Well-posedness of a Model for the Growth of Tree Stems and Vines

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Abstract

The paper studies a PDE model for the growth of a tree stem or a vine, having the form of a differential inclusion with state constraints. The equations describe the elongation due to cell growth, and the response to gravity and to external obstacles.

The main theorem shows that the evolution problem is well posed, until a specific "breakdown configuration" is reached. A formula is proved, characterizing the reaction produced by unilateral constraints. At a.e. time t, this is determined by the minimization of an elastic energy functional under suitable constraints.

1 Introduction

We consider a PDE model, recently introduced in [1], describing the growth of a plant stem or a vine.

The position of the stem at time t is described by a curve $\gamma(t,\cdot)$. For $s \in [0,t]$, we think of $\gamma(t,s)$ as the position at time t of the cell born at time s. The model takes into account:

(1) the linear elongation,

(2) the upward bending, as a response to gravity,

(3) an additional bending, in case of a vine clinging to branches of other plants,

(4) the reaction produced by obstacles, such as rocks, trunks or branches of other trees.

For simplicity, we rescale time and assume that the map $s \mapsto \gamma(t,s)$ parameterizes the curve by arc-length. Without loss of generality, one can assume that $\gamma(t,0) = 0 \in \mathbb{R}^3$, so that

$$\gamma(t,s) \;=\; \int_0^s \mathbf{k}(t,\sigma)\,d\sigma\,, \qquad\qquad \mathbf{k}(t,s) \;\doteq\; \gamma_s(t,s). \tag{1.1}$$

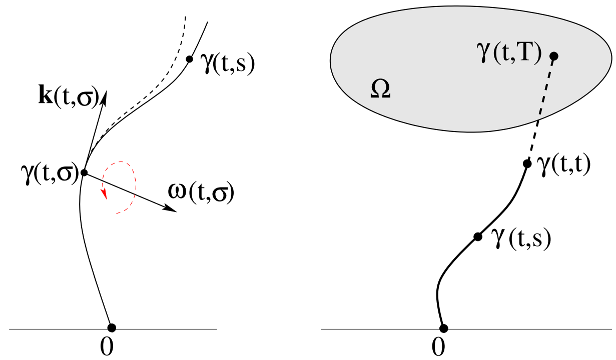

Figure 1: Left: at any point $\gamma(t,\sigma)$ along the stem, an infinitesimal change in curvature is produced as a response to gravity (or stems of other plants). The angular velocity is given by the vector $\omega(\sigma)$. This affects the position of all higher points along the stem. Right: At a given time t, the curve $\gamma(t,\cdot)$ is parameterized by $s\in[0,t]$. It is convenient to prolong this curve by adding a segment of length $T-t$ at its tip (dotted line, possibly entering inside the obstacle). This yields an evolution equation on a fixed functional space $H^2([0,T];\mathbb{R}^3)$.

The change in the position of points on the stem is described by

$$\gamma_t(t,s) \;=\; \int_0^s \omega(t,\sigma)\times\big(\gamma(t,s)-\gamma(t,\sigma)\big)\,d\sigma \;\doteq\; F(t,s)\,. \tag{1.2}$$

Here ω represents an angular velocity (see Fig. 1). According to (1.2), portions of the stem can slightly change their curvature in time, as a response to gravity or (in the case of vines) to branches of other plants. Notice that the infinitesimal change in curvature at the point $\gamma(t,\sigma)$ affects all the upper portion of the stem, i.e. all points $\gamma(t,s)$ with $s\in[\sigma,t]$. In our model,

$$\omega(t,s) \;=\; \Psi\big(t,s,\gamma(t,s),\gamma_s(t,s)\big)$$

depends on the position and on the orientation of the stem, at a given point. For example, to model the bending of the stem in the upward direction (as a response to gravity), one can take

$$\Psi(t,s,\gamma,\mathbf{k}) \;\doteq\; e^{-\beta(t-s)}\mathbf{k}\times\mathbf{e}_3\,. \tag{1.3}$$

Here $\beta>0$ is a stiffness constant, while $\mathbf{e}_3\in\mathbb{R}^3$ is a unit vector, oriented in the upward vertical direction. Notice that $t-s$ is the age at time t of the cell born at time s. The factor $e^{-\beta(t-s)}$ accounts for the fact that older portions of the stem become more stiff, hence their curvature changes more slowly. As shown in [1], a second term can be added to the right hand side of (1.3) to describe a vine curling around branches of other trees. To cover all the models in [1], our present results will be stated for a general function $\Psi=\Psi(t,s,\gamma,\mathbf{k})$.

In addition, we consider an obstacle $\Omega\subset\mathbb{R}^3$, whose presence imposes the unilateral constraint

$$\gamma(t,s)\;\notin\;\Omega \qquad\qquad \text{for all}\quad s\in[0,t]\,. \tag{1.4}$$

As in [1], the evolution of the stem can be described by an equation of the form

$$\gamma_t(t,s) \;=\; F(t,s)+\mathbf{v}(t,s), \qquad\qquad \mathbf{v}(t,\cdot)\in\Gamma(t), \tag{1.5}$$

where $\Gamma(t)$ is a cone of admissible velocities determined by the constraint reaction.

Under natural assumptions, the main theorem in [1] provides the existence of a solution to (1.5). This solution is defined up to the first time where a "breakdown configuration" is reached, characterized at (2.21)-(2.22). Examples are shown in Fig. 3. The theorem is proved by writing the evolution equation for γ in the form of a differential inclusion with closed convex right hand side, in the functional space $H^2([0,T];\mathbb{R}^3)$. The uniqueness of these solutions, however, had remained an open question.

We remark that most of the literature on differential inclusions with constraints has been concerned with problems of the form

$$\frac{d}{dt}x(t) \in F(x(t)) - N_S(x(t)), \qquad\qquad x(t) \in S,$$

where $N_S(x)$ is the outer normal cone to the set S at the point x. When the set $S = S(t)$ is allowed to depend on time, this is called a "perturbed sweeping process", see [3, 4, 6, 7]. In this setting, the Cauchy problem usually has a unique solution, continuously depending on the initial data.

On the contrary, in the present case the cone Γ of admissible velocities in (1.5) bears no relation to the normal cone. In fact, as the stem reaches a "breakdown configuration" illustrated in Fig. 2, the cone Γ becomes tangent to the boundary of the admissible set S. For this reason, the well-posedness of the Cauchy problem for (1.5) is a delicate issue.

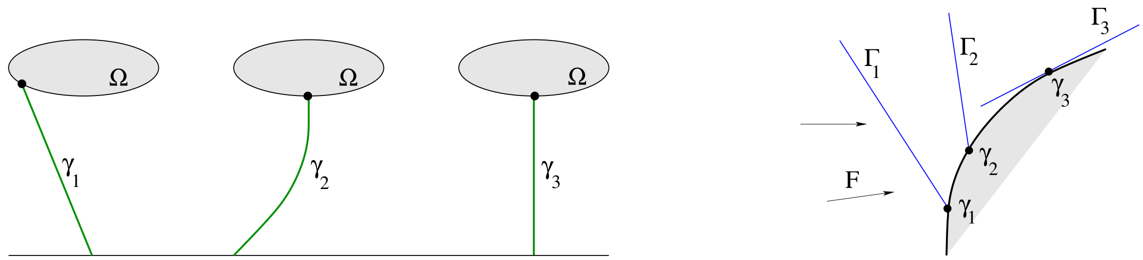

Figure 2: Left: three configurations of the stem, relative to the obstacle. Right: in an abstract space, the first two configurations are represented by points γ_1, γ_2 on the boundary of the admissible set S where the corresponding cones Γ_1, Γ_2 are transversal. On the other hand, γ_3 is a "breakdown configuration", satisfying all assumptions (2.21)-(2.22). Its corresponding cone Γ_3 is tangent to the boundary of the set S. Here the shaded region is the complement of S.

The aim of the present paper is twofold:

(i) Prove the uniqueness and continuous dependence on initial data, for solutions to (1.5).

(ii) Provide a characterization of the velocity $\mathbf{v}(t,\cdot) \in \Gamma(t)$ in (1.5) determined by the obstacle reaction.

Following [1], a solution $t \mapsto \gamma(t,\cdot)$ is regarded as a map taking values in the Hilbert space $H^2([0,T];\mathbb{R}^3)$. Unfortunately, a study of the H^2 distance between two solutions does not lead to any useful estimate. In the present paper, the distance between two solutions $\gamma_1(t,\cdot)$, $\gamma_2(t,\cdot)$ will be estimated by constructing a family of rotations, transforming a unit tangent vector $\mathbf{k}_1(t,s)$ to γ_1 into the corresponding tangent vector $\mathbf{k}_2(t,s)$ to γ_2, for every $s \in [0,t]$. By estimating how the norm of these rotation vectors grows in time, we shall provide a bound on the distance between the two solutions γ_1, γ_2 for all times t.

Next, by further developing the analysis in [1] we will show that, for a.e. time t, the vector $\mathbf{v}(t, \cdot)$ is uniquely determined by the solution of a variational problem. Indeed, $\mathbf{v}$ can be recovered by the formula (2.28), where $\bar{\omega}(\cdot)$ is the minimizer of an elastic deformation energy, subject to the unilateral constraints posed by the obstacle Ω.

The remainder of the paper is organized as follows. In Section 2 we review the model equations and all the main definitions and assumptions. We then recall the existence theorem proved in [1], and state the main results of the paper; namely, the uniqueness and characterization of solutions, stated in Theorems 2 and 3, respectively. Section 3 contains some preliminary lemmas, on the existence of rotation vectors transforming one curve into another one. The uniqueness of solutions is proved in Section 4, while the representation formula (2.28) is proved in Section 5.

For the general theory of optimal control, also in the presence of state constraints, we refer to [2, 8]. A description of plant development from a biological point of view can be found in [5].

2 Statement of the main results

We start with a brief review of the model considered in [1].

At each time t, the position of the stem is described by a map $s \mapsto \gamma(t, s)$ from $[0, t]$ into $\mathbb{R}^3$. Clearly, the domain of this map grows with time. It is convenient to reformulate the model as an evolution problem on a functional space independent of t. For this purpose, we fix $T > t_0$ and consider the Hilbert-Sobolev space $H^2([0, T]; \mathbb{R}^3)$. Any function $\gamma(t, \cdot) \in H^2([0, t]; \mathbb{R}^3)$ will be canonically extended to $H^2([0, T]; \mathbb{R}^3)$ by setting (see Fig. 1, right)

$$\gamma(t, s) \doteq \gamma(t, t) + (s - t)\gamma_s(t, t) \qquad \text{for} \quad s \in [t, T]. \tag{2.1}$$

Notice that the above extension is well defined because $\gamma(t, \cdot)$ and $\gamma_s(t, \cdot)$ are continuous functions. Throughout the following, we shall study functions defined on a domain of the form

$$\mathcal{D}_T \doteq \big\{(t, s)\,; \quad 0 \leq s \leq t, \quad t \in [t_0, T]\big\}, \tag{2.2}$$

and extended to the rectangle $[t_0, T] \times [0, T]$ as in (2.1). In particular, the partial derivative $\gamma_s(t, s)$ will be constant for $s \in [t, T]$.

Adopting the notation $a \wedge b \doteq \min\{a, b\}$, we consider an evolution problem on the space $H^2([0, T]; \mathbb{R}^3)$, having the form

$$\gamma_t(t, s) = \int_0^{s \wedge t} \Psi\Big(t, \sigma, \gamma(t, \sigma), \gamma_s(t, \sigma)\Big) \times \Big(\gamma(t, s) - \gamma(t, \sigma)\Big) d\sigma + \mathbf{v}(t, s). \tag{2.3}$$

Here $s \in [0, T]$, $\Psi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ is a smooth function, and $\mathbf{v}(t, \cdot)$ is an admissible velocity field produced by the constraint reaction. More precisely, let $\Omega \subset \mathbb{R}^3$ be an open set with $\mathcal{C}^2$ boundary. Given the configuration $\gamma(t, \cdot)$ of the stem at time t, let

$$\chi(t) \doteq \Big\{s \in [0, t]\,; \quad \gamma(t, s) \in \partial\Omega\Big\} \tag{2.4}$$

be the set where the stem touches the obstacle. For $s \in \chi(t)$, let $\mathbf{n}(t, s)$ be the unit outer normal to the boundary $\partial\Omega$ at the point $\gamma(t, s)$. The **cone of admissible velocities** produced by the obstacle reaction is defined to be the set of velocity fields

$$\Gamma(t) \doteq \Big\{ \mathbf{v} : [0,T] \mapsto \mathbb{R}^3 ; \text{ there exists a positive measure } \mu, \text{ supported on the coincidence set } \chi(t) \text{ in (2.4), such that for every } s \in [0,T] \text{ one has}$$

$$\mathbf{v}(s) = -\int_0^s e^{-\beta(t-\sigma)} \left(\int_{[\sigma,t]} \Big(\mathbf{n}(t,s') \times \big(\gamma(t,s') - \gamma(t,\sigma)\big)\Big) d\mu(s') \right) \times \big(\gamma(t,s) - \gamma(t,\sigma)\big)\, d\sigma \Big\}. \tag{2.5}$$

Here and in the sequel, $\mathbf{n}(t,s')$ denotes the unit outer normal vector to the set Ω at the boundary point $\gamma(t,s') \in \partial\Omega$.

Remark 1. As in [1], the definition of the cone $\Gamma(t)$ in (2.5) is motivated by the following considerations. At any point $P = \gamma(t,s') \in \chi(t)$ where the stem touches the obstacle, an outward pointing force acting on the stem at P can produce an infinitesimal deformation described by

$$\gamma^\varepsilon(t,s) = \gamma(t,s) + \varepsilon \mathbf{v}(s),$$

with

$$\mathbf{v}(s) = \int_0^s \omega(\sigma) \times \big(\gamma(t,s) - \gamma(t,\sigma)\big)\, d\sigma\,. \tag{2.6}$$

Here $\omega(\sigma)$ describes the infinitesimal bending of the stem at the point $\gamma(t,\sigma)$. The elastic energy of the corresponding deformation can be described as

$$\mathcal{E}(\omega) = \frac{1}{2}\int_0^t e^{\beta(t-\sigma)} |\omega(\sigma)|^2\, d\sigma. \tag{2.7}$$

Notice that the weight $e^{\beta(t-\sigma)}$ accounts for the fact that older cells are stiffer, and offer more resistance to bending. It is natural to choose ω in order to minimize the total energy $\mathcal{E}$, subject to a linear constraint of the form

$$\mathbf{n}(t,s') \cdot \mathbf{v}(s') = c_0$$

for some $c_0 > 0$. Necessary conditions for optimality yield the representation

$$\omega(\sigma) = \begin{cases} -\lambda e^{-\beta(t-\sigma)} \mathbf{n}(t,s') \times \big(\gamma(t,s') - \gamma(t,\sigma)\big) & \text{for } 0 \le \sigma \le s', \\ 0 & \text{for } s' < \sigma \le t, \end{cases} \tag{2.8}$$

for some Lagrange multiplier $\lambda > 0$. Inserting (2.8) in (2.6) and integrating over the set $\chi(t)$ where the stem touches the obstacle, one formally obtains (2.5).

We remark that, in (2.7), the factor $e^{\beta(t-\sigma)}$ could be replaced more generally by any smooth, strictly positive function $\phi(t,\sigma)$. However, since all the models proposed in [1] contain this exponential factor, we choose to keep it in the same form also in the present analysis.

The equation (2.3) will be solved on a domain of the form

$$\mathcal{D} \doteq \big\{(t,s);\ \ t \in [t_0,T],\ \ s \in [0,T]\big\}, \tag{2.9}$$

with initial and boundary conditions

$$\gamma(t_0,s) = \overline{\gamma}(s), \qquad s \in [0,t_0], \tag{2.10}$$

$$\gamma_{ss}(t,s) \;=\; 0, \qquad\qquad t\in[t_0,T],\quad s\in]t,T]\,, \tag{2.11}$$

and the constraint

$$\gamma(t,s)\;\notin\;\Omega \qquad \text{for all}\quad t\in[t_0,T],\quad s\in[0,t]\,. \tag{2.12}$$

Differentiating w.r.t. s, one obtains an equivalent evolution equation for the unit tangent vector $\mathbf{k}$, namely

$$\mathbf{k}_t(t,s) \;=\; \left(\int_0^{s\wedge t}\Psi\Big(t,\sigma,\gamma(t,\sigma),\gamma_s(t,\sigma)\Big)d\sigma\right)\times\mathbf{k}(t,s)+\mathbf{h}(t,s). \tag{2.13}$$

Here $\mathbf{h}(t,\cdot)$ is any element of the cone

$$\begin{aligned}\Gamma'(t) \;&\doteq\; \Bigg\{\mathbf{h}:[0,t]\mapsto\mathbb{R}^3\,;\;\; \text{there exists a positive measure } \mu \text{ supported on } \chi(t) \text{ such that}\\ \mathbf{h}(s) \;&=\; -\int_0^s\left(\int_{[\sigma,t]} e^{-\beta(t-\sigma)}\mathbf{n}(t,s')\times\big(\gamma(t,s')-\gamma(t,\sigma)\big)\,d\mu(s')\right)d\sigma\times\mathbf{k}(t,s)\Bigg\}.\end{aligned} \tag{2.14}$$

The equation (2.13) should be solved on the domain $\mathcal{D}$ in (2.9), with initial and boundary conditions

$$\mathbf{k}(t_0,s) \;=\; \overline{\mathbf{k}}(s) \;=\; \overline{\gamma}_s(s), \qquad\qquad s\in[0,t_0]\,, \tag{2.15}$$

$$\mathbf{k}_s(t,s) \;=\; 0, \qquad\qquad t\in[t_0,T],\quad s\in[t,T]\,, \tag{2.16}$$

together with the state constraint (2.12). Notice that the right hand side of (2.13) is always perpendicular to the tangent vector $\mathbf{k}(t,s)\doteq\gamma_s(t,s)$. As a consequence, the identities

$$|\mathbf{k}(t,s)| \;=\; |\gamma_s(t,s)| \;=\; 1$$

remain always valid, provided they hold at the initial time $t=t_0$.

Definition 1. *We say that a function $\gamma=\gamma(t,s)$, defined for $(t,s)\in[t_0,T]\times[0,T]$ is a solution to the equation (2.3)-(2.5) with initial and boundary conditions (2.10)–(2.12) if the following holds.*

(i) The map $t\mapsto\gamma(t,\cdot)$ is Lipschitz continuous from $[t_0,T]$ into $H^2([0,T];\mathbb{R}^3)$.

(ii) For every t,s one has

$$\begin{aligned}\gamma(t,s) \;&=\; \gamma(t_0,s)+\int_{t_0}^t\int_0^{s\wedge t}\Psi\Big(\tau,\sigma,\gamma(\tau,\sigma),\gamma_s(\tau,\sigma)\Big)\times\Big(\gamma(\tau,s)-\gamma(\tau,\sigma)\Big)d\sigma\,d\tau\\ &\qquad+\int_0^t\mathbf{v}(\tau,s)\,d\tau\,,\end{aligned} \tag{2.17}$$

where each $\mathbf{v}(\tau,\cdot)$ is an element of the cone $\Gamma(\tau)$ defined as in (2.5).

(iii) The initial conditions hold:

$$\gamma(t_0,s) \;=\; \begin{cases}\overline{\gamma}(s) & \textit{if}\;\; s\in[0,t_0],\\ \overline{\gamma}(t_0)+(s-t_0)\overline{\gamma}'(t_0) & \textit{if}\;\; s\in[t_0,T].\end{cases} \tag{2.18}$$

(iv) The pointwise constraints hold:

$$\gamma(t,s) \ \notin \ \Omega \qquad \textit{for all} \ \ t\in[t_0,T], \quad s\in[0,t]. \tag{2.19}$$

$$\gamma(t,s) \ = \ \gamma(t,t)+(s-t)\gamma_s(t,t) \qquad \textit{for all} \ \ t\in[t_0,T], \quad s\in[t,T]. \tag{2.20}$$

Notice that the conditions (2.18) and (2.20) imply that (2.10)-(2.11) are satisfied. Given an initial data $\gamma(t_0,s)=\overline{\gamma}(s)$, the result in [1] provides the existence of a solution as long as the following breakdown configuration is not attained (see Fig. 3).

(B) *The tip of the stem touches the obstacle perpendicularly, namely*

$$\overline{\gamma}(t_0) \ \in \ \partial\Omega\,, \qquad \overline{\gamma}_s(t_0) \ = \ -\,\mathbf{n}(\overline{\gamma}(t_0)). \tag{2.21}$$

Moreover,

$$\overline{\gamma}_{ss}(s) \ = 0 \qquad \textit{for all } s\in \ ]0,t[\ \textit{ such that } \overline{\gamma}(s)\notin\partial\Omega\,. \tag{2.22}$$

Here $\mathbf{n}(x)$ denotes the unit outer normal to Ω at a boundary point $x\in\partial\Omega$.

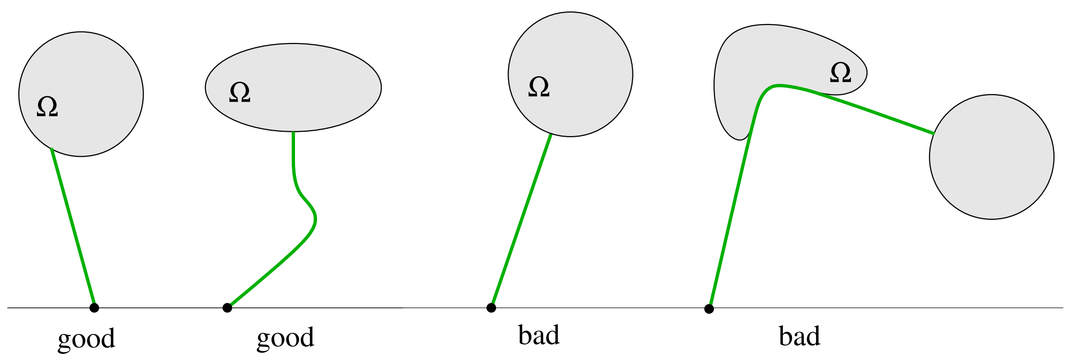

Figure 3: For the two initial configurations on the left, the constrained growth equation (2.3) admits a unique solution. On the other hand, the two configurations on the right satisfy both (2.21) and (2.22) in **(B)**. In such cases, the Cauchy problem is ill posed.

Theorem 1. *Let Ψ in (2.3) be a $\mathcal{C}^2$ function, and let $\Omega\subset\mathbb{R}^3$ be a bounded open set with $\mathcal{C}^2$ boundary. At time t_0, consider the initial data (2.10), where the curve $s\mapsto\overline{\gamma}(s)$ is in $H^2([0,t_0];\mathbb{R}^3)$ and satisfies*

$$\overline{\gamma}(0)=0\notin\partial\Omega, \qquad \overline{\gamma}(s)\notin\Omega \qquad \textit{for all} \ \ s\in[0,t_0]. \tag{2.23}$$

Moreover, assume that the condition **(B)** *does NOT hold.*

Then there exists $T>t_0$ such that the equations (2.3)-(2.5) with initial and boundary conditions (2.10)–(2.12) admit at least one solution for $t\in[t_0,T]$.

Either (i) the solution is globally defined for all times $t\geq t_0$, or else (ii) the solution can be extended to a maximal time interval $[0,T]$, where $\gamma(T,\cdot)$ satisfies all conditions in **(B)**.

In the present paper we prove that the above solution is unique. Moreover, for a.e. time t the velocity $\mathbf{v}(t,\cdot)$ determined by the constraint reaction can be computed as follows. Using the shorter notation $\Psi(\sigma) = \Psi\Big(t, \sigma, \gamma(t, \sigma), \gamma_s(t, \sigma)\Big)$ and $\mathbf{n}(t, s) = \mathbf{n}(\gamma(t, s))$ whenever $\gamma(t, s) \in \partial\Omega$, consider the minimization problem

$$\text{minimize:} \quad \mathcal{E}(\omega) \doteq \frac{1}{2}\int_0^t e^{\beta(t-\sigma)}|\omega(\sigma)|^2\, d\sigma, \tag{2.24}$$

subject to the unilateral constraint

$$\left\langle \int_0^s \big(\Psi(\sigma) + \omega(\sigma)\big) \times \big(\gamma(t, s) - \gamma(t, \sigma)\big)\, d\sigma\,,\ \mathbf{n}(t, s)\right\rangle \geq 0 \qquad \text{for all} \quad s \in \chi(t). \tag{2.25}$$

If the tip of the stem touches the obstacle, then we also impose that it does not penetrate, namely

$$\left\langle \gamma_s(t, t) + \int_0^t \big(\Psi(\sigma) + \omega(\sigma)\big) \times \big(\gamma(t, t) - \gamma(t, \sigma)\big)\, d\sigma\,,\ \mathbf{n}(t, t)\right\rangle \geq 0. \tag{2.26}$$

We will show that, at a.e. time t, the evolution equation (2.3) is satisfied with

$$\gamma_t(t, s) = \int_0^s \Big(\Psi(\sigma) + \bar{\omega}(t, \sigma)\Big) \times \Big(\gamma(t, s) - \gamma(t, \sigma)\Big) d\sigma, \tag{2.27}$$

where $\bar{\omega}(\cdot)$ is the unique minimizer for (2.24)–(2.26). In other words, for a.e. time t, among all possible choices of $\mathbf{v} \in \Gamma(t)$, the equation (2.3) is satisfied precisely with

$$\mathbf{v}(t, s) = \int_0^s \bar{\omega}(t, \sigma) \times \big(\gamma(t, s) - \gamma(t, \sigma)\big)\, d\sigma\,. \tag{2.28}$$

Theorem 2 (uniqueness). *In the same setting as Theorem 1, the solution to the evolution equation (2.3)-(2.5) with initial and boundary conditions (2.10)–(2.12) is unique.*

Theorem 3 (representation of solutions). *For a.e. $t \in [0, T]$ the time derivative γ_t of the solution constructed in Theorem 1 is given by (2.27), where $\bar{\omega}(t, \cdot)$ is the unique minimizer of (2.24), subject to (2.25)-(2.26).*

3 Preliminary lemmas

In the following, given a vector $\mathbf{w} = (w_1, w_2, w_3)^T$, we shall denote by $R[\mathbf{w}]$ the 3×3 rotation matrix

$$R[\mathbf{w}] \doteq e^A \doteq \sum_{k=0}^{\infty} \frac{A^k}{k!}, \qquad A \doteq \begin{pmatrix} 0 & -w_3 & -w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}. \tag{3.1}$$

Notice that, for every $\bar{\mathbf{v}} \in \mathbb{R}^3$, the image $R[\mathbf{w}]\bar{\mathbf{v}}$ is the value at time $t = 1$ of the solution to

$$\dot{\mathbf{v}}(t) = \mathbf{w} \times \mathbf{v}(t), \qquad \mathbf{v}(0) = \bar{\mathbf{v}}.$$

Next, consider two time-dependent unit vectors: $\mathbf{k}_1(t), \mathbf{k}_2(t)$. We seek rotation vectors $\mathbf{w}(t)$ such that

$$\mathbf{k}_2(t) = R[\mathbf{w}(t)]\,\mathbf{k}_1(t) \qquad \text{for all} \quad t \geq 0. \tag{3.2}$$

In particular, we seek an equation relating the time derivatives $\mathbf{w}_t$ and $\mathbf{k}_{i,t}$, $i = 1, 2$. Differentiating (3.2) w.r.t. time, one obtains

$$\mathbf{k}_{2,t}(t) \;=\; \left(\frac{d}{d\mathbf{w}} R[\mathbf{w}(t)]\mathbf{w}_t\right) \mathbf{k}_1(t) + R[\mathbf{w}(t)]\, \mathbf{k}_{1,t}(t). \tag{3.3}$$

Assume that

$$\mathbf{k}_{i,t}(t) \;=\; \omega_i(t) \times \mathbf{k}_i(t), \qquad\qquad i = 1, 2, \tag{3.4}$$

for some angular velocities ω_1, ω_2. At a time τ where $\mathbf{w}(\tau) = 0$, and hence $R[\mathbf{w}(\tau)] = I$ is the identity matrix, (3.3) reduces to

$$\omega_2(\tau) \times \mathbf{k}_2(\tau) \;=\; \mathbf{w}_t(\tau) \times \mathbf{k}_1(\tau) + \omega_1(\tau) \times \mathbf{k}_1(\tau). \tag{3.5}$$

Hence, since $\mathbf{k}_2(\tau) = \mathbf{k}_1(\tau)$, one has $\mathbf{w}_t = \omega_2 - \omega_1$. We now study the more general case where $\mathbf{w}(\tau)$ is small but nonzero.

Lemma 1. *Assume that the unit vectors $\mathbf{k}_1(\cdot), \mathbf{k}_2(\cdot)$ satisfy (3.4) for some continuous angular velocities $\omega_i(\cdot)$. Moreover, assume that, at some time τ, one has*

$$\mathbf{k}_2(\tau) \;=\; R[\mathbf{w}(\tau)]\, \mathbf{k}_1(\tau),$$

with $|\mathbf{w}(\tau)| < \delta$ sufficiently small. Then there exists $T > \tau$, a constant C, and an absolutely continuous map $t \mapsto \mathbf{w}(t)$ such that (3.2) holds for all $t \in [\tau, T]$, and moreover

$$\left|\frac{d}{dt}\mathbf{w}(t) - (\omega_2(t) - \omega_1(t))\right| \;\leq\; C \cdot \Big(|\omega_1(t)| + |\omega_2(t)|\Big)\, |\mathbf{w}(t)|. \tag{3.6}$$

Proof. For a fixed τ, choose two additional vectors $\mathbf{v}_1, \mathbf{v}_2$ so that $\{\mathbf{k}_2(\tau), \mathbf{v}_1, \mathbf{v}_2\}$ is a (positively oriented) orthonormal basis of $\mathbb{R}^3$. Consider the function[1]

$$\begin{aligned}
F(c_1, c_2) \;&\doteq\; \frac{d}{dt}\left\{ R\Big[\mathbf{w}(\tau) + (t-\tau)\big(\omega_2(\tau) - \omega_1(\tau) + c_1\mathbf{v}_1 + c_2\mathbf{v}_2\big)\Big]\mathbf{k}_1(t) - \mathbf{k}_2(t)\right\}_{t=\tau} \\
&=\; \int_0^1 R\big[(1-\xi)\mathbf{w}(\tau)\big]\Big(\big(\omega_2(\tau) - \omega_1(\tau) + c_1\mathbf{v}_1 + c_2\mathbf{v}_2\big) \times R\big[\xi\mathbf{w}(\tau)\big]\mathbf{k}_1(\tau)\Big)\, d\xi \\
&\qquad + R[\mathbf{w}(\tau)]\,\big(\omega_1(\tau) \times \mathbf{k}_1(\tau)\big) - \omega_2(\tau) \times \mathbf{k}_2(\tau) \\
&=\; \int_0^1 \big(R\big[(1-\xi)\mathbf{w}(\tau)\big](\omega_2(\tau) - \omega_1(\tau) + c_1\mathbf{v}_1 + c_2\mathbf{v}_2)\big)d\xi \times \mathbf{k}_2(\tau) \\
&\qquad + \big(R\big[\mathbf{w}(\tau)\big]\omega_1(\tau) - \omega_2(\tau)\big) \times \mathbf{k}_2(\tau).
\end{aligned} \tag{3.7}$$

Notice that the vector $F(c_1, c_2)$ is always perpendicular to $\mathbf{k}_2(\tau)$. Hence the vector equation

$$F(c_1, c_2) \;=\; 0 \in \mathbb{R}^3 \tag{3.8}$$

is equivalent to the system of two scalar equations

$$F_1(c_1, c_2) \;=\; \mathbf{v}_1 \cdot F(c_1, c_2) \;=\; 0, \qquad\qquad F_2(c_1, c_2) \;=\; \mathbf{v}_2 \cdot F(c_1, c_2) \;=\; 0, \tag{3.9}$$

[1] To differentiate the exponential matrix $R[\cdot]$, we use the formula $\frac{d}{d\epsilon}e^{A+\epsilon B}\big|_{\epsilon=0} \;=\; \int_0^1 e^{(1-\xi)A} B\, e^{\xi A}\, d\xi$.

where the dot indicates a scalar product. The partial derivatives of the map $(c_1, c_2) \mapsto (F_1, F_2)$ are computed by

$$
\begin{aligned}
\frac{\partial F_i}{\partial c_j} &= \frac{\partial}{\partial c_j}\left(\mathbf{v}_i \cdot \int_0^1 \big(R\big[\xi \mathbf{w}(\tau)\big](\omega_2(\tau) - \omega_1(\tau) + c_1\mathbf{v}_1 + c_2\mathbf{v}_2)\big)\, d\xi \times \mathbf{k}_2(\tau)\right) \\
&= \mathbf{v}_i \cdot \int_0^1 \Big(R\big[\xi \mathbf{w}(\tau)\big]\mathbf{v}_j\Big) d\xi \ \times \mathbf{k}_2(\tau) \ = \ (\mathbf{v}_j \times \mathbf{k}_2(\tau)) \cdot \mathbf{v}_i + \mathcal{O}(1) \cdot |\mathbf{w}(\tau)|\,.
\end{aligned}
\tag{3.10}
$$

Hence the Jacobian matrix is

$$
\left(\frac{\partial F_i}{\partial c_j}\right)_{i,j=1,2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mathcal{O}(1) \cdot |\mathbf{w}(\tau)|\,. \tag{3.11}
$$

As usual, here the Landau symbol $\mathcal{O}(1)$ denotes a uniformly bounded quantity. In particular, for $|\mathbf{w}(\tau)|$ sufficiently small this Jacobian matrix is invertible.

We now observe that the right hand side of (3.7) is linear w.r.t. the vectors $\omega_1(\tau), \omega_2(\tau)$. Moreover:

(i) When $\mathbf{w}(\tau) = 0$ we have $\mathbf{k}_2(\tau) = \mathbf{k}_1(\tau)$ and $R[\mathbf{w}(\tau)] = I$. In this case, for arbitrary $\omega_1(\tau), \omega_2(\tau) \in \mathbb{R}^3$, the equation (3.8) is satisfied by taking $c_1 = c_2 = 0$.

(ii) When $\omega_1(\tau) = \omega_2(\tau) = 0 \in \mathbb{R}^3$, for an arbitrary $\mathbf{w}(\tau)$ the equation (3.8) is again satisfied by taking $c_1 = c_2 = 0$.

By an application of the implicit function theorem, we obtain the existence of a unique vector, say

$$
\omega^\sharp(\tau) \ = \ c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \ = \ \Phi\big(\mathbf{k}_2(t), \mathbf{w}(t), \omega_1(t), \omega_2(t)\big), \tag{3.12}
$$

satisfying

$$
\omega^\sharp(\tau) \ \in \ \mathbf{k}_2(\tau)^\perp, \tag{3.13}
$$

$$
\begin{aligned}
&\int_0^1 \big(R\big[(1-\xi)\mathbf{w}(\tau)\big](\omega_2(\tau) - \omega_1(\tau) + \omega^\sharp(\tau))\big) d\xi \times \mathbf{k}_2(\tau) \\
&\qquad + \Big(R\big[\mathbf{w}(\tau)\big]\omega_1(\tau) - \omega_2(\tau)\Big) \times \mathbf{k}_2(\tau) \ = \ 0.
\end{aligned}
\tag{3.14}
$$

The above identities (i)-(ii) imply

$$
|\omega^\sharp(\tau)| \ = \ \mathcal{O}(1) \cdot \Big(|\omega_1(\tau)| + |\omega_2(\tau)|\Big)\, |\mathbf{w}(\tau)|. \tag{3.15}
$$

By the continuity of the angular velocities ω_1, ω_2, the above construction can be repeated for every $t \in [\tau, T]$, as long as the rotation vector $\mathbf{w}(t)$ remains sufficiently small. This yields an evolution equation for $\mathbf{w}$, of the form

$$
\frac{d}{dt}\mathbf{w}(t) \ = \ \omega_2(t) - \omega_1(t) + \Phi\big(\mathbf{k}_2(t), \mathbf{w}(t), \omega_1(t), \omega_2(t)\big), \tag{3.16}
$$

where Φ is the function implicitly defined in (3.12), providing the unique solution to (3.13)-(3.14). By the regularity of Φ, given the functions $\omega_1(\cdot)$, $\omega_2(\cdot)$, $\mathbf{k}_2(\cdot)$ and the initial condition $\mathbf{w}(\tau)$, the evolution equation (3.16) has a unique local solution, defined as long as the vector $\mathbf{w}$ remains small enough. This completes the proof of the lemma. □

Toward a proof of Theorem 2 we need an integral version of Lemma 1. As before, we consider two curves, growing in time: $\gamma_i(t,s)$, $s\in[0,t]$. We denote by $\mathbf{k}_i(t,s)=\gamma_{i,s}(t,s)$ the unit tangent vectors.

Lemma 2. *Assume that, for $i=1,2$,*

$$\mathbf{k}_{i,t}(t,s) \;=\; \left(\int_0^s \omega_i(t,\sigma)\,d\sigma\right)\times \mathbf{k}_i(t,s), \qquad\qquad s\in[0,t]. \tag{3.17}$$

Moreover, assume that at time τ one has

$$\mathbf{k}_2(\tau,s) \;=\; R\left[\int_0^s \mathbf{w}(\tau,\sigma)d\sigma\right]\mathbf{k}_1(\tau,s), \tag{3.18}$$

with $\|\mathbf{w}(\tau,\cdot)\|_{\mathbf{L}^2([0,\tau])}\le\delta$ sufficiently small. Then there exists $T>\tau$ such that, for all $t\in[\tau,T]$ one has the representation

$$\mathbf{k}_2(t,s) \;=\; R\left[\int_0^s \mathbf{w}(t,\sigma)d\sigma\right]\mathbf{k}_1(t,s), \qquad\qquad s\in[0,t]. \tag{3.19}$$

Here the rotation vectors $\mathbf{w}(t,\cdot)$ can be chosen so that

$$\left|\int_0^s \mathbf{w}_t(t,\sigma)-\omega_2(t,\sigma)+\omega_1(t,\sigma)d\sigma\right| \;=\; \mathcal{O}(1)\cdot\left(\left|\int_0^s |\omega_1(t,\sigma)d\sigma\right|+\left|\int_0^s \omega_2(t,\sigma)d\sigma\right|\right)\|\mathbf{w}(t,\cdot)\|_{\mathbf{L}^1([0,t])} \tag{3.20}$$

for $s\in[0,t]$.

Proof. We repeat the construction of Lemma 1. For every $s\in[0,\tau]$ we have

$$|\mathbf{k}_1(\tau,s)| \;=\; |\mathbf{k}_2(\tau,s)| \;=\; 1, \qquad\qquad \big|\mathbf{k}_2(\tau,s)-\mathbf{k}_1(\tau,s)\big| \;=\; \mathcal{O}(1)\cdot\|\mathbf{w}(\tau,\cdot)\|_{\mathbf{L}^1([0,\tau])}\,.$$

For each $s\in[0,\tau]$, choose unit vectors $\mathbf{v}_1(s),\mathbf{v}_2(s)$ so that $\{\mathbf{k}_2(\tau,s),\mathbf{v}_1(s),\mathbf{v}_2(s)\}$ is a (positively oriented) orthonormal basis of $\mathbb{R}^3$. Notice that $s\mapsto\mathbf{v}_1(s),\mathbf{v}_2(s)$ are in $H^1([0,\tau])$.

Given two scalar functions $c_1(s)$, $c_2(s)$, for each $s\in[0,\tau]$ define

$$\begin{aligned}
&F(s,c_1(s),c_2(s))\\
&\doteq\; \frac{d}{dt}\left\{R\left[\int_0^s \mathbf{w}(\tau,\sigma)+(t-\tau)\big(\omega_2(\tau,\sigma)-\omega_1(\tau,\sigma)+c_1(\sigma)\mathbf{v}_1(\sigma)+c_2(\sigma)\mathbf{v}_2(\sigma)\big)\,d\sigma\right]\mathbf{k}_1(t,s)-\mathbf{k}_2(t,s)\right\}_{t=\tau}\\
&=\; \int_0^1 R\left[(1-\xi)\int_0^s \mathbf{w}(\tau,\sigma)\,d\sigma\right]\left(\int_0^s \big(\omega_2(\tau,\sigma)-\omega_1(\tau,\sigma)+c_1(\sigma)\mathbf{v}_1(\sigma)+c_2(\sigma)\mathbf{v}_2(\sigma)\big)d\sigma\right)\\
&\qquad\qquad \times R\left[\xi\int_0^s \mathbf{w}(\tau,\sigma)\,d\sigma\right]\mathbf{k}_1(\tau,s)\,d\xi\\
&\qquad\qquad +R\left[\int_0^s \mathbf{w}(\tau,\sigma)\,d\sigma\right]\left(\int_0^s \omega_1(\tau,\sigma)\,d\sigma\right)\times\mathbf{k}_1(\tau,s)-\left(\int_0^s \omega_2(\tau,\sigma)\,d\sigma\right)\times\mathbf{k}_2(\tau,s)\\
&=\; \int_0^1 R\left[\xi\int_0^s \mathbf{w}(\tau,\sigma)\,d\sigma\right]\left(\int_0^s \big(\omega_2(\tau,\sigma)-\omega_1(\tau,\sigma)+c_1(\sigma)\mathbf{v}_1(\sigma)+c_2(\sigma)\mathbf{v}_2(\sigma)\big)d\sigma\right)d\xi\times\mathbf{k}_2(\tau,s)\\
&\qquad\qquad +\left(R\left[\int_0^s \mathbf{w}(\tau,\sigma)\,d\sigma\right]\int_0^s \omega_1(\tau,\sigma)\,d\sigma\right)\times\mathbf{k}_2(\tau,s)-\int_0^s \omega_2(\tau,\sigma)\,d\sigma\times\mathbf{k}_2(\tau,s).
\end{aligned} \tag{3.21}$$

Notice that the vector $F(s,c_1(s),c_2(s))$ is always perpendicular to $\mathbf{k}_2(\tau,s)$. Hence the vector equation

$$F(s,c_1(s),c_2(s)) \;=\; 0 \in \mathbb{R}^3 \tag{3.22}$$

is equivalent to the system of two scalar equations

$$\begin{cases} F_1(s,c_1(s),c_2(s)) \;\doteq\; \mathbf{v}_1(s)\cdot F(s,c_1(s),c_2(s)) \;=\; 0, \\ F_2(s,c_1(s),c_2(s)) \;\doteq\; \mathbf{v}_2(s)\cdot F(s,c_1(s),c_2(s)) \;=\; 0. \end{cases} \tag{3.23}$$

These should hold for all $s \in [0,\tau]$.

For $s = 0$ the equations (3.23) are trivially satisfied. Hence it suffices to solve the equations for the derivatives:

$$\frac{d}{ds}F_i(s,c_1(s),c_2(s)) \;=\; 0, \qquad\qquad i=1,2, \qquad s\in[0,\tau]. \tag{3.24}$$

Notice also that

$$\mathbf{v}_1\cdot(\mathbf{a}\times\mathbf{k}_2) = \mathbf{a}\cdot(\mathbf{k}_2\times\mathbf{v}_1) = \mathbf{v}_2\cdot\mathbf{a},$$
$$\mathbf{v}_2\cdot(\mathbf{a}\times\mathbf{k}_2) = \mathbf{a}\cdot(\mathbf{k}_2\times\mathbf{v}_2) = -\mathbf{v}_1\cdot\mathbf{a},$$

and that, in view of (3.22),

$$\frac{d}{ds}F_i(s,c_1(s),c_2(s)) = \Big(\frac{d}{ds}F(s,c_1(s),c_2(s))\Big)\cdot\mathbf{v}_i(s). \tag{3.25}$$

Taking these observations into account, we then compute:

$$\begin{aligned}
&\frac{d}{ds}F_1(s,c_1(s),c_2(s)) \\
&= \int_0^1\int_0^1 \Big(\alpha\,\xi\,R\Big[\int_0^s \mathbf{w}(\tau,\sigma)d\sigma\Big]\cdot\mathbf{w}(\tau,s)\Big) \\
&\qquad\quad \times\Big(\int_0^s(\omega_2(\tau,\sigma)-\omega_1(\tau,\sigma)+c_1(\sigma)\mathbf{v}_1(\sigma)+c_2(\sigma)\mathbf{v}_2(\sigma))d\sigma\Big)d\alpha\,d\xi\cdot\mathbf{v}_2(s) \\
&\quad + \int_0^1 R\Big[\xi\int_0^s\mathbf{w}(\tau,\sigma)d\sigma\Big](\omega_2(\tau,s)-\omega_1(\tau,s)+c_1(s)\mathbf{v}_1(s)+c_2(s)\mathbf{v}_2(s))\,d\xi\cdot\mathbf{v}_2(s) \\
&\quad + \int_0^1\Big(R\Big[\xi\int_0^s\mathbf{w}(\tau,\sigma)d\sigma\Big]\cdot\mathbf{w}(\tau,s)\,\times\Big(\int_0^s\omega_1(\tau,\sigma)\,d\sigma\Big)\Big)d\xi\cdot\mathbf{v}_2(s) \\
&\quad + \Big(R\Big[\int_0^s\mathbf{w}(\tau,\sigma)d\sigma\Big]\cdot\omega_1(\tau,s)\Big)\cdot\mathbf{v}_2(s) - \omega_2(\tau,s)\cdot\mathbf{v}_2(s) \;=\; 0.
\end{aligned} \tag{3.26}$$

A similar relation holds true for $\frac{d}{ds}F_2(s,c_1(s),c_2(s))$. Notice that both relations together lead to a system of equations

$$
\begin{aligned}
c_1(s) \;=\;& -\int_0^1\int_0^1 \Big(\alpha\,\xi\, R\Big[\int_0^s \mathbf{w}(\tau,\sigma)d\sigma\Big]\cdot \mathbf{w}(\tau,s)\Big) \\
&\quad\times \Big(\int_0^s (\omega_2(\tau,\sigma)-\omega_1(\tau,\sigma)+c_1(\sigma)\mathbf{v}_1(\sigma)+c_2(\sigma)\mathbf{v}_2(\sigma))d\sigma\Big)d\alpha\, d\xi\cdot \mathbf{v}_1(s) \\
&+\Big(\mathcal{O}(1)\cdot\Big|\int_0^s \mathbf{w}(\tau,\sigma)d\sigma\Big|(\omega_2(\tau,s)-\omega_1(\tau,s)+c_1(s)\mathbf{v}_1(s)+c_2(s)\mathbf{v}_2(s))\Big)\cdot \mathbf{v}_1(s) \\
&+\Psi_1(s,\omega_1,\omega_2,\mathbf{w},\mathbf{v}_1) \;\doteq\; P_1(\mathbf{c})
\end{aligned} \tag{3.27}
$$

$$
\begin{aligned}
c_2(s) \;=\;& -\int_0^1\int_0^1 \Big(\alpha\,\xi\, R\Big[\int_0^s \mathbf{w}(\tau,\sigma)d\sigma\Big]\cdot \mathbf{w}(\tau,s)\Big) \\
&\quad\times \Big(\int_0^s (\omega_2(\tau,\sigma)-\omega_1(\tau,\sigma)+c_1(\sigma)\mathbf{v}_1(\sigma)+c_2(\sigma)\mathbf{v}_2(\sigma))d\sigma\Big)d\alpha\, d\xi\cdot \mathbf{v}_2(s) \\
&+\Big(\mathcal{O}(1)\cdot\Big|\int_0^s \mathbf{w}(\tau,\sigma)d\sigma\Big|(\omega_2(\tau,s)-\omega_1(\tau,s)+c_1(s)\mathbf{v}_1(s)+c_2(s)\mathbf{v}_2(s))\Big)\cdot \mathbf{v}_2(s) \\
&+\Psi_2(s,\omega_1,\omega_2,\mathbf{w},\mathbf{v}_2) \;\doteq\; P_2(\mathbf{c})
\end{aligned} \tag{3.28}
$$

where Ψ_i for $i=1,2$, are smooth functions which do not depend on $c_i(s)$. Now denote with $\mathbf{c}=(c_1,c_2)$ and consider the operator $\mathcal{P}[\mathbf{c}] \doteq (P_1(\mathbf{c}),P_2(\mathbf{c}))$ such that $\mathbf{c}=\mathcal{P}[\mathbf{c}]$. We now aim to show that the system (3.27), (3.28) admits a unique solution proving that $\mathcal{P}[\cdot]$ is a contraction on $\mathbf{L}^2[0,\tau]$ for δ small enough. Indeed, for any $\mathbf{c},\tilde{\mathbf{c}}\in\mathbf{L}^2[0,\tau]$, one has

$$
\begin{aligned}
\big\|\mathcal{P}[\mathbf{c}]-\mathcal{P}[\tilde{\mathbf{c}}]\big\|_{\mathbf{L}^2[0,\tau]} \;\leq\;& \int_0^\tau \Big|(I+\mathcal{O}(1)\cdot\delta)\,\mathbf{w}(\tau,s)\Big|^2 \\
&\times\Big|\int_0^s (c_1(\sigma)-\tilde{c}_1(\sigma))\mathbf{v}_1(\sigma)+(c_2(\sigma)-\tilde{c}_2(\sigma))\mathbf{v}_2(\sigma)d\sigma\Big|^2 \\
&+K_2\delta\int_0^\tau \Big|(c_1(\sigma)-\tilde{c}_1(\sigma))\mathbf{v}_1(\sigma)+(c_2(\sigma)-\tilde{c}_2(\sigma))\mathbf{v}_2(\sigma)\Big|^2 d\sigma \\
\leq\;& (1+K_1\delta)^2\int_0^\tau |\mathbf{w}(\tau,s)|^2 ds \\
&\times\int_0^\tau |(c_1(\sigma)-\tilde{c}_1(\sigma))\mathbf{v}_1(\sigma)+(c_2(\sigma)-\tilde{c}_2(\sigma))\mathbf{v}_2(\sigma)|^2 d\sigma \\
&+K_2\delta\int_0^\tau \Big|(c_1(\sigma)-\tilde{c}_1(\sigma))\mathbf{v}_1(\sigma)+(c_2(\sigma)-\tilde{c}_2(\sigma))\mathbf{v}_2(\sigma)\Big|^2 d\sigma \\
\leq\;& \delta\Big((1+K_1\delta)^2+K_2\Big)\|\mathbf{c}-\tilde{\mathbf{c}}\|_{\mathbf{L}^2[0,\tau]},
\end{aligned} \tag{3.29}
$$

for some constants K_1,K_2 independent of $\mathbf{c},\tilde{\mathbf{c}}$. When $\delta>0$ is sufficiently small, (3.29) shows that $\mathcal{P}[\cdot]$ is a strict contraction. As a consequence, the system of equations (3.27)-(3.28) admits a unique solution, which we denote by $\big(\bar{c}_1(\cdot),\,\bar{c}_2(\cdot)\big)$. Then $\bar{c}_1$, $\bar{c}_2$ will also satisfy the relations (3.23). Moreover

$$
\int_0^s (\bar{c}_1(\sigma)\mathbf{v}_1(\sigma)+\bar{c}_2(\sigma)\mathbf{v}_2(\sigma))d\sigma \;=\; \Phi\big(\mathbf{k}_2(\tau,s),\mathbf{W}(\tau,s),\Omega_1(\tau,s),\Omega_2(\tau,s)\big), \tag{3.30}
$$

where

$$\mathbf{W}(\tau,s) \;=\; \int_0^s \mathbf{w}(\tau,\sigma)d\sigma, \qquad \Omega_1(\tau,s)=\int_0^s \omega_1(\tau,\sigma)d\sigma, \qquad \Omega_2(\tau,s)=\int_0^s \omega_2(\tau,\sigma)d\sigma, \tag{3.31}$$

and Φ is a smooth function satisfying

$$\Big|\Phi\big(\mathbf{k}_2(\tau,s),\mathbf{W}(\tau,s),\Omega_1(\tau,s),\Omega_2(\tau,s)\big)\Big| \;=\; \mathcal{O}(1)\cdot\big(\|\omega_1(\tau,\cdot)\|_{\mathbf{L}^1}+\|\omega_2(\tau,\cdot)\|_{\mathbf{L}^1}\big)\|\mathbf{w}(\tau,\cdot)\|_{\mathbf{L}^1} \tag{3.32}$$

Again, by the continuity of the integrated angular velocities Ω_1,Ω_2, the above construction can be repeated for every $t\in[\tau,T]$, as long as the rotation vector $\|\mathbf{w}(\tau,\cdot)\|_{\mathbf{L}^2}$ remains sufficiently small. This yields an evolution equation for $\mathbf{W}$, of the form

$$\frac{d}{dt}\mathbf{W}(t,s) \;=\; \Omega_2(t,s)-\Omega_1(t,s)+\Phi\big(\mathbf{k}_2(t,s),\mathbf{W}(t,s),\Omega_1(t,s),\Omega_2(t,s)\big), \qquad s\in[0,t]. \tag{3.33}$$

By the regularity of $\Phi(\cdot,\cdot,\cdot,\cdot)$, given the functions Ω_1, Ω_2, $\mathbf{k}_2$, and the initial condition $\mathbf{W}(\tau,s)$, the evolution equation (3.33) has a unique local solution for every $s\in[0,t]$, defined as long as the vector $\mathbf{W}(t,s)$ remains small enough. This completes the proof of the lemma. □

4 Uniqueness of solutions

Consider two solutions γ_1,γ_2 of (2.3)-(2.5), and call $\mathbf{k}_i(t,s)=\gamma_{i,s}(t,s)$ the corresponding tangent vectors. For each t, we shall construct a rotation vector $\mathbf{w}(t,\cdot)$ such that

$$\mathbf{k}_2(t,s) \;=\; R\left[\int_0^s \mathbf{w}(t,\sigma)\,d\sigma\right]\mathbf{k}_1(t,s), \qquad s\in[0,t]\,. \tag{4.1}$$

To measure the size of this vector $\mathbf{w}$, for any $t>0$ on the space $\mathbf{L}^2([0,t])$ we shall use the equivalent inner product and norm

$$\langle \mathbf{v},\mathbf{w}\rangle \;\doteq\; \int_0^t e^{-\beta s}\langle \mathbf{v}(s),\,\mathbf{w}(s)\rangle\,ds\,, \qquad \|\mathbf{v}\| \;\doteq\; \langle \mathbf{v},\mathbf{v}\rangle^{1/2}. \tag{4.2}$$

Using this equivalent norm, we shall prove the key inequality

$$\Big\langle \mathbf{w}_t(t,\cdot),\,\mathbf{w}(t,\cdot)\Big\rangle \;\leq\; C\cdot\|\mathbf{w}(t,\cdot)\|^2, \tag{4.3}$$

for a suitable constant C. In turn, this yields the estimate

$$\frac{d}{dt}\|\mathbf{w}(t,\cdot)\|^2 \;\leq\; 2C\cdot\|\mathbf{w}(t,\cdot)\|^2\,. \tag{4.4}$$

In particular, if $\mathbf{w}(t_0,\cdot)\equiv 0$, this will imply $\mathbf{w}(t,\cdot)\equiv 0$ for all $t\geq t_0$, proving uniqueness.

Toward a proof of (4.3) we use the representation

$$\mathbf{k}_{i,t}(t,s) \;=\; \left(\int_0^s \omega_i(t,\sigma)\,d\sigma\right)\times\mathbf{k}_i(t,s), \qquad s\in[0,t],\quad i=1,2, \tag{4.5}$$

where the angular velocities ω_i satisfy

$$\begin{aligned}\omega_i(t,s) \;&=\; \Psi\big(t,s,\gamma_i(t,s),\gamma_{i,s}(t,s)\big)\\ &\qquad - \int_0^s \left(\int_{[\sigma,t]} e^{-\beta(t-\sigma)}\mathbf{n}(t,s')\times\big(\gamma_i(t,s')-\gamma_i(t,\sigma)\big)\,d\mu_i(s')\right)d\sigma\,,\end{aligned} \tag{4.6}$$

where μ_i is a positive measure, supported on the contact set $\{s\in[0,t]\,;\;\;\gamma_i(t,s)\in\partial\Omega\}$.

Thanks to Lemma 2, since we know that ω_1,ω_2 are uniformly bounded, we have

$$\big\langle \mathbf{w}_t(t,\cdot)\,,\;\mathbf{w}(t,\cdot)\big\rangle \;=\; \big\langle \omega_2(t,\cdot)-\omega_1(t,\cdot)\,,\;\mathbf{w}(t,\cdot)\big\rangle + \mathcal{O}(1)\cdot\|\mathbf{w}(t,\cdot)\|^2. \tag{4.7}$$

To estimate the first term on the right hand side of (4.7), we write

$$\begin{aligned}&\big\langle \omega_2(t,\cdot)-\omega_1(t,\cdot)\,,\;\mathbf{w}(t,\cdot)\big\rangle\\ &=\;\Big\langle \Psi\big(t,\cdot,\gamma_2(t,\cdot),\gamma_{2,s}(t,\cdot)\big)-\Psi\big(t,\cdot,\gamma_1(t,\cdot),\gamma_{1,s}(t,\cdot)\big)\,,\;\mathbf{w}(t,\cdot)\Big\rangle\\ &\quad+\left\langle \int_0^\cdot \left(\int_{[\sigma,t]} e^{-\beta(t-\sigma)}\mathbf{n}(t,s')\times\big(\gamma_1(t,s')-\gamma_1(t,\sigma)\big)\,d\mu_1(s')\right)d\sigma\,,\;\mathbf{w}(t,\cdot)\right\rangle\\ &\quad-\left\langle \int_0^\cdot \left(\int_{[\sigma,t]} e^{-\beta(t-\sigma)}\mathbf{n}(t,s')\times\big(\gamma_2(t,s')-\gamma_2(t,\sigma)\big)\,d\mu_2(s')\right)d\sigma\,,\;\mathbf{w}(t,\cdot)\right\rangle\\ &\doteq\; J_0+J_1+J_2\,.\end{aligned} \tag{4.8}$$

The regularity properties of Ψ immediately imply

$$\begin{aligned}|J_0| \;&\le\; \int_0^t \Big|\Psi\big(t,s,\gamma_2(t,s),\gamma_{2,s}(t,s)\big)-\Psi\big(t,s,\gamma_1(t,s),\gamma_{1,s}(t,s)\big)\Big|\,|\mathbf{w}(t,s)|\,ds\\ &\le C_0\cdot\|\mathbf{w}(t,\cdot)\|^2,\end{aligned} \tag{4.9}$$

for some constant C_0.

It remains to estimate the last two terms in (4.6). To fix the ideas, consider a point $s'\in\chi_1(t)$, so that $\gamma_1(t,s')\in\partial\Omega$. This point will contribute to the angular velocity ω_1 through a term of the form

$$\begin{cases} e^{-\beta(t-\sigma)}\Big(\big(\gamma_1(t,s')-\gamma_1(t,\sigma)\big)\times\mathbf{n}_1(t,s')\Big) & \text{if}\;\;\sigma\in[0,s'],\\ \qquad\qquad 0 & \text{if}\;\;\sigma>s'.\end{cases} \tag{4.10}$$

By assumption,

$$\begin{aligned}\gamma_2(t,s')-\gamma_1(t,s') \;&=\; \int_0^{s'}\big(\mathbf{k}_2(t,s)-\mathbf{k}_1(t,s)\big)\,ds\\ &=\;\int_0^{s'}\left(R\left[\int_0^s \mathbf{w}(t,\sigma)\,d\sigma\right]-I\right)\mathbf{k}_1(t,s)\,ds\\ &=\;\int_0^{s'}\left(\int_0^s \mathbf{w}(t,\sigma)\,d\sigma\right)\times\mathbf{k}_1(t,s)\,ds+\mathcal{O}(1)\cdot\|\mathbf{w}(t,\cdot)\|^2\\ &=\;\int_0^{s'}\mathbf{w}(t,s)\big(\gamma_1(t,s')-\gamma_1(t,s)\big)\,ds+\mathcal{O}(1)\cdot\|\mathbf{w}(t,\cdot)\|^2\\ &=\;\mathcal{O}(1)\cdot\|\mathbf{w}(t,\cdot)\|^2.\end{aligned} \tag{4.11}$$

Using (4.11) and the properties of the triple product, we now compute

$$
\begin{aligned}
\int_0^{s'} & e^{-\beta s}\, e^{-\beta(t-s)} \Big\langle \big(\gamma_1(t,s') - \gamma_1(t,s)\big) \times \mathbf{n}_1(t,s')\,,\ \mathbf{w}(t,s) \Big\rangle\, ds \\
&= e^{-\beta t} \mathbf{n}_1(t,s') \cdot \int_0^{s'} \mathbf{w}(t,s) \times \big(\gamma_1(t,s') - \gamma_1(t,s)\big)\, ds \\
&= e^{-\beta t} \mathbf{n}_1(t,s') \cdot \big(\gamma_2(t,s') - \gamma_1(t,s')\big) + \mathcal{O}(1) \cdot \|\mathbf{w}(t,\cdot)\|^2 \\
&\leq \mathcal{O}(1) \cdot \big|\gamma_2(t,s') - \gamma_1(t,s')\big|^2 + \mathcal{O}(1) \cdot \|\mathbf{w}(t,\cdot)\|^2 \\
&= \mathcal{O}(1) \cdot \|\mathbf{w}(t,\cdot)\|^2 .
\end{aligned}
\tag{4.12}
$$

Recalling that the total mass of the measure μ_1 is uniformly bounded, the second term on the right hand side of (4.8) can thus be estimated by

$$
J_1 \ \leq\ \mathcal{O}(1) \cdot\ \|\mathbf{w}(t,\cdot)\|^2 \cdot \int_{[0,t]} \mu_1(s')\, ds' \ \leq\ C_1\, \|\mathbf{w}(t,\cdot)\|^2, \tag{4.13}
$$

for some constant C_1. Similarly,

$$
J_2 \ \leq\ \mathcal{O}(1) \cdot\ \|\mathbf{w}(t,\cdot)\|^2 \cdot \int_{[0,t]} \mu_2(s')\, ds' \ \leq\ C_2\, \|\mathbf{w}(t,\cdot)\|^2. \tag{4.14}
$$

By (4.8) together (4.9), (4.13), and (4.14), in view of (4.7) we achieve a proof of (4.3). By Gronwall's lemma, this proves the uniqueness of solutions to (2.3) and (2.11)–(2.12), and continuous dependence of solutions on the initial data (2.10). □

5 Proof of the representation formula

In this section we give a proof of Theorem 3, showing that any solution to (2.3)-(2.5) has the form (2.27).

For any time t, consider the contact set $\chi(t)$ of points $s \in [0,t]$ where the stem touches the obstacle. Observe that the map $t \mapsto \chi(t)$ is an upper semicontinuous multifunction with compact values.

Lemma 3. *There exists a set of times $\mathcal{N}$ of measure zero such that, for each $t \in [t_0,T] \backslash \mathcal{N}$ the partial derivative $\gamma_t(t,s)$ exists for all $s \in [0,T]$. Moreover, the map $s \mapsto \gamma_t(\tau,s)$ is Lipschitz continuous.*

Proof. We use the representation

$$
\begin{aligned}
\gamma(t,s) \ &=\ \int_0^s \mathbf{k}(t,\sigma)\, d\sigma, \\
\gamma(t+\varepsilon,s) - \gamma(t,s) \ &=\ \int_t^{t+\varepsilon} \int_0^s \mathbf{k}_t(\tau,\sigma)\, d\sigma\, d\tau\,.
\end{aligned}
\tag{5.1}
$$

By the regularity of the solution γ, proved in Theorem 1 of [1], the partial derivative $\mathbf{k}_t$ is well defined for a.e. $(\tau,\sigma) \in [t_0,T] \times [0,T]$. Moreover, it satisfies a uniform bound $|\mathbf{k}_t(\tau,\sigma)| \leq C$.

Therefore, there exists a set of times $\mathcal{N} \subset [t_0, T]$ of measure zero such that, for $t \notin \mathcal{N}$, the partial derivative $\mathbf{k}_t(t, \sigma)$ exists for a.e. $\sigma \in [0, T]$.

Using (5.1) and the Lebesgue dominated convergence theorem, for every $t \notin \mathcal{N}$ we obtain

$$\gamma_t(t,s) \;=\; \lim_{\varepsilon\to 0} \frac{\gamma(t+\varepsilon,s)-\gamma(t,s)}{\varepsilon} \;=\; \lim_{\varepsilon\to 0}\int_0^s \frac{\mathbf{k}(t+\varepsilon,\sigma)-\mathbf{k}(t,\sigma)}{\varepsilon}\,d\sigma \;=\; \int_0^s \mathbf{k}_t(t,\sigma)\,d\sigma.$$

This achieves the proof. □

Corollary 1. *Consider any time $\tau \in [t_0, T] \setminus \mathcal{N}$. Then, calling $\mathbf{n}(\tau, s)$ the unit outer normal to the obstacle at the boundary point $\gamma(\tau, s) \in \partial\Omega$, one has*

$$\big\langle \gamma_t(\tau,s),\, \mathbf{n}(\tau,s)\big\rangle \;=\; 0 \qquad \textit{for all } \; s \in \chi(\tau)\setminus\{\tau\}. \tag{5.2}$$

In addition, if the tip of the stem touches the obstacle, i.e. if $\tau \in \chi(\tau)$, then

$$\big\langle \gamma_t(\tau,\tau) + \mathbf{k}(\tau,\tau),\, \mathbf{n}(\tau,\tau)\big\rangle \;=\; 0. \tag{5.3}$$

Proof. Denote by

$$\Phi(x) \;\doteq\; \begin{cases} \mathrm{dist}(x,\Omega) & \text{if } \; x \notin \Omega,\\ -\mathrm{dist}(x,\partial\Omega) & \text{if } \; x \in \Omega,\end{cases}$$

the signed distance of a point x to the boundary of Ω. Since Ω has a $\mathcal{C}^2$ boundary, the function Φ is twice continuously differentiable in a neighborhood of $\partial\Omega$.

If (5.2) fails for some $s \in \chi(\tau) \setminus \{\tau\}$, then

$$\Phi(\gamma(\tau,s)) \;=\; 0, \qquad\qquad \frac{d}{dt}\Phi(\gamma(t,s))\bigg|_{t=\tau} \;=\; \big\langle \gamma_t(\tau,s),\, \mathbf{n}(\tau,s)\big\rangle \;\neq\; 0.$$

This yields a contradiction, because $\Phi(\gamma(t,s)) \;\geq\; 0$ for all t.

Similarly, if $\tau \in \chi(\tau)$ but (5.3) fails, then

$$\Phi(\gamma(\tau,\tau)) \;=\; 0, \qquad\qquad \frac{d}{dt}\Phi(\gamma(t,t))\bigg|_{t=\tau} \;=\; \big\langle \gamma_t(\tau,\tau) + \mathbf{k}_s(\tau,\tau),\, \mathbf{n}(\tau,\tau)\big\rangle \;\neq\; 0.$$

This yields a contradiction, because $\Phi(\gamma(t,t)) \;\geq\; 0$ for all t. □

Proof of Theorem 3.

We will show that the representation formula (2.27) is valid at every time $\tau \in [t_0, T] \setminus \mathcal{N}$, where the conclusions (5.2)-(5.3) of Corollary 1 hold. Notice that, since condition **(B)** does NOT hold, the set of ω satisfying the constraints (2.25), (2.26) is non empty.

1. Fix a time $\tau \in [t_0, T] \setminus \mathcal{N}$ and let $\mathbf{v} \in \Gamma(\tau)$ be a velocity field for which the bilateral constraints are satisfied:

$$\left\langle \int_0^s \Psi(\sigma)\times\big(\gamma(\tau,s)-\gamma(\tau,\sigma)\big)\,d\sigma + \mathbf{v}(s)\,,\; \mathbf{n}(\tau,s)\right\rangle \;=\; 0 \qquad\qquad \text{for all } \; s \in \chi(\tau)\setminus\{\tau\}, \tag{5.4}$$

together with

$$\left\langle \gamma_s(\tau,\tau) + \int_0^\tau \Psi(\sigma)\times\big(\gamma(\tau,\tau)-\gamma(\tau,\sigma)\big)\,d\sigma + \mathbf{v}(\tau)\,,\ \mathbf{n}(\tau,\tau)\right\rangle \;=\; 0 \tag{5.5}$$

if $\gamma(\tau,\tau)\in\partial\Omega$. By (2.5), $\mathbf{v}$ has the form

$$\mathbf{v}(s) \;=\; \int_0^s \bar\omega(\sigma)\times\big(\gamma(t,s)-\gamma(t,\sigma)\big)\big)\,d\sigma\,, \tag{5.6}$$

where the angular velocity is

$$\bar\omega(\sigma) \;=\; -\,e^{-\beta(\tau-\sigma)}\int_{[\sigma,\tau]}\Big(\mathbf{n}(\tau,s')\times\big(\gamma(\tau,s')-\gamma(t,\sigma)\big)\Big)d\mu(s')\,, \tag{5.7}$$

for some positive measure μ supported on the set $\chi(\tau)$. To achieve the proof we need to show that $\bar\omega(\cdot)$ provides the global minimizer for the optimization problem (2.24) subject to the unilateral constraints (2.25)-(2.26).

2. Consider any other field of angular velocities, say $\bar\omega+\omega$. The optimality of $\bar\omega$ will be proved by showing that

- either $\mathcal{E}(\bar\omega+\omega)\geq\mathcal{E}(\bar\omega)$,
- or else, replacing $\bar\omega$ with $\bar\omega+\omega$, the constraints (2.25)-(2.26) are no longer satisfied.

By the convexity of the integrand in (2.24) it follows

$$\frac{1}{2}\int_0^\tau e^{\beta(\tau-\sigma)}|\bar\omega(\sigma)+\omega(\sigma)|^2\,d\sigma \;\geq\; \frac{1}{2}\int_0^\tau e^{\beta(\tau-\sigma)}|\bar\omega(\sigma)|^2\,d\sigma+\int_0^\tau e^{\beta(\tau-\sigma)}\big\langle\bar\omega(\sigma),\,\omega(\sigma)\big\rangle\,d\sigma\,. \tag{5.8}$$

The last term on the right hand side of (5.8) is computed by

$$\begin{aligned}
&\int_0^\tau e^{\beta(\tau-\sigma)}\big\langle\bar\omega(\sigma),\,\omega(\sigma)\big\rangle\,d\sigma\\
&\quad=\; -\int_0^\tau e^{\beta(\tau-\sigma)}\omega(\sigma)\cdot e^{-\beta(\tau-\sigma)}\left(\int_{[\sigma,\tau]}\Big(\mathbf{n}(\tau,s')\times\big(\gamma(\tau,s')-\gamma(t,\sigma)\big)\Big)d\mu(s')\right)d\sigma\\
&\quad=\; \int_0^\tau\left(\int_{[\sigma,\tau]}\Big(\omega(\sigma)\times\big(\gamma(\tau,s')-\gamma(t,\sigma)\big)\Big)\cdot\mathbf{n}(\tau,s')\,d\mu(s')\right)d\sigma.
\end{aligned} \tag{5.9}$$

If $\bar\omega+\omega$ is admissible, then by (5.4)-(5.5) it follows

$$\left(\int_0^{s'}\omega(\sigma)\times\big(\gamma(\tau,s')-\gamma(t,\sigma)\big)\,d\sigma\right)\cdot\mathbf{n}(\tau,s')\;\geq\;0\qquad\text{for all}\quad s'\in\chi(\tau).$$

Integrating w.r.t. μ and exchanging the order of integration one obtains

$$\begin{aligned}
0 \;&\leq\; \int_{[0,\tau]}\left(\int_0^{s'}\omega(\sigma)\times\big(\gamma(\tau,s')-\gamma(t,\sigma)\big)\,d\sigma\right)\cdot\mathbf{n}(\tau,s')\,d\mu(s')\\
&=\; \int_0^\tau\left(\int_{[\sigma,\tau]}\omega(\sigma)\times\big(\gamma(\tau,s')-\gamma(t,\sigma)\big)\cdot\mathbf{n}(\tau,s')d\mu(s')\right)d\sigma\,.
\end{aligned} \tag{5.10}$$

Hence the right hand side of (5.9) is nonnegative.

This shows that $\bar{\omega}(\cdot)$ in (5.7) provides the global minimizer to the constrained optimization problem (2.24)-(2.26). Since this minimization problem has strictly convex cost and convex constraints, we conclude that $\bar{\omega}(\cdot)$ is the unique minimizer, as claimed in Theorem 3. $\square$

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References

[1] A. Bressan, M. Palladino, and W. Shen, Growth models for tree stems and vines. *J. Differential Equations*, to appear.

[2] L. Cesari, *Optimization - Theory and Applications*, Springer-Verlag, 1983.

[3] G. Colombo and V. Goncharov, The sweeping processes without convexity. *Set-Valued Anal.* **7** (1999), 357–374.

[4] G. Colombo and M. Monteiro Marques, Sweeping by a continuous prox-regular set. *J. Differential Equations* **187** (2003), 46–62.

[5] O. Leyser and S. Day, *Mechanisms in Plant Development.* Blackwell Publishing, 2003.

[6] J. J. Moreau, Evolution problems associated with a moving convex set in a Hilbert space, *J. Differential Equations* **26** (1977), 347–374.

[7] R. Rossi and U. Stefanelli, An order approach to a class of quasivariational sweeping processes. *Adv. Diff. Equat.* **10** (2005), 527–552.

[8] R. B. Vinter, *Optimal Control*, Birkhäuser, Boston, 2000.