma(s))\right\rangle \geq 1. \tag{4.23}$$

Proof. Let $\mathbf{w} \in \mathcal{C}^2(\mathbb{R}^3;\mathbb{R}^3)$ be a function which satisfies

$$\mathbf{w}(x) = \begin{cases} \nabla\Phi(x) & \text{if } |\Phi(x)| \leq \delta_0\,, \\ 0 & \text{if } |\Phi(x)| \geq 2\delta_0\,. \end{cases} \tag{4.24}$$

Since we are assuming that the boundary $\partial\Omega$ is $\mathcal{C}^3$, such a function exists, provided that $\delta_0 > 0$ is chosen sufficiently small. Three cases will be considered.

CASE 1: Either $\gamma_0(t_0) \notin \partial\Omega$, or else $\gamma(t_0) \in \partial\Omega$ and $\langle\gamma'(t_0), \mathbf{n}(\gamma(t_0))\rangle = 0$ (see Fig. 5, left).

In this case we define

$$\tilde{\mathbf{v}}(s) \doteq 2\mathbf{w}(\gamma(s)), \tag{4.25}$$

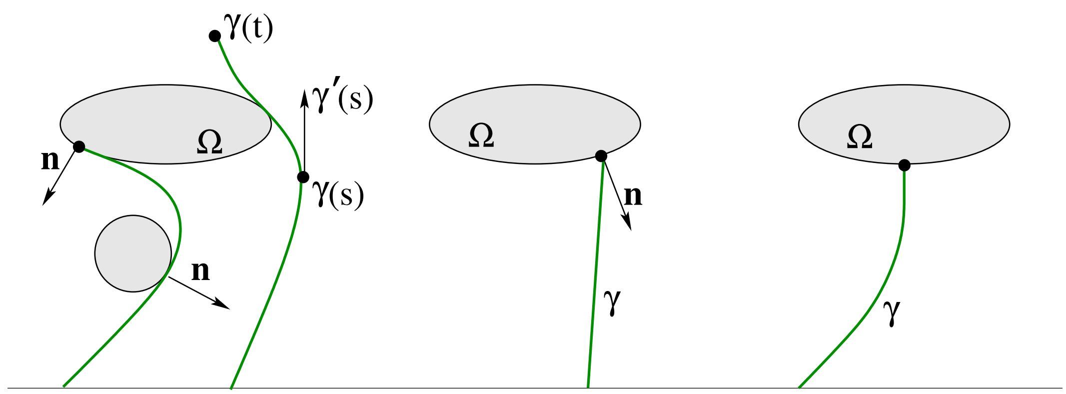


Figure 5: The three cases considered in the proof of Lemma 2.

$$\mathbf{v}(s) \doteq \tilde{\mathbf{v}}(s) - \left(\int_0^s \Big\langle \gamma'(\sigma), \tilde{\mathbf{v}}'(\sigma) \Big\rangle \, d\sigma\right) \gamma'(s). \tag{4.26}$$

Observing that

$$\langle \gamma'(s), \mathbf{v}'(s)\rangle = 0 \qquad \text{for all } \ s \in [0,t], \tag{4.27}$$

by Lemma 1 we can find $\omega \in \mathbf{L}^2([0,t])$ such that (4.11)-(4.13) hold.

To prove (4.23), we first observe that

$$\left|\int_0^s \Big\langle \gamma'(\sigma), \tilde{\mathbf{v}}'(\sigma)\Big\rangle d\sigma\right| \leq \int_0^s |\tilde{\mathbf{v}}'(\sigma)|\, d\sigma \leq \int_0^t \left|\frac{d}{d\sigma} 2\mathbf{w}(\gamma(\sigma))\right| d\sigma \leq 2t \cdot \|\mathbf{w}\|_{\mathcal{C}^2}. \tag{4.28}$$

We also recall that, by assumption,

$$\Big\langle \gamma_0'(s)\,,\ \mathbf{n}(\gamma_0(s))\Big\rangle = 0$$

for all $s \in [0, t_0]$ such that $\gamma_0(s) \in \partial\Omega$. For any given $\varepsilon_0 > 0$, choosing $\delta > 0$ sufficiently small we achieve the implication

$$\Big|\Phi(\gamma(s))\Big| \leq 2\delta \qquad \Longrightarrow \qquad \Big|\Big\langle \gamma'(s)\,,\ \nabla\Phi(\gamma(s))\Big\rangle\Big| \leq \varepsilon_0 \tag{4.29}$$

for all $s \in [0,t]$. Choosing $\varepsilon_0 = \Big(4(t_0+\delta)\|\mathbf{w}\|_{\mathcal{C}^2}\Big)^{-1}$, if $\Big|\Phi(\gamma(s))\Big| \leq \delta$, by (4.28) and (4.29) we now have

$$\begin{aligned}
&\Big\langle \int_0^s \omega(\sigma) \times \big(\gamma(s) - \gamma(\sigma)\big)\, d\sigma\,,\ \nabla\Phi(\gamma(s))\Big\rangle = \Big\langle \mathbf{v}(s)\,,\ \nabla\Phi(\gamma(s))\Big\rangle \\
&\quad = \Big\langle 2\nabla\Phi(\gamma(s))\,,\ \nabla\Phi(\gamma(s))\Big\rangle - \Big\langle \Big(\int_0^s \Big\langle \gamma'(\sigma), \tilde{\mathbf{v}}'(\sigma)\Big\rangle\, d\sigma\Big) \gamma'(s)\,,\ \nabla\Phi(\gamma(s))\Big\rangle \\
&\quad \geq 2 - 2t\|\mathbf{w}\|_{\mathcal{C}^2} \cdot \varepsilon_0 \geq 1.
\end{aligned} \tag{4.30}$$

CASE 2: $\gamma_0(t_0) \in \partial\Omega$ and $-1 < \langle \gamma_0'(t_0), \mathbf{n}(\gamma_0(t_0))\rangle < 0$. In other words, the tip of the stem touches the obstacle, but is neither tangent nor perpendicular to the boundary $\partial\Omega$ (see Fig. 5, center).

By choosing $\delta > 0$ sufficiently small, we can find $0 < \delta_1 < \delta_2$ such that

$$\Phi(\gamma(s)) \begin{cases} \leq\ 2\delta_0 & \text{if } \ s \in [t-\delta_1,\, t]\,, \\ \geq\ 2\delta_0 & \text{if } \ s \in [t-\delta_2,\, t-\delta_1]\,, \end{cases} \tag{4.31}$$

and moreover, for some $\varepsilon_1 > 0$,

$$-1 + 2\varepsilon_1 \ < \ \Big\langle \gamma'(s)\,,\ \nabla\Phi(\gamma(s))\Big\rangle \ < \ 0, \qquad \text{for} \ \ s \in [t-\delta_1,\ t].$$

Consider the function

$$\tilde{\mathbf{v}}(s) \ \doteq \ \begin{cases} 2\mathbf{w}(\gamma(s)) & \text{if } \ s \in [0, t-\delta_1], \\ c_2\mathbf{w}(\gamma(s)) & \text{if } \ s \in [t-\delta_1,\, t], \end{cases} \tag{4.32}$$

where c_2 is a suitably large constant, whose precise value will be determined later. Since $\mathbf{w}(\gamma(s)) = 0$ for $s \in [t-\delta_2,\, t-\delta_1]$, the function $\tilde{\mathbf{v}}$ has the same regularity as γ. With this choice of $\tilde{\mathbf{v}}$, we then define $\mathbf{v}(s)$ as in (4.26).

We claim that (4.23) holds. Indeed, for $s \in [0, t-\delta_1]$ the estimates (4.29)-(4.30) remain valid. To handle the case $s \in [t-\delta_1, t]$, we begin with the estimate

$$\begin{aligned} \left|\int_{t-\delta_1}^{s} \Big\langle \gamma'(\sigma),\, \frac{d}{d\sigma}\mathbf{w}(\gamma(\sigma))\Big\rangle\, d\sigma\right| \ &\leq \ \left|\int_{t-\delta_1}^{s} \Big\langle \gamma''(\sigma),\, \mathbf{w}(\gamma(\sigma))\Big\rangle\, d\sigma\right| + \Big|\langle \gamma'(s),\, \mathbf{w}(\gamma(s))\rangle\Big| \\ &\leq \ \int_{t-\delta_1}^{t} \big|\gamma''(\sigma)\big|\, d\sigma + (1-2\varepsilon_1) \ \leq \ 1-\varepsilon_1\,. \end{aligned} \tag{4.33}$$

We used here the fact that $\mathbf{w}(\gamma(t-\delta_1)) = 0$, and that $\varepsilon_1 > 0$ is a constant depending only on γ_0, while δ_1 can be rendered as arbitrarily small by choosing $\delta > 0$ small enough. Using the above estimate, one obtains

$$\begin{aligned} &\Big\langle \int_0^s \omega(\sigma) \times \big(\gamma(s) - \gamma(\sigma)\big)\, d\sigma\,,\ \nabla\Phi(\gamma(s))\Big\rangle \ = \ \Big\langle \mathbf{v}(\gamma(s))\,,\ \nabla\Phi(\gamma(s))\Big\rangle \\ &\quad = \ \Big\langle c_2\,\nabla\Phi(\gamma(s))\,,\ \nabla\Phi(\gamma(s))\Big\rangle - \Big\langle \Big(\int_0^{t-\delta_1} \Big\langle \gamma'(\sigma),\, \tilde{\mathbf{v}}'(\sigma)\Big\rangle\, d\sigma\Big)\, \gamma'(s)\,,\ \nabla\Phi(\gamma(s))\Big\rangle \\ &\qquad - \Big\langle \Big(\int_{t-\delta_1}^{s} \Big\langle \gamma'(\sigma),\, \tilde{\mathbf{v}}'(\sigma)\Big\rangle\, d\sigma\Big)\, \gamma'(s)\,,\ \nabla\Phi(\gamma(s))\Big\rangle \\ &\quad \geq \ c_2 - 2t\|\mathbf{w}\|_{\mathcal{C}^2}\, \varepsilon_0 - c_2(1-\varepsilon_1) \ \geq \ 1, \end{aligned} \tag{4.34}$$

provided that c_2 was chosen sufficiently large.

As before, since (4.27) holds, by Lemma 1 we can find $\omega \in \mathbf{L}^2([0,t])$ such that (4.11)-(4.13) hold.

CASE 3: $\gamma_0(t_0) \in \partial\Omega$ and $\gamma'(t_0) = -\mathbf{n}(\gamma_0(t_0))$. Hence the tip of the stem touches the obstacle perpendicularly (Fig. 5, right).

Define $\mathbf{w}$ as in (4.24). By assumption, (4.2) fails. By choosing $\delta > 0$ small enough, we can thus find $0 < a < b < t_0$ such that

$$\gamma'(a) \ \neq \ \gamma'(b), \qquad\qquad \Phi(\gamma(s)) \ \geq \ 2\delta \qquad \text{for all} \ \ s \in [a,b]\,. \tag{4.35}$$

In turn, we can find a smooth function $\mathbf{z} : \mathbb{R} \mapsto \mathbb{R}^3$, supported on $[a, b]$, such that

$$\int_a^b \langle \gamma'(s), \mathbf{z}'(s) \rangle \, ds \;=\; 1. \tag{4.36}$$

Then we set

$$\tilde{\mathbf{v}}(s) \;\doteq\; 2\mathbf{w}(\gamma(s)) - c_3 \mathbf{z}(s), \tag{4.37}$$

for some constant c_3 large enough. Finally, we define $\mathbf{v}(\cdot)$ as in (4.26). As in the previous cases, by the identity (4.27) we can use Lemma 1 and obtain a function $\omega \in \mathbf{L}^2([0, t])$ such that (4.11)-(4.13) hold.

It remains to prove that (4.23) holds, provided that $\delta > 0$ was chosen sufficiently small. Indeed, if $\delta > 0$ is small enough, we can find $0 < \delta_1 < \delta_2$ such that (4.31) holds and moreover

$$-1 \;\leq\; \Big\langle \gamma'(s)\,,\ \nabla\Phi(\gamma(s)) \Big\rangle \;<\; -\frac{1}{2}, \qquad \text{for}\ \ s \in [t - \delta_1,\ t].$$

For $s \in [0, t - \delta_1]$, for $\delta > 0$ small we again have the bound in (4.29). Hence

$$\begin{aligned}
\Big\langle \int_0^s \omega(\sigma) \times \big(\gamma(s) - \gamma(\sigma)\big)\, d\sigma\,,\ \nabla\Phi(\gamma(s)) \Big\rangle \;&=\; \Big\langle \mathbf{v}(\gamma(s))\,,\ \nabla\Phi(\gamma(s)) \Big\rangle \\
&= \Big\langle 2\nabla\Phi(\gamma(s))\,,\ \nabla\Phi(\gamma(s)) \Big\rangle - \Big\langle \Big(\int_0^s \Big\langle \gamma'(\sigma), \tilde{\mathbf{v}}'(\sigma) \Big\rangle\, d\sigma \Big) \gamma'(s)\,,\ \nabla\Phi(\gamma(s)) \Big\rangle \\
&\geq\; 2 - C\,\varepsilon_0 \;\geq\; 1,
\end{aligned} \tag{4.38}$$

Notice that here C is a constant that depends only on γ_0, while $\varepsilon_0 > 0$ can be rendered arbitrarily small by choosing $\delta > 0$ small enough.

Finally, when $s \in [t - \delta_1,\ t]$ we have

$$\begin{aligned}
\Big\langle \int_0^s \omega(\sigma) \times \big(\gamma(s) - \gamma(\sigma)\big)\, d\sigma\,,\ \nabla\Phi(\gamma(s)) \Big\rangle \;&=\; \Big\langle \mathbf{v}(\gamma(s))\,,\ \nabla\Phi(\gamma(s)) \Big\rangle \\
&= \Big\langle 2\nabla\Phi(\gamma(s))\,,\ \nabla\Phi(\gamma(s)) \Big\rangle - \Big\langle \Big(\int_0^s \Big\langle \gamma'(\sigma), \tilde{\mathbf{v}}'(\sigma) \Big\rangle\, d\sigma \Big) \gamma'(s)\,,\ \nabla\Phi(\gamma(s)) \Big\rangle \\
&= 2 - \Big(\int_0^a + \int_b^s \Big) \Big\langle \gamma'(\sigma), \tilde{\mathbf{v}}'(\sigma) \Big\rangle\, d\sigma \ \cdot \Big\langle \gamma'(s)\,,\ \nabla\Phi(\gamma(s)) \Big\rangle \\
&\qquad - \int_a^b \Big\langle \gamma'(\sigma), \tilde{\mathbf{v}}'(\sigma) \Big\rangle\, d\sigma \ \cdot \Big\langle \gamma'(s)\,,\ \nabla\Phi(\gamma(s)) \Big\rangle \\
&\geq\; 2 - \Big(\int_0^a + \int_b^s \Big) |\tilde{\mathbf{v}}'(\sigma)|\, d\sigma + c_3 \int_a^b \langle \gamma'(\sigma), \mathbf{z}'(\sigma) \rangle\, d\sigma \Big\langle \gamma'(s)\,,\ \nabla\Phi(\gamma(s)) \Big\rangle \\
&\geq\; 2 - C + \frac{c_3}{2}\,.
\end{aligned} \tag{4.39}$$

We observe that here the constant C depends only on γ_0, while c_3 can be chosen sufficiently large so that the right hand side of (4.39) is ≥ 1. This completes the proof. □

Remark 5. If all conditions (4.2)-(4.3) hold, then the conclusion of Lemma 2 can fail. For example, assume that γ_0 is a segment, with $\gamma_0(t_0) \in \partial\Omega$ and $\gamma_0'(s) = \mathbf{e}_0 = -\mathbf{n}(\gamma(t_0))$ for all

$s \in [0, t_0]$. Then there is no field of angular velocities $\omega(\cdot)$ which satisfies (4.23) at $s = t_0$. Indeed, in this case for every $\omega(\cdot)$ one has

$$\left\langle \int_0^{t_0} \omega(\sigma) \times \big(\gamma_0(t_0) - \gamma_0(\sigma)\big)\, d\sigma\,,\ \nabla\Phi(\gamma(t_0))\right\rangle \;=\; \left\langle \int_0^{t_0} \omega(\sigma) \times (t_0 - \sigma)\mathbf{e}_0\, d\sigma\,,\ -\mathbf{e}_0\right\rangle \;=\; 0.$$

Given a curve $\gamma : [0, T] \mapsto \mathbb{R}^3$ and $t \in [0, T]$, we introduce the quantity

$$E(t, \gamma, \Omega) \;\doteq\; \sup\Big\{d(\gamma(s), \partial\Omega)\,; \quad s \in [0, t]\,,\ \gamma(s) \in \Omega\Big\}, \tag{4.40}$$

measuring the maximum depth at which the initial portion of γ (i.e., for $s \in [0, t]$) penetrates inside the obstacle Ω.

Lemma 3. *For a given path $\gamma_0 : [0, t_0] \mapsto \mathbb{R}^3$, assume that at least one of the conditions in (4.2)-(4.3) fails. Then there exist $T > t_0$ and $\rho > 0$ such that the following holds.*

For every $\gamma \in \mathcal{V}_\rho$ and $t \in [0, T]$, the problem (4.8)-(4.9) has a unique solution $\bar\omega$. Moreover, for some constant C_0 one has the estimate

$$\|\bar\omega\|_{\mathbf{L}^2} \;\leq\; 2C_0 \cdot E(t, \gamma, \Omega), \tag{4.41}$$

for some constant C_0 independent of $\gamma \in \mathcal{V}_\rho$.

Proof. **1.** By choosing $T - t_0$ and $\rho > 0$ small enough, for any $\gamma \in \mathcal{V}_\rho$ an application of Lemma 2 yields the existence of some angular velocity $\omega(\cdot)$ such that (4.22)-(4.23) hold, for some uniform constant C_0. Set

$$\hat\omega \;\doteq\; 2E(t, \gamma, \Omega) \cdot \omega\,.$$

Then, by choosing $\delta, \rho > 0$ small enough we obtain

$$\gamma_{\hat\omega}(s) \notin \Omega \qquad\qquad \text{for all}\ \ s \in [0, t]\,.$$

2. Now take a minimizing sequence $(\omega_n)_{n\geq1}$. By the previous analysis, we can assume that

$$\|\omega_n\| \;\leq\; 2C_0 \cdot E(t, \gamma, \Omega) \tag{4.42}$$

for all $n \geq 1$. Moreover, it is not restrictive to assume that $\omega_n(s) = 0$ for $t < s \leq T$.

We then extract a subsequence that converges weakly in $\mathbf{L}^2([0, t])$, say $\omega_n \rightharpoonup \tilde\omega$. Clearly

$$\|\tilde\omega\| \;\leq\; \liminf_{n\to\infty} \|\omega_n\| \;\leq\; 2C_0 \cdot E(t, \gamma, \Omega).$$

Hence $\tilde\omega$ achieves the minimum. Finally, the condition

$$\gamma_{\tilde\omega}(s) \notin \Omega \qquad \text{for all} \qquad s \in [0, t]$$

follows by the uniform convergence of the sequence γ_{ω_n} on $[0, t]$.

3. To prove uniqueness, consider two minimizers: $\omega_1 \neq \omega_2$, and set $\omega = (\omega_1 + \omega_2)/2$. Observe that

$$\|\omega\|^2 \;=\; \|\omega_i\|^2 - \frac{1}{4}\|\omega_1 - \omega_2\|^2\,, \qquad\qquad i = 1,2\,. \tag{4.43}$$

If γ_ω lies entirely outside the obstacle, we already reach a contradiction. In general, the definition (4.6) implies that

$$\gamma_\omega(s) \;=\; \frac{\gamma_{\omega_1}(s) + \gamma_{\omega_2}(s).}{2}$$

is the midpoint of a segment with vertices outside Ω. Observe that, for every $s \in [0,t]$, one has

$$\bigl|\gamma_{\omega_1}(s) - \gamma_{\omega_2}(s)\bigr| \;=\; \mathcal{O}(1)\cdot\bigl(\|\omega_1\| + \|\omega_2\|\bigr) \;=\; \mathcal{O}(1)\cdot E(t,\gamma,\Omega)\,.$$

Since the boundary $\partial\Omega$ is smooth, we conclude

$$E(t,\gamma_\omega,\Omega) \;=\; \mathcal{O}(1)\cdot E^2(\gamma,\Omega).$$

By the argument used in Lemma 3,, we can construct a perturbation $\hat\omega$ with

$$\|\hat\omega\| \;=\; \mathcal{O}(1)\cdot\|\gamma_{\omega_1} - \gamma_{\omega_2}\|^2_{\mathbf{L}^\infty} \;=\; \mathcal{O}(1)\cdot E^2(t,\gamma,\Omega),$$

such that $\gamma_{\omega+\hat\omega}(s) \notin \Omega$ for all $s \in [0,t]$. We now reach a contradiction by observing that

$$\begin{aligned}\|\omega + \hat\omega\|^2_\beta \;&\leq\; \|\omega\|^2 \,+ \mathcal{O}(1)\cdot\|\omega\|\,\|\gamma_{\omega_1} - \gamma_{\omega_2}\|^2_{\mathbf{L}^\infty}\\ &\leq\; \|\omega_i\|^2 - \tfrac{1}{4}\|\omega_1-\omega_2\|^2 + \mathcal{O}(1)\cdot\|\omega\|\,\|\omega_1-\omega_2\|^2 \;<\; \|\omega_i\|^2\,,\end{aligned}$$

provided that $\|\omega\|$ is sufficiently small. □

The next lemma derives some necessary conditions for optimality, and provides a useful representation of the minimizing function $\bar\omega(\cdot)$.

Lemma 4. *In the same setting as Lemma 3, let $\bar\omega$ be a minimizer for the problem (4.8)-(4.9). Then, choosing $\delta,\rho > 0$ small enough, the following holds. For all $s \in [0,t]$ one has*

$$\bar\omega(s) \;=\; -\int_s^t \int_{[\sigma,t]} e^{-\beta(t-s)}\nabla\Phi(\gamma_{\bar\omega}(s'))d\mu(s') \times \gamma'(\sigma)d\sigma, \tag{4.44}$$

where μ is a positive measure, supported on the contact set $\chi \doteq \Bigl\{s \in [0,t]\,;\quad \gamma_{\bar\omega}(s) \in \partial\Omega\Bigr\}$.

Proof. 1. Calling $\Phi(\cdot)$ the signed distance from the boundary $\partial\Omega$, as in (2.12), the minimization (4.8)-(4.9) can be reformulated as a standard problem of optimal control with state constraints. Here $\omega(\cdot)$ is the control function. The state of the system is $y(s) = (y_1,y_2,y_3)(s) \in \mathbb{R}^{3+3+1}$, with dynamics

$$\left\{\begin{array}{rcl} y_1'(s) &=& \omega(s),\\ y_2'(s) &=& \gamma'(s) + y_1(s)\times\gamma'(s),\\ y_3'(s) &=& \frac{1}{2}e^{-\beta s}|\omega(s)|^2,\end{array}\right. \qquad\qquad \left\{\begin{array}{rcl} y_1(0) &=& 0,\\ y_2(0) &=& 0,\\ y_3(0) &=& 0,\end{array}\right. \tag{4.45}$$

and the cost function is

$$g(y_3(t)) = e^{\beta t}y_3(t).$$

Here $y_2(s) = \gamma_\omega(s)$. Indeed, an integration by parts yields

$$y_2(s) \;=\; \gamma(s) + \int_0^s \left(\int_0^\sigma \omega(s')\, ds' \right) \times \gamma'(\sigma) d\sigma \;=\; \gamma(s) + \int_0^s \omega(\sigma) \times (\gamma(s) - \gamma(\sigma)) d\sigma.$$

The state constraint is

$$\Phi(y_2(s)) \;\geq\; 0 \qquad\qquad \text{for all} \quad s \in [0,t]. \tag{4.46}$$

Necessary conditions for optimality are provided in [11], Theorem 9.5.1. Namely, there exists a Lagrange multiplier $\lambda \geq 0$, an absolutely continuous adjoint vector $\mathbf{p}(\cdot) = (p_1, p_2, p_3)(\cdot)$ and a non-negative Radon measure μ, not all identically zero, such that the following holds.

(i) The vector $\mathbf{p}$ *provides a Carathéodory solution to the linear Cauchy problem on* $[0,t]$

$$\begin{cases} p_1'(s) &=\; q_2(s) \times \gamma'(s), \\ p_2'(s) &=\; 0, \\ p_3'(s) &= 0, \end{cases} \qquad\qquad \begin{cases} p_1(t) &=\; 0, \\ q_2(t) &=\; 0, \\ p_3(t) &= -\lambda e^{\beta t}, \end{cases} \tag{4.47}$$

where

$$q_2(s) \;\doteq\; \begin{cases} p_2(s) - \int_{[0,s[} \nabla\Phi(y_2(\sigma)) d\mu(\sigma) & \text{if} \;\; 0 \leq s < t\,, \\ p_2(t) - \int_{[0,t]} \nabla\Phi(y_2(\sigma)) d\mu(\sigma) & \text{if} \;\; s = t\,. \end{cases}$$

(ii) For a.e. $s \in [0,t]$*, one has the optimality condition*

$$p_1(s)\bar\omega(s) + \frac{1}{2} e^{-\beta s} p_3(s) |\bar\omega(s)|^2 \;=\; \sup_{\omega \in \mathbb{R}^3} \left\{ p_1(s)\omega + \frac{1}{2} e^{-\beta s} p_3(s) |\omega|^2 \right\}. \tag{4.48}$$

(iii) The positive measure μ *is supported on the region where the curve touches the obstacle:*

$$\mathrm{Supp}(\mu) \;\subseteq\; \big\{ s \in [0,t]\,; \quad \Phi(y_2(s)) = 0 \big\}. \tag{4.49}$$

2. We claim that, if at least one of the conditions (4.8)-(4.9) is not satisfied, then the above necessary conditions hold with $\lambda = 1$. Indeed, assume on the contrary that $\lambda = 0$. Then the optimality condition (4.48) can be satisfied only if $p_1(s) \equiv 0$. In turn this implies

$$p_1'(s) \;=\; \int_{[s,t]} \nabla\Phi(\gamma_{\bar\omega}(\sigma)) d\mu(\sigma) \times \gamma'(s) \;=\; 0 \qquad \text{for all } s \in [0,t]. \tag{4.50}$$

Furthermore, since λ, $\mathbf{p}$ and μ cannot all be zero, this implies $\mu \neq 0$.

Now consider the integral

$$\int_{[0,t]} \Phi(\gamma_{\bar\omega + \varepsilon\hat\omega}(\tau)) d\mu(\tau),$$

where $\varepsilon > 0$, $\hat{\omega}$ is a control which satisfies (4.22) and (4.23) and μ is a measure satisfying the necessary conditions. Taking the first variation w.r.t. ε, one obtains

$$\begin{aligned}
\frac{d}{d\varepsilon}\int_{[0,t]}\Phi(\gamma_{\bar{\omega}+\varepsilon\hat{\omega}}(\tau))d\mu(\tau)\Big|_{\varepsilon=0} &= \int_{[0,t]}\nabla\Phi(\gamma_{\bar{\omega}}(\tau))\cdot\int_0^\tau \hat{\omega}(\sigma)\times(\gamma(\tau)-\gamma(\sigma))d\sigma d\mu(\tau) \\
= & -\int_0^t \hat{\omega}(\sigma)\cdot\int_{[\sigma,t]}\nabla\Phi(\gamma_{\bar{\omega}}(\tau))\times(\gamma(\tau)-\gamma(\sigma))d\mu(\tau)d\sigma \\
= & \int_0^t\int_0^\sigma \hat{\omega}(\tau)d\tau\cdot\int_{[\sigma,t]}\nabla\Phi(\gamma_{\bar{\omega}}(\tau))d\mu(\tau)\times\gamma'(\sigma)d\sigma = \int_0^t\left(\int_{[0}^\sigma \hat{\omega}(\tau)d\tau\right)\cdot p_1'(\sigma)d\sigma \;=\; 0.
\end{aligned} \tag{4.51}$$

On the other hand, integrating with respect to μ on $[0,t]$, it follows from equation (4.23) that

$$\frac{d}{d\varepsilon}\int_{[0,t]}\Phi(\gamma_{\bar{\omega}+\varepsilon\hat{\omega}}(\tau))d\mu(\tau)\Big|_{\varepsilon=0} \;\geq\; \mu([0,t]) \;>\; 0,$$

which is a contradiction. This proves that $\lambda = 1$.

3. By the previous step, the necessary conditions are satisfied with $\lambda = 1$. In particular, from the conditions (ii)-(iii) it follows that

$$p_1(s) \;=\; -\int_s^t\int_{[\sigma,t]}\nabla\Phi(\gamma_{\bar{\omega}}(s'))d\mu(s')\times\gamma'(\sigma)d\sigma\,,$$

while

$$\bar{\omega}(s) \;=\; e^{-\beta(t-s)}p_1(s) \;=\; -\int_s^t\int_{[\sigma,t]}e^{-\beta(t-s)}\nabla\Phi(\gamma_{\bar{\omega}}(s'))d\mu(s')\times\gamma'(\sigma)d\sigma\,.$$

This concludes the proof. □

The above construction allows us to define a nonlinear "push-out" operator $\mathcal{P}$ as follows.

For convenience, given an angular velocity $\omega \in \mathbb{R}^3$, we shall denote by $R[\omega]$ the 3×3 rotation matrix

$$R[\omega] \;\doteq\; e^A \;\doteq\; \sum_{k=0}^{\infty}\frac{A^k}{k!}, \qquad\qquad A \;\doteq\; \begin{pmatrix} 0 & -\omega_3 & -\omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{4.52}$$

Notice that, for every $\bar{\mathbf{v}} \in \mathbb{R}^3$, the image $R[\omega]\bar{\mathbf{v}}$ is the value at time $t = 1$ of the solution to

$$\dot{\mathbf{v}}(t) \;=\; \omega\times\mathbf{v}(t), \qquad\qquad \mathbf{v}(0) = \bar{\mathbf{v}}.$$

Next, given $\gamma \in H^2([0,t];\mathbb{R}^3)$, let $\bar{\omega}$ be the minimizer for the corresponding problem (4.8)-(4.9). Using the the notation (4.52), then define

$$\mathcal{P}[\gamma](s) \;\doteq\; \int_0^s R\left[\int_0^\sigma \bar{\omega}(\zeta)d\zeta\right]\gamma'(\sigma)\,d\sigma\,. \tag{4.53}$$

Here R^ω is the rotation operator, defined as in (4.52). The difference is still $\mathcal{O}(1)\cdot\|\bar{\omega}\|^2_{\mathbf{L}^2}$, but it may be easier to estimate the difference $\|\mathcal{P}[\gamma]-\gamma\|_{H^2}$. We refer to $\mathcal{P}$ as the "push-out operator".

Remark 6. In general, $\mathcal{P}[\gamma] \neq \gamma_{\bar\omega}$. Therefore, it may not be true that $\mathcal{P}[\gamma](s) \notin \Omega$ for all $s \in [0,t]$. However, the two curves are very close. Indeed, differentiating (4.6) we obtain

$$\gamma'_{\bar\omega}(s) \;=\; \gamma'(s) + \left(\int_0^s \omega(\sigma)\,d\sigma\right) \times \omega'(s) \;=\; R\left[\int_0^s \omega(\sigma)\,d\sigma\right]\gamma'(s) + \mathcal{O}(1)\cdot \|\bar\omega\|^2_{\mathbf{L}^1}\,. \tag{4.54}$$

Therefore

$$\Big|\gamma_{\bar\omega}(s) - \mathcal{P}[\gamma](s)\Big| \;\leq\; \mathcal{O}(1)\cdot\|\bar\omega\|^2_{\mathbf{L}^1([0,t])} \;=\; \mathcal{O}(1)\cdot\|\bar\omega\|^2_{\mathbf{L}^2([0,t])}\,. \tag{4.55}$$

In particular, by (4.41) the curve $\mathcal{P}[\gamma]$ can penetrate inside the obstacle at most by the amount

$$E\Big(\mathcal{P}[\gamma],\,\Omega\Big) \;=\; \mathcal{O}(1)\cdot\|\bar\omega\|^2_{\mathbf{L}^2} \;\leq\; C\cdot E^2(\gamma,\,\Omega) \tag{4.56}$$

for some uniform constant C.

5 Proof of Theorem 1

1. Differentiating (3.3) w.r.t. s, one obtains an equation for the tangent vector $\mathbf{k}(t,\cdot) \in H^1([0,T];\,\mathbb{R}^3)$. After an integration w.r.t. t, this can be written as

$$\mathbf{k}(t,s) \;=\; \mathbf{k}(t_0,s) + \int_0^t \left(\int_0^{\tau\wedge s} \Psi\Big(\tau,\sigma,P(\tau,\sigma),\mathbf{k}(\tau,\sigma)\Big)d\sigma\right)\times \mathbf{k}(\tau,s)\,d\tau + \int_{t_0}^t \mathbf{h}(\tau,s)\,d\tau. \tag{5.1}$$

Here $P(t,s)$ is given by (2.6), while $\mathbf{h}(\tau,\cdot)$ is an element of the cone

$$\Gamma'(\tau) \;\doteq\; \Bigg\{\mathbf{h}:[0,T]\mapsto\mathbb{R}^3\,;\;\; \text{there exists a positive measure } \mu, \text{ supported on the contact set } \chi(\tau)\subseteq[0,\tau] \text{ in (2.25), such that for every } s\in[0,T] \text{ one has}$$
$$\mathbf{h}(s) \;=\; -\int_0^s\left(\int_{[\sigma,\tau]} e^{-\beta(\tau-\sigma)}\mathbf{n}(\tau,s')\times\big(P(\tau,s')-P(\tau,\sigma)\big)\,d\mu(s')\right)d\sigma\times\mathbf{k}(\tau,s)\Bigg\}. \tag{5.2}$$

This integral equation should be solved for $(t,s)\in[t_0,T]\times[0,T]$, for some $T>t_0$, with initial data

$$\mathbf{k}(t_0,s) \;=\; \bar{\mathbf{k}}(s) \;=\; \begin{cases} \overline{P}\,'(s) & \text{if } \ s\in[0,t_0],\\ \overline{P}\,'(t_0) & \text{if } \ s\in[t_0,T].\end{cases} \tag{5.3}$$

Notice that $\mathbf{k}_t(t,s)$ is always perpendicular to $\mathbf{k}(t,s)$. Therefore $|\mathbf{k}(t,s)|\equiv 1$ for all t,s. Moreover, since $\chi(t)\subset[0,t]$, every element $\mathbf{h}(t,\cdot)\in\Gamma'(t)$ is constant for $s\in[t,T]$. By (5.1), this confirms that $\mathbf{k}(t,\cdot)$ is constant for $s\in[t,T]$.

2. To construct a sequence of approximate solutions, we use an operator-splitting approximation scheme. Let $\gamma_0(\cdot)=\overline{P}(\cdot)$ be the curve of initial data. As in (4.5), consider a small neighborhood $\mathcal{V}_\rho$ of γ_0 for which the conclusions of Lemmas 1–3 hold.

Fix a time step $\varepsilon>0$ and set $t_k=t_0+\varepsilon k$. Assume that the approximate solution $\mathbf{k}=\mathbf{k}(t,s)$ has been constructed for all times $t\in[0,t_{k-1}]$ and $s\in[0,t]$. To extend the solution on the next time interval $[t_{k-1},t_k]$ we proceed as follows.

First, we construct an approximate solution of the problem without obstacle. Recalling the notation introduced at (4.52) for a rotation matrix, we define

$$\mathbf{k}(t,s) \;=\; R\left[(t-t_{k-1})\int_0^{s\wedge t_{k-1}} \Psi\big(t_{k-1},\sigma,P(t_{k-1},\sigma),\mathbf{k}(t_{k-1},\sigma)\big)d\sigma\right]\mathbf{k}(t_{k-1},s), \tag{5.4}$$

for $t\in[t_{k-1},t_k[$ and $s\in[0,T]$. Taking $t=t_k$, the above construction produces a curve

$$s \;\mapsto\; \gamma(s) \;=\; P(t_k-,s) \;=\; \int_0^s \mathbf{k}(t,\sigma)\,d\sigma.$$

In general, the physically meaningful portion of this curve:

$$\gamma(s) \;\doteq\; P(t_k-,s)\,, \qquad\qquad s\in[0,t_k], \tag{5.5}$$

may well partly lie inside the obstacle. Using the push-out operator (4.53), we then replace $P(t_k-,\cdot)$ by a new curve, by setting

$$P(t_k,s) \;=\; \mathcal{P}[P(t_k-,\cdot)](s) \;=\; \int_0^s R\left[\int_0^\sigma \bar\omega_k(\zeta)d\zeta\right]P_s(t_k-,\sigma)\,d\sigma, \qquad s\in[0,T]\,. \tag{5.6}$$

Equivalently:

$$\mathbf{k}(t_k,s) \;=\; R\left[\int_0^s \bar\omega_k(\zeta)d\zeta\right]\mathbf{k}(t_k-,s)\,. \tag{5.7}$$

As in Lemma 3, here $\bar\omega_k$ is the unique minimizer for the problem (4.8)-(4.9), with $\gamma:[0,t_k]\mapsto\mathbb{R}^3$ as in (5.5). According to Lemma 4, we have the representation

$$\bar\omega_k(s) \;=\; -\,e^{-\beta(t_k-s)}\int_s^{t_k}\left(\int_{[\sigma,t_k]}\nabla\Phi(\gamma_{\bar\omega_k}(s'))d\mu_k(s')\right)\times P_s(t_k-,\sigma)d\sigma, \tag{5.8}$$

where μ_k is a positive measure supported on the contact set

$$\chi_k \;=\; \{s\in[0,t_k]:\ \gamma_{\bar\omega_k}(t,s)\in\Omega\}. \tag{5.9}$$

Applying Fubini's theorem, one can exchange the order of integration in (5.8) and obtain

$$\begin{aligned}\omega_k(s) \;&=\; -\,e^{-\beta(t_k-s)}\int_{[s,t_k]}\nabla\Phi(\gamma_{\bar\omega_k}(s'))\times\Big(\int_s^{s'}P_s(t_k-,\sigma)d\sigma\Big)d\mu_k(s')\\ &=\; -\,e^{-\beta(t_k-s)}\int_{[s,t_k]}\nabla\Phi(\gamma_{\bar\omega_k}(s'))\times\Big(P(t_k-,s')-P(t_k-,s)\Big)\,d\mu_k(s').\end{aligned} \tag{5.10}$$

By (4.41) and (4.56), as long as the approximation remains inside the neighborhood $\mathcal{V}_\rho$, there exists a constant C_3 such that, recalling the definition (4.40) one has

$$\|\bar\omega_k\|_{\mathbf{L}^2([0,t_k])} \;\le\; C_3\,E(t_k,\,P(t_k-,\cdot),\,\Omega)\,, \tag{5.11}$$

$$E(t_k,\,P(t_k,\cdot),\,\Omega) \;\le\; C_3\,E^2(t_k,\,P(t_k-,\cdot),\,\Omega). \tag{5.12}$$

Moreover, the depth at which the curve $P(t_k-,\cdot)$ penetrates the obstacle is estimated by

$$E\big(t_k,\,P(t_k-,\cdot),\,\Omega\big) \;\le\; E(t_{k-1},\,P(t_{k-1},\cdot),\,\Omega)+C_4\varepsilon\,, \tag{5.13}$$

for some constant C_4. For every $\varepsilon > 0$ sufficiently small, the above estimates (5.12)-(5.13) yield the implication

$$E(t_{k-1},\, P(t_{k-1},\cdot),\, \Omega) \;\leq\; \varepsilon \qquad \Longrightarrow \qquad E(t_k,\, P(t_k,\cdot),\, \Omega) \;\leq\; \varepsilon\,.$$

By assumption, the initial condition lies outside the obstacle, i.e.

$$E(t_k,\, P(t_0,\cdot),\, \Omega) \;=\; 0. \tag{5.14}$$

By induction, for all $\varepsilon > 0$ small enough and all $k \geq 1$ we thus conclude

$$E(t_k,\, P(t_k,\cdot),\, \Omega) \;\leq\; \varepsilon\,. \tag{5.15}$$

In turn, by (5.11)-(5.12) this implies an estimate of the form

$$\|\bar\omega_k\|_{\mathbf{L}^2([0,t_k])} \;\leq\; C_5\,\varepsilon\,. \tag{5.16}$$

3. We claim that, for every $\varepsilon > 0$ and $k \geq 1$, the total mass of the positive measure μ_k in (5.8) is bounded by

$$\|\mu_k\| \;=\; \mu_k([0,t_k]) \;\leq\; C_6\,\varepsilon, \tag{5.17}$$

for some constant C_6, independent of ε, k. Indeed, (4.41) implies

$$|P(t_k-,s) - \gamma_{\bar\omega_k}(s)| \;=\; \mathcal{O}(1)\cdot\|\bar\omega_k\|_{\mathbf{L}^1} \;=\; \mathcal{O}(1)\cdot\varepsilon.$$

Taking $\gamma : [0,t_k] \mapsto \mathbb{R}^3$ as in (5.5) and integrating the left hand side of (4.23) w.r.t. the measure μ_k we obtain

$$\begin{aligned}
&\mu_k([0,t_k]) \;\leq\; \int_{[0,t_k]} \left\langle \int_0^s \omega(\sigma)\times\big(P(t_k-,s)-P(t_k-,\sigma)\big)\,d\sigma\,,\; \nabla\Phi(P(t_k-,s))\right\rangle d\mu_k(s)\\
&=\; -\int_0^{t_k}\omega(\sigma)\cdot\Big(\int_{[\sigma,t_k]}\big(\nabla\Phi(P(t_k-,s))-\nabla\Phi(\gamma_{\bar\omega_k}(s))\big)\times\big(P(t_k-,s)-P(t_k-,\sigma)\big)d\mu_k(s)\Big)d\sigma\\
&\qquad\qquad -\int_0^{t_k}\omega(\sigma)\cdot\Big(\int_{[\sigma,t_k]}\nabla\Phi(\gamma_{\bar\omega_k}(s))\times\big(P(t_k-,s)-P(t_k-,\sigma)\big)d\mu_k(s)\Big)d\sigma\\
&\leq\; \int_0^{t_k}|\omega(\sigma)|\,C\varepsilon\,\|\mu_k\|\,d\sigma + \left|\int_0^t \omega(\sigma)\cdot\bar\omega_k(\sigma)d\sigma\right|\\
&\leq\; C\varepsilon\,\|\omega(\sigma)\|_{\mathbf{L}^1}\,\|\mu_k\| + \|\omega\|_{\mathbf{L}^2}\|\bar\omega_k\|_{\mathbf{L}^2}\,,
\end{aligned} \tag{5.18}$$

for some constant C independent of ε, k. By (4.22) we have

$$\|\omega\|_{\mathbf{L}^1([0,t_k])} \;\leq\; \sqrt{t_k}\,\|\omega\|_{\mathbf{L}^2([0,t_k]} \;\leq\; \sqrt{T}\,C_0\,,$$

while (4.41) implies $\|\bar\omega_k\|_{\mathbf{L}^1},\ \|\bar\omega_k\|_{\mathbf{L}^2} \;=\; \mathcal{O}(1)\cdot\varepsilon$. Choosing $\varepsilon > 0$ small enough so that $C\varepsilon\|\omega\|_{\mathbf{L}^1} < 1/2$, from (5.18) we deduce

$$\frac{1}{2}\,\|\mu_k\| \;\leq\; \|\omega\|_{\mathbf{L}^2}\|\bar\omega_k\|_{\mathbf{L}^2} \;=\; \mathcal{O}(1)\cdot\varepsilon.$$

This yields the estimate (5.17).

Using (5.17) in (5.10) we can refine the estimate (5.16) and conclude that

$$\|\bar{\omega}_k\|_{\mathbf{L}^\infty([0,t_k])} \;\leq\; C_7\varepsilon, \tag{5.19}$$

for some constant C_7 independent of k, ε.

4. By the previous arguments, for every time step $\varepsilon > 0$ sufficiently small, we obtain a piecewise continuous approximate solution $\mathbf{k}_\varepsilon = \mathbf{k}_\varepsilon(t,s)$ defined for $s \in [0,t]$ and $t \in [t_0, T_\varepsilon]$. Here T_ε is the supremum of all times τ for which the corresponding curve $P_\varepsilon(\tau,s) = \int_0^s \mathbf{k}_\varepsilon(\tau,\sigma)d\sigma$ remains in the neighborhood $\mathcal{V}_\rho$.

We now observe that, for every $\varepsilon > 0$ small enough and $k \geq 1$, as long as the approximation $P_\varepsilon(t,\cdot)$ remains inside $\mathcal{V}_\rho$ one has

$$\|\mathbf{k}_\varepsilon(t,\cdot) - \mathbf{k}_\varepsilon(t',\cdot)\|_{H^1([0,t_{k-1}])} \;\leq\; C_8\,|t-t'| \qquad \text{for all } \; t,t' \in [t_{k-1}, t_k[\,, \tag{5.20}$$

$$\|\mathbf{k}_\varepsilon(t_k,\cdot) - \mathbf{k}_\varepsilon(t_k-,\cdot)\|_{H^1([0,t_k])} \;\leq\; C_8\,\varepsilon\,, \tag{5.21}$$

for some constant C_8 independent of k, ε. Indeed, the first estimate is an immediate consequence of the boundedness of Ψ. The second estimate follows from (5.16). As long as the approximations remain inside $\mathcal{V}_\rho$, the bounds (5.20)-(5.21) imply an estimate of the form

$$\|\mathbf{k}_\varepsilon(t,\cdot) - \mathbf{k}_\varepsilon(\tau,\cdot)\|_{H^1([0,\,t])} \;\leq\; C_9(\varepsilon + \tau - t)\,, \tag{5.22}$$

for some constant C_9 and all $t < \tau$. Since by construction

$$P_\varepsilon(t,0) = 0, \qquad\qquad \mathbf{k}_\varepsilon(t,0) \;=\; \bar{\mathbf{k}}(0) \qquad \text{for all } \; t \geq 0,$$

from (5.22) we deduce

$$P_\varepsilon(t_k,\cdot) \;\in\; \mathcal{V}_\rho \qquad\qquad \text{for all } \; t \in [0,T], \tag{5.23}$$

for some $T > 0$ independent of ε.

Thanks to the above estimates, we conclude that for $\varepsilon > 0$ sufficiently small all the approximations $\mathbf{k}_\varepsilon = \mathbf{k}_\varepsilon(t,s)$ are well defined. To achieve the convergence of a subsequence, two observations are in order:

- As maps from $[t_0,T]$ into $H^1([0,T])$, all functions $t \mapsto \mathbf{k}_\varepsilon(t,\cdot)$ have uniformly bounded total variation.
- For every t, ε, consider the difference of the partial derivatives

$$\mathbf{w}(t,s) \;\doteq\; \frac{\partial}{\partial s}\mathbf{k}_\varepsilon(t,s) - \frac{\partial}{\partial s}\bar{\mathbf{k}}(s).$$

  Then, for every $t \in [t_0,T]$, the map $s \mapsto \mathbf{w}(t,s)$ is uniformly Lipschitz continuous. Indeed, this is because (i) the integral in (5.4) is uniformly bounded, for all k, ε, and (ii) the function $\bar{\omega}_k$ in (5.7) satisfies the uniform bound (5.19). As a consequence, all functions $\mathbf{k}_\varepsilon(t,\cdot)$ remain within a fixed compact subset $\mathcal{K} \subset H^1([0,T];\mathbb{R}^3)$.

Thanks to the above properties, we can thus extract a subsequence $\varepsilon_n \to 0$ and achieve the H^1 convergence $\mathbf{k}_{\varepsilon_n}(t,\cdot) \to \mathbf{k}(t,\cdot)$, uniformly for $t \in [t_0, T]$.

5. In this step we study the convergence of the measures μ_k^ε. For any fixed $\varepsilon > 0$, let μ^ε be the positive measure on $[t_0, T] \times \mathbb{R}$ whose restriction to $]t_{k-1}, t_k] \times \mathbb{R}$ coincides with the product $\mathcal{L} \otimes \mu_k^\varepsilon$. Here $\mathcal{L}$ denotes the Lebesgue measure. In other words, for every subinterval $[a,b] \subseteq [t_{k-1}, t_k]$ and every open set $V \subset \mathbb{R}$, we have

$$\mu^\varepsilon\big([a,b] \times V\big) \;=\; (b-a)\cdot \varepsilon^{-1} \mu_k^\varepsilon(V). \tag{5.24}$$

In view of the uniform bound (5.18), we can extract a weakly convergent subsequence, so that $\mu^\varepsilon \rightharpoonup \mu$. More precisely (see [1]) , there exists a measurable family of uniformly bounded positive measures $\{\mu^t\,;\;\; t \in [t_0, T]\}$ such that, for every continuous function $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ one has

$$\lim_{\varepsilon\to 0} \int_{[t_0,T]\times[0,T]} \varphi(t,s)\, d\mu^\varepsilon(t,s) \;=\; \int_{[t_0,T]\times[0,T]} \varphi(t,s)\, d\mu(t,s) = \;\int_{t_0}^T \left(\int_{[0,T]} \varphi(t,s) d\mu^t(s)\right) dt. \tag{5.25}$$

For each $\varepsilon > 0$ and $t \in [t_0, T]$ we define

$$\chi_\varepsilon(t) \;\doteq\; \{s \in [0,t]\,;\, \gamma_{\bar\omega_k}(s) \in \partial\Omega\} \;\supseteq\; \mathrm{Supp}\{\mu_k^\varepsilon\}, \tag{5.26}$$

where k is the unique index such that $t \in]t_{k-1}, t_k] \doteq]\varepsilon(k-1), \varepsilon k]$. By (4.55), (5.16), and (5.22) it follows that $\gamma_{\omega_k}(t,s) \to P(t,s)$ uniformly on $\mathcal{D}_T$. Now fix any $\delta > 0$ and consider the set

$$V^\delta(t) \;\doteq\; \big\{s \in [0,t]\,;\;\; d(P(t,s), \partial\Omega) \le \delta\big\}.$$

In view of the uniform convergence $\mathbf{k}_\varepsilon \to \mathbf{k}$ and $P_\varepsilon \to P$ as $\varepsilon \to 0$, it follows that $\chi_\varepsilon(t) \subseteq V^\delta(t)$ for all $\varepsilon > 0$ sufficiently small. By the weak convergence, one has

$$\mathrm{Supp}\{\mu^t\} \;\subseteq\; \limsup_{\varepsilon\to 0} \chi_\varepsilon(t) \;\subseteq\; V^\delta(t)$$

for any $\delta > 0$ and $t \in [t_0, T]$. Since $\delta > 0$ was arbitrary, this implies

$$\mathrm{supp}\big\{\mu^t\big\} \;\subseteq\; \chi(t) \;=\; \{s \in [0,t]\,;\;\; P(t,s) \in \partial\Omega\big\} \tag{5.27}$$

for all $t \in [t_0, T]$.

6. We complete the proof by showing that $\mathbf{k}$ provides the desired solution. By (5.15), letting $\varepsilon \to 0$ it is clear that $P(t,s) \notin \Omega$, for all $t \in [t_0, T]$ and $s \in [0,t]$.

It remains to prove that, for every $t \in [t_0, T]$, the identity (5.1) holds, with

$$\begin{aligned}\mathbf{h}(\tau,s) \;&=\; -\int_0^s \left(\int_{[\sigma,\tau]} e^{-\beta(\tau-\sigma)} \mathbf{n}(\tau,s') \times \big(P(\tau,s') - P(\tau,\sigma)\big)\, d\mu^\tau(s')\right) d\sigma \times \mathbf{k}(\tau,s)\\ &=\; -\int_0^s \left(\int_{[\sigma,\tau]} e^{-\beta(\tau-\sigma)} \nabla\Phi(P(\tau,s')) \times \big(P(\tau,s') - P(\tau,\sigma)\big)\, d\mu^\tau(s')\right) d\sigma \times \mathbf{k}(\tau,s).\end{aligned} \tag{5.28}$$

Here $\{\mu^\tau;\; \tau \in [t_0, T]\}$ is the family of measures constructed in step **5.**

The identity (5.1) will be obtained by taking the limit as $\varepsilon \to 0$ of the identities (5.4) and (5.7) satisfied by $\mathbf{k}^\varepsilon$. By the previous analysis, as $\varepsilon \to 0$ we have

$$\mathbf{k}^\varepsilon(t,s) \;\to\; \mathbf{k}(t,s), \qquad\qquad P^\varepsilon(t,s) \;\doteq\; \int_0^s \mathbf{k}^\varepsilon(t,\sigma)\,d\sigma \;\to\; \int_0^s \mathbf{k}(t,\sigma)\,d\sigma \;=\; P(t,s), \tag{5.29}$$

uniformly for $(t,s) \in [t_0,T] \times [0,T]$.

To handle the right hand side of (5.1), we start with matrix estimate

$$\Big|(I + A_1 + \cdots + A_n)\mathbf{v} - e^{A_1} \circ \cdots \circ e^{A_n}\mathbf{v}\Big| \;=\; \mathcal{O}(1) \cdot \left(\sum_i |A_i|\right)^2 |\mathbf{v}|,$$

which implies, as a special case:

$$\Big|\mathbf{v} + (\omega_1 + \cdots + \omega_n) \times \mathbf{v} - R[\omega_1] \circ \cdots \circ R[\omega_n]\mathbf{v}\Big| \;=\; \mathcal{O}(1) \cdot \left(\sum_i |\omega_i|\right)^2 |\mathbf{v}|. \tag{5.30}$$

Using the notation

$$t_k^\varepsilon \doteq t_0 + k\varepsilon\,, \qquad\qquad k_\varepsilon(t) \;\doteq\; \max\{k \geq 0\,;\;\; t_k^\varepsilon \leq t\},$$

in view of (5.30) we can write

$$\begin{aligned}\mathbf{k}(t,s) - \mathbf{k}(t_0,s) \;&=\; \lim_{\varepsilon\to 0} \sum_{k=0}^{k^\varepsilon(t)} \left(\varepsilon \int_0^{t_k \wedge s} \Psi\big(t_k^\varepsilon, \sigma, P^\varepsilon(t_k^\varepsilon,\sigma), \mathbf{k}^\varepsilon(t_k^\varepsilon,\sigma)\big)d\sigma\right) \times \mathbf{k}^\varepsilon(t_k^\varepsilon, s)\\ &+ \lim_{\varepsilon\to 0} \sum_{k=0}^{k^\varepsilon(t)} \Big(\int_{[s,t_k]} e^{-\beta(t_k^\varepsilon - s)} \nabla\Phi(P^\varepsilon(t_k^\varepsilon, s')) \times (P^\varepsilon(t_{k-1}^\varepsilon, s') - P(t_{k-1}^\varepsilon, s))d\mu_k^\varepsilon(s')\Big) \times \mathbf{k}^\varepsilon(t_k, s).\end{aligned} \tag{5.31}$$

By the uniform convergence (5.29) and the smoothness of the function Φ, the first term on the right hand side of (5.31) converges to the corresponding term in (5.1). Moreover, recalling (5.24), and using the weak convergence $\mu^\varepsilon \rightharpoonup \mu$, we conclude that the second term on the right hand side of (5.31) converges to the corresponding term in (5.1). This completes the proof. □

6 More general models

All previous results can be extended to the case where $0 < \alpha < +\infty$, so that all sections of the stem undergo a linear elongation, exponentially decreasing in time. In addition, following Remark 1, one can also consider deformations which minimize the more general deformation energy (2.27), where the twist and bending components are given different weights. We describe below the minor differences in the analysis, required by these extensions.

When $\alpha < +\infty$, the unit tangent vector $\mathbf{k}$ to the stem is given by (2.3), and the formula (2.6) is replaced by (2.4). The evolution of $\mathbf{k}(t,s)$ is still described by (5.1), replacing (3.5) with

$$\Psi(t,\sigma,P,\mathbf{k}) \;\doteq\; \big(1 - e^{-\alpha(t-\sigma)}\big)e^{-\beta(t-\sigma)}\Big(\kappa(\mathbf{k} \times \mathbf{e}_3) + (\nabla\psi(P) \times \mathbf{k})\Big). \tag{6.1}$$

Next, we derive the appropriate replacements for the cones Γ in (3.4) and Γ' in (5.2).

As in (2.3), call $\mathbf{k}(t,\sigma)$ the unit tangent vector to the curve $P(t,\cdot)$ at the point σ. Given any vector function $\omega(\cdot)$, we define the orthogonal decomposition

$$\omega(\sigma) \;=\; \omega^{twist}(\sigma)+\omega^{bend}(\sigma),$$

where

$$\omega^{twist}(\sigma) \;\doteq\; \Pi_{\mathbf{k}(t,\sigma)}\omega(\sigma), \qquad\qquad \omega^{bend}(\sigma) \;\doteq\; \Pi_{\mathbf{k}^\perp(t,\sigma)}\omega(\sigma),$$

denote the components parallel and orthogonal to the vector $\mathbf{k}(t,\sigma)$, respectively.

Consider the constrained optimization problem

$$\text{minimize:}\quad \mathcal{E} \;\doteq\; \frac{1}{2}\int_0^{s'} \big(1-e^{-\alpha(t-\sigma)}\big)e^{\beta(t-\sigma)}\Big(c_1\left|\omega^{twist}(\sigma)\right|^2+c_2\left|\omega^{bend}(\sigma)\right|^2\Big)\,d\sigma \tag{6.2}$$

subject to the constraint

$$\nabla\Phi(P(t,s'))\cdot\left(\int_0^{s'}\big(1-e^{-\alpha(t-\sigma)}\big)\omega(\sigma)\times\big(P(t,s')-P(t,\sigma)\big)\,d\sigma\right)+\Phi(P(t,s')) \;=\; 0. \tag{6.3}$$

To derive the appropriate necessary conditions, consider a family of perturbations of the form

$$\omega_\epsilon(\sigma) \;=\; \omega(\sigma)+\epsilon\tilde\omega(\sigma) \;=\; \Big(\omega^{twist}(\sigma)+\omega^{bend}(\sigma)\Big)+\Big(\epsilon\tilde\omega^{twist}(\sigma)+\epsilon\tilde\omega^{bend}(\sigma)\Big).$$

Arguing as in (2.17), we differentiate $\mathcal{E}$ w.r.t. ϵ at $\epsilon=0$, and eventually obtain

$$\begin{aligned}&\int_0^{s'}(1-e^{-\alpha(t-\sigma)})e^{\beta(t-\sigma)}\Big(c_1\omega^{twist}(\sigma)\cdot\tilde\omega^{twist}(\sigma)+c_2\omega^{bend}(\sigma)\cdot\tilde\omega^{bend}(\sigma)\Big)d\sigma\\ &\qquad=\;\lambda\int_0^{s'}(1-e^{-\alpha(t-\sigma)})\tilde\omega(\sigma)\cdot\big(\nabla\Phi(P(t,s'))\times(P(t,s')-P(t,\sigma))\big)d\sigma\end{aligned} \tag{6.4}$$

for some Lagrange multiplier λ. Notice that, by orthogonality, the inner products satisfy

$$\omega^{twist}(\sigma)\cdot\tilde\omega^{twist}(\sigma) \;=\; \omega^{twist}(\sigma)\cdot\tilde\omega(\sigma) \qquad\qquad \omega^{bend}(\sigma)\cdot\tilde\omega^{bend}(\sigma) \;=\; \omega^{bend}(\sigma)\cdot\tilde\omega(\sigma).$$

Since the relation (6.4) holds true for every perturbation $\tilde\omega$, one obtains

$$c_1\omega^{twist}(\sigma)+c_2\omega^{bend}(\sigma) \;=\; \lambda e^{-\beta(t-\sigma)}\nabla\Phi(P(t,s'))\times(P(t,s')-P(t,\sigma)). \tag{6.5}$$

Projecting the above equation on the subspaces parallel and perpendicular to $\mathbf{k}(t,\sigma)$ respectively, with obvious meaning of notation we finally obtain

$$\begin{cases}\omega^{twist}(\sigma) \;=\; \dfrac{\lambda}{c_1}e^{-\beta(t-\sigma)}\Big(\nabla\Phi(P(t,s'))\times(P(t,s')-P(t,\sigma))\Big)^{twist},\\[2mm] \omega^{bend}(\sigma) \;=\; \dfrac{\lambda}{c_2}e^{-\beta(t-\sigma)}\Big(\nabla\Phi(P(t,s'))\times(P(t,s')-P(t,\sigma))\Big)^{bend}.\end{cases} \tag{6.6}$$

In place of (2.26), the **cone of admissible velocities** can now be defined as

$$\begin{aligned}\Gamma(t) \;\doteq\; \Bigg\{&\mathbf{v}:[0,t]\mapsto\mathbb{R}^3;\;\;\text{there exists a positive measure }\mu\text{ supported on }\chi(t)\text{ such that}\\ &\mathbf{v}(s)=-\int_0^s(1-e^{-\alpha(t-\sigma)})e^{-\beta(t-\sigma)}\int_{[\sigma,t]}\Bigg[\Big(\frac{\mathbf{n}(t,s'))}{c_1}\times(P(t,s')-P(t,\sigma))\Big)^{twist}\\ &\qquad+\Big(\frac{\mathbf{n}(t,s')}{c_2}\times(P(t,s')-P(t,\sigma))\Big)^{bend}\Bigg]d\mu(s')\times\big(P(t,s)-P(t,\sigma)\big)\,d\sigma\Bigg\}.\end{aligned} \tag{6.7}$$

Accordingly, the cone $\Gamma'(\tau)$ in (5.2) is now replaced by

$$\Gamma'(\tau) \doteq \Bigg\{ \mathbf{h} : [0,T] \mapsto \mathbb{R}^3 \,; \quad \text{there exists a positive measure } \mu, \text{ supported on } \chi(\tau), \text{ such that}$$
$$\mathbf{h}(s) = -\int_0^s (1-e^{-\alpha(t-\sigma)})e^{-\beta(t-\sigma)} \Bigg(\int_{[\sigma,t]} \Bigg[\Big(\frac{\mathbf{n}(t,s')}{c_1} \times (P(t,s') - P(t,\sigma)) \Big)^{twist}$$
$$+ \Big(\frac{\mathbf{n}(t,s')}{c_2} \times (P(t,s') - P(t,\sigma)) \Big)^{bend} \Bigg] d\mu(s') \Bigg) d\sigma \times \mathbf{k}(t,s)\, d\sigma \Bigg\}. \tag{6.8}$$

All the previous arguments can be applied to this more general situation, with minor changes.

7 Numerical simulations

We present a couple of numerical simulations for the model (2.37) in two dimensional space. Finite difference discretization is used, where the ODE (2.37) is solved with forward Euler method, and the integrations are carried out with trapezoid rule. The stem/vine is discretized with uniform arc-length Δs, and the time step is also uniform with $\Delta t = \Delta s$. Simulations are carried out in Matlab. All the Matlab codes used, together with many figures and simulations can be found in [10].

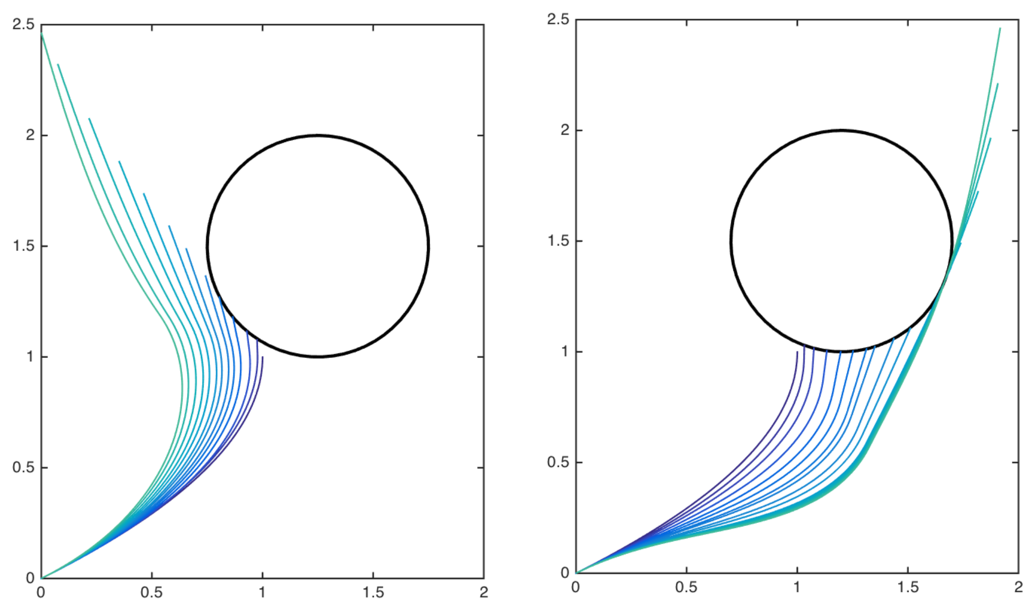


Figure 6: Numerical simulation for testing the bifurcation for avoiding obstacles.

Simulation 1. We simulate the growth of a tree stem avoiding an obstacle along the way. In this model we neglect the second term on the right hand side of (2.37). The following parameters are used:

$$\beta = 0.5, \quad \kappa = 1.$$

The stem is initiated at the origin, with the initial shape $x = 1 - (y-1)^2$ for $0 \leq y \leq 1$. The obstacle is a circle, centered at (a, b) with radius $r = 0.5$. Two slightly different locations of (a, b) are chosen, and the results are shown in Figure 6. For the left plot, we use $(a, b) = (1.2, 1.5)$, and the stem bends to the left to avoid the obstacle. For the right plot, we use $(a, b) = (1.25, 1.5)$, and the stem bends to the right to avoid the obstacle.

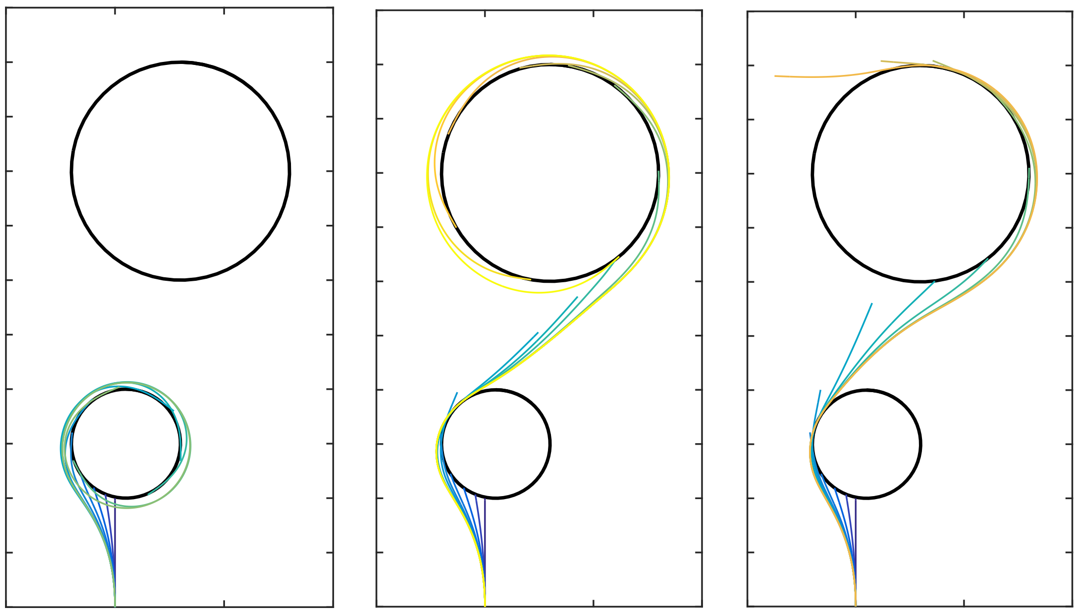


Figure 7: Numerical simulation for vine growth with different bending parameter.

Simulation 2. We simulate a growing vine, in the presence of two obstacles. The full equation (2.37) is now used.

The function $\eta(d)$ in (2.9) is chosen to be

$$\eta(d) = \begin{cases} \gamma\left(1 - e^{-d}\right), & 0 \leq d < \delta_0, \\ \gamma\left(1 - e^{-\delta_0}\right), & d \geq \delta_0. \end{cases}$$

The parameter γ measures the strength of the feedback response, in the presence of an obstacle.

We simulate various cases, highlighting the differences in the solution caused by this bending factor γ. The vine is initiated at the origin, initially growing straight up. Two circular obstacles are placed, one centered at $(0.1, 1.5)$ with smaller radius $r_1 = 0.5$, the other centered at $(0.6, 4)$ with larger radius $r_2 = 1$. We use the following parameters:

$$\beta = 2, \qquad \kappa = 1, \qquad \delta_0 = 0.05.$$

Numerical results with three different values of γ are plotted in Figure 7:

- For the left plot, we use $\gamma = 7$, a rather large value. The vine already curls around the smaller disc.
- For the middle plot, we choose the smaller value $\gamma = 4$. The vine now fails to cling to the smaller disc, but manages to curl around the larger disc.

- For the right plot, we choose an even smaller parameter value: $\gamma = 3$. In this case, the response to gravity prevails and the vine fails to cling even to the larger disc.

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