Structurally Stable Singularities for a Nonlinear Wave Equation

Alberto Bressan, Tao Huang, and Fang Yu

Department of Mathematics, Penn State University
University Park, Pa. 16802, U.S.A.

e-mails: bressan@math.psu.edu , txh35@psu.edu , fuy3@psu.edu

March 29, 2015

Dedicated to Tai Ping Liu in the occasion of his 70-th birthday

Abstract

For the nonlinear wave equation $u_{tt}-c(u)\big(c(u)u_x\big)_x = 0$, it is well known that solutions can develop singularities in finite time. For an open dense set of initial data, the present paper provides a detailed asymptotic description of the solution in a neighborhood of each singular point, where $|u_x| \to \infty$. The different structure of conservative and dissipative solutions is analyzed.

1 Introduction

The nonlinear wave equation

$$u_{tt} - c(u)\big(c(u)u_x\big)_x \;=\; 0\,, \tag{1.1}$$

provides a mathematical model for the behavior of nematic liquid crystals. Solutions have been studied by several authors [1, 2, 3, 4, 7, 10, 12, 13]. We recall that, even for smooth initial data

$$u(x,0) \;=\; u_0(x)\,, \qquad\qquad u_t(x,0) \;=\; u_1(x)\,, \tag{1.2}$$

regularity can be lost in finite time. More precisely, the H^1 norm of the solution $u(\cdot,t)$ remains bounded, hence u is always Hölder continuous, but the norm of the gradient $\|u_x(\cdot,t)\|_{\mathbf{L}^\infty}$ can blow up in finite time.

The paper [4] introduced a nonlinear transformation of variables that reduces (1.1) to a semilinear system. In essence, it was shown that the quantities

$$w \;\doteq\; 2\arctan\big(u_t + c(u)u_x\big), \qquad\qquad z \;\doteq\; 2\arctan\big(u_t - c(u)u_x\big),$$

satisfy a first order semilinear system of equations, w.r.t. new independent variables X, Y constant along characteristics. Going back to the original variables x, t, u, one obtains a global solution of the wave equation (1.1).

Based on this representation and using ideas from [5, 6, 8, 9], in [1] it was recently proved that, for generic initial data, the conservative solution is smooth outside a finite number of points and curves in the t-x plane. Moreover, conditions were identified which guarantee the *structural stability* of the set of singularities. Namely, when these generic conditions hold, the topological structure of the singular set is not affected by a small $\mathcal{C}^3$ perturbation of the initial data.

Aim of the present paper is to derive a detailed asymptotic description of these structurally stable solutions, in a neighborhood of each singular point. This is achieved both for conservative and for dissipative solutions of (1.1). We recall that conservative solutions satisfy an additional conservation law for the energy, so that the total energy

$$\mathcal{E}(t) \;=\; \frac{1}{2}\int [u_t^2 + c^2(u)u_x^2]\,dx$$

coincides with a constant for a.e. time t. On the other hand, for dissipative solutions the total energy is a monotone decreasing function of time. A representation of dissipative solutions in terms of a suitable semilinear system in characteristic coordinates can be found in [3].

The remainder of this paper is organized as follows. In Section 2 we review the variable transformations introduced in [4] and the conditions for structural stability derived in [1]. Section 3 is concerned with conservative solutions. In this case, for smooth initial data the map

$$(X,Y) \;\mapsto\; (x,t,u,w,z)(X,Y) \tag{1.3}$$

remains globally smooth, on the entire X-Y plane. To recover the singularities of the solution $u(x,t)$ of (1.1), it suffices to study the Taylor approximation of (1.3) at points where $w=\pi$ or $z=\pi$. In Section 4 we perform a similar analysis in the case of dissipative solutions. This case is technically more difficult, because the corresponding semilinear system has discontinuous source terms.

We remark that, for conservative solutions, a general uniqueness theorem has been recently established in [2]. On the other hand, for dissipative solutions no general result on uniqueness or continuous dependence is yet known. Whether structurally stable dissipative solutions are *generic*, arising from an open dense set of $\mathcal{C}^3$ initial data, is also an open problem.

2 Review of the equations

Throughout the following, on the wave speed c we assume

(A) The map $c:\mathbb{R}\mapsto\mathbb{R}_+$ is smooth and uniformly positive. The quotient $c'(u)/c(u)$ is uniformly bounded. Moreover, the following generic condition is satisfied:

$$c'(u) \;=\; 0 \qquad \Longrightarrow \qquad c''(u) \;\neq\; 0. \tag{2.1}$$

Because of (2.1), the derivative $c'(u)$ can vanish only at isolated points.

In (1.2) we consider initial data (u_0,u_1) in the product space $H^1(\mathbb{R})\times\mathbf{L}^2(\mathbb{R})$. It is convenient to introduce the variables

$$\left\{\begin{array}{rcl} R &\doteq& u_t + c(u)u_x\,,\\ S &\doteq& u_t - c(u)u_x\,. \end{array}\right. \tag{2.2}$$

In a smooth solution, R^2 and S^2 satisfy the balance laws

$$\begin{cases} (R^2)_t - (cR^2)_x &= \frac{c'}{2c}(R^2S - RS^2)\,, \\ (S^2)_t + (cS^2)_x &= \frac{c'}{2c}(S^2R - SR^2)\,. \end{cases} \tag{2.3}$$

As a consequence, the energy is conserved:

$$E \doteq \frac{1}{2}\left(u_t^2 + c^2u_x^2\right) = \frac{R^2+S^2}{4}\,. \tag{2.4}$$

One can think of R^2 and S^2 as the energy of backward and forward moving waves, respectively. Notice that these are not separately conserved. Indeed, by (2.3) energy can be exchanged between forward and backward waves.

A major difficulty in the analysis of (1.1) is the possible breakdown of regularity of solutions. Indeed, even for smooth initial data, the quantities u_x, u_t can blow up in finite time. To deal with possibly unbounded values of R, S, following [4] we introduce a new set of dependent variables:

$$w \doteq 2\arctan R\,, \qquad\qquad z \doteq 2\arctan S\,. \tag{2.5}$$

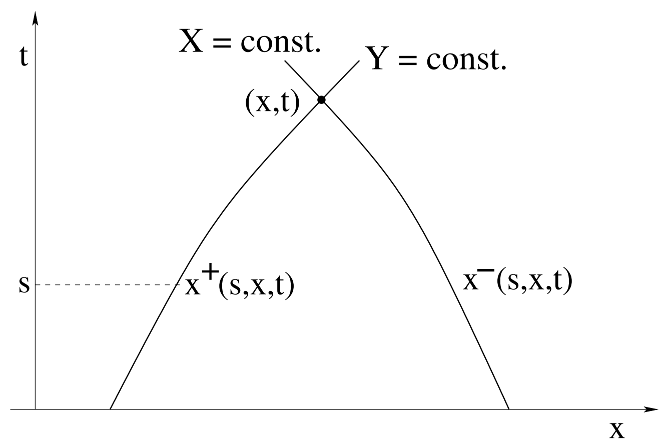

Figure 1: The backward and forward characteristic through the point (x,t).

To reduce the equation (1.1) to a semilinear one, it is convenient to perform a further change of independent variables (Fig. 1). Consider the equations for the forward and backward characteristics:

$$\dot{x}^+ = c(u)\,, \qquad\qquad \dot{x}^- = -c(u)\,. \tag{2.6}$$

The characteristics passing through the point (x,t) will be denoted by

$$s \mapsto x^+(s,x,t)\,, \qquad\qquad s \mapsto x^-(s,x,t)\,,$$

respectively. As coordinates (X,Y) of a point (x,t) we shall use the quantities

$$X \doteq x^-(0,x,t)\,, \qquad\qquad Y \doteq -x^+(0,x,t)\,. \tag{2.7}$$

For future use, we now introduce the further variables

$$p \doteq \frac{1+R^2}{X_x}\,, \qquad\qquad q \doteq \frac{1+S^2}{-Y_x}\,. \tag{2.8}$$

Starting with the nonlinear equation (1.1), using X, Y as independent variables one obtains a semilinear hyperbolic system with smooth coefficients for the variables u, w, z, p, q, x, t, namely

$$\begin{cases} u_X = \frac{\sin w}{4c}\, p\,, \\ u_Y = \frac{\sin z}{4c}\, q\,, \end{cases} \tag{2.9}$$

$$\begin{cases} w_Y = \frac{c'}{8c^2}\,(\cos z - \cos w)\, q\,, \\ z_X = \frac{c'}{8c^2}\,(\cos w - \cos z)\, p\,, \end{cases} \tag{2.10}$$

$$\begin{cases} p_Y = \frac{c'}{8c^2}\,(\sin z - \sin w)\, pq\,, \\ q_X = \frac{c'}{8c^2}\,(\sin w - \sin z)\, pq\,, \end{cases} \tag{2.11}$$

$$\begin{cases} x_X = \frac{(1+\cos w)\, p}{4}\,, \\ x_Y = -\frac{(1+\cos z)\, q}{4}\,, \end{cases} \tag{2.12}$$

$$\begin{cases} t_X = \frac{(1+\cos w)\, p}{4c}\,, \\ t_Y = \frac{(1+\cos z)\, q}{4c}\,. \end{cases} \tag{2.13}$$

See [4] for detailed computations. Boundary data can be assigned on the line $\gamma_0 = \{(X, Y)\,;\ X + Y = 0\}$, by setting

$$\begin{cases} u(s, -s) = \overline{u}(s)\,, \\ x(s, -s) = \overline{x}(s)\,, \\ t(s, -s) = \overline{t}(s)\,, \end{cases} \qquad \begin{cases} w(s, -s) = \overline{w}(s)\,, \\ z(s, -s) = \overline{z}(s)\,, \end{cases} \qquad \begin{cases} p(s, -s) = \overline{p}(s)\,, \\ q(s, -s) = \overline{q}(s)\,, \end{cases} \tag{2.14}$$

for suitable smooth functions $\overline{u}, , \overline{x}, \overline{t}, \overline{w}, \overline{z}, \overline{p}, \overline{q}$.

Remark 1. The above system is clearly invariant w.r.t. the addition of an integer multiple of 2π to the variables w, z. Taking advantage of this property, in the following we shall regard w, z as points in the quotient manifold $\mathbb{T} \doteq \mathbb{R}/2\pi\mathbb{Z}$. As a consequence, we have the implications

$$\begin{aligned} w \neq \pi \qquad &\Longrightarrow \qquad \cos w > -1\,, \\ z \neq \pi \qquad &\Longrightarrow \qquad \cos z > -1\,. \end{aligned} \tag{2.15}$$

Remark 2. The system (2.9)–(2.13) is overdetermined. Indeed, the functions u, x, t can be computed by using either one of the equations in (2.9), (2.12), (2.13), respectively. As shown in [1], in order that all the above equations be simultaneously satisfied along the line γ_0 one needs the additional compatibility conditions

$$\frac{d}{ds}\overline{u}(s) = \frac{\sin \overline{w}(s)}{4c(\overline{u}(s))}\, \overline{p}(s) - \frac{\sin \overline{z}(s)}{4c(\overline{u}(s))}\, \overline{q}(s)\,, \tag{2.16}$$

$$\frac{d}{ds}\overline{x}(s) \;=\; \frac{(1+\cos\overline{q}(s))\overline{p}(s)+(1+\cos\overline{z}(s))\overline{q}(s)}{4}\,, \tag{2.17}$$

$$\frac{d}{ds}\overline{t}(s) \;=\; \frac{(1+\cos\overline{w}(s))\overline{p}(s)-(1+\cos\overline{z}(s))\overline{q}(s)}{4c(\overline{u}(s))}\,. \tag{2.18}$$

In turn, if (2.16)–(2.18) hold along γ_0, then a unique solution to the system (2.9)–(2.13) can be constructed, on the entire X-Y plane.

Given initial data (u_0, u_1) in (1.2), we assign boundary data (2.14) on the line γ_0, by setting

$$\left\{\begin{array}{rcl}\overline{u}(x) &=& u_0(x)\,,\\ \overline{t}(x) &=& 0\,,\\ \overline{x}(x) &=& x\,,\end{array}\right. \qquad \left\{\begin{array}{rcl}\overline{w}(x) &=& 2\arctan R(x,0)\,,\\ \overline{z}(x) &=& 2\arctan S(x,0)\,,\end{array}\right. \qquad \left\{\begin{array}{rcl}\overline{p}(x) &\equiv& 1+R^2(x,0)\,,\\ \overline{q}(x) &\equiv& 1+S^2(x,0)\,.\end{array}\right. \tag{2.19}$$

We recall that, at time $t=0$, by (1.2) one has

$$\begin{aligned} R(x,0) &= (u_t + c(u)u_x)(x,0) = u_1(x) + c(u_0(x))u_{0,x}(x),\\ S(x,0) &= (u_t - c(u)u_x)(x,0) = u_1(x) - c(u_0(x))u_{0,x}(x).\end{aligned}$$

As proved in [1], for any choice of u_0, u_1 in (2.19) the compatibility conditions (2.16)–(2.18) are automatically satisfied.

The following theorems summarize the main results on conservative solutions, proved in [4, 1, 2]. As before, $\mathcal{U}$ denotes the product space in (2.21).

Theorem 1. *Let the wave speed $c(\cdot)$ satisfy the assumptions* **(A)**.

Given initial data $(u_0, u_1) \in H^1(\mathbb{R}) \times \mathbf{L}^2(\mathbb{R})$, there exists a unique solution $(X,Y) \mapsto (u,w,z,p,q,x,t)(X,Y)$ to the system (2.9)–(2.13) with boundary data (2.14), (2.19) assigned along the line γ_0. Moreover, the set

$$\text{Graph}(u) \;\doteq\; \Big\{\big(x(X,Y),\; t(X,Y),\; u(X,Y)\big)\,;\quad (X,Y)\in\mathbb{R}^2\Big\} \tag{2.20}$$

is the graph of the unique conservative solution $u = u(x,t)$ of the Cauchy problem (1.1)-(1.2).

Theorem 2. *Let the assumptions* **(A)** *be satisfied and let $T > 0$ be given. Then there exists an open dense set*

$$\mathcal{D} \;\subset\; \mathcal{U} \;\doteq\; \Big(\mathcal{C}^3(\mathbb{R})\cap H^1(\mathbb{R})\Big)\times\Big(\mathcal{C}^2(\mathbb{R})\cap \mathbf{L}^2(\mathbb{R})\Big) \tag{2.21}$$

such that the following holds.

For every initial data $(u_0, u_1) \in \mathcal{D}$, the corresponding solution (u,w,z,p,q,z,t) of (2.9)–(2.13) with boundary data (2.14), (2.19) has level sets $\{w=\pi\}$ and $\{z=\pi\}$ in generic position. More precisely, none of the values

$$\left\{\begin{array}{rcl}(w, w_X, w_{XX}) &=& (\pi,0,0),\\[2mm] (z, z_Y, z_{YY}) &=& (\pi,0,0),\end{array}\right. \tag{2.22}$$

$$\begin{cases} (w, z, w_X) &= (\pi, \pi, 0), \\ (w, z, z_Y) &= (\pi, \pi, 0), \end{cases} \tag{2.23}$$

$$\begin{cases} (w, w_X, c'(u)) &= (\pi, 0, 0), \\ (z, z_Y, c'(u)) &= (\pi, 0, 0), \end{cases} \tag{2.24}$$

is ever attained, at any point (X, Y) *for which*

$$\big(x(X, Y),\ t(X, Y)\big) \in \mathbb{R} \times [0, T]\,. \tag{2.25}$$

The singularities of the solution u in the x-t plane correspond to the image of the level sets $\{w = \pi\}$ and $\{z = \pi\}$ w.r.t. the map

$$\Lambda : (X, Y) \mapsto \big(x(X, Y),\ t(X, Y)\big). \tag{2.26}$$

If none of the values in (2.22)-(2.24) is ever attained, by the implicit function theorem the above level sets are the union of a locally finite family of $\mathcal{C}^2$ curves in the X-Y plane. In turn, restricted to the domain $\mathbb{R} \times [0, T]$, the singularities of u are located along finitely many $\mathcal{C}^2$ curves in the x-t plane.

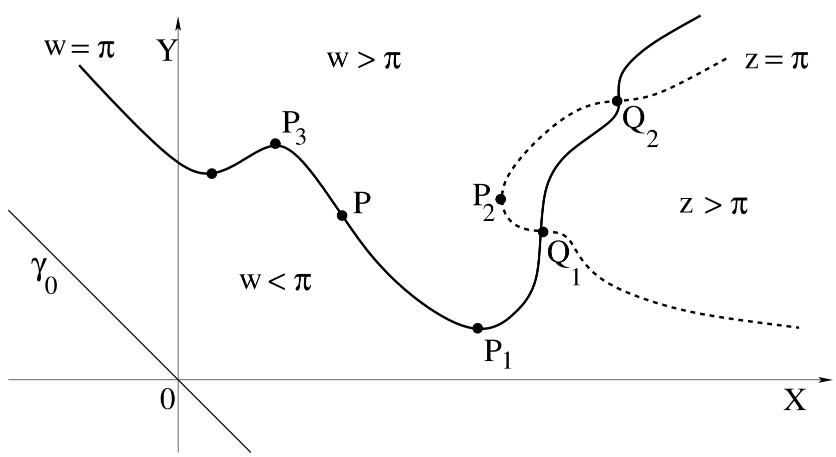

Figure 2: Two level sets $\{w = \pi\}$ and $\{z = \pi\}$, in a generic conservative solution of (2.9)–(2.11). Here P is a singular point of Type 1, while P_1, P_2, P_3 are points of Type 2, and Q_1, Q_2 are points of Type 3. Notice that at P_1, structural stability requires that the function $Y(X)$ implicitly defined by $w(X, Y(X)) = \pi$ has strictly positive second derivative. At the points Q_1, Q_2, by (2.10) one has $w_Y = z_X = 0$. Hence the two curves $\{w = \pi\}$ and $\{z = \pi\}$ have a perpendicular intersection.

3 Singularities of conservative solutions

For smooth data $u_0, u_1 \in \mathcal{C}^\infty(\mathbb{R})$, the solution $(X, Y) \mapsto (x, t, u, w, z, p, q)(X, Y)$ of the semilinear system (2.9)–(2.13), with initial data as in (2.14), (2.19), remains smooth on the entire X-Y plane. Yet, the solution $u = u(x, t)$ of (1.1) can have singularities because the coordinate

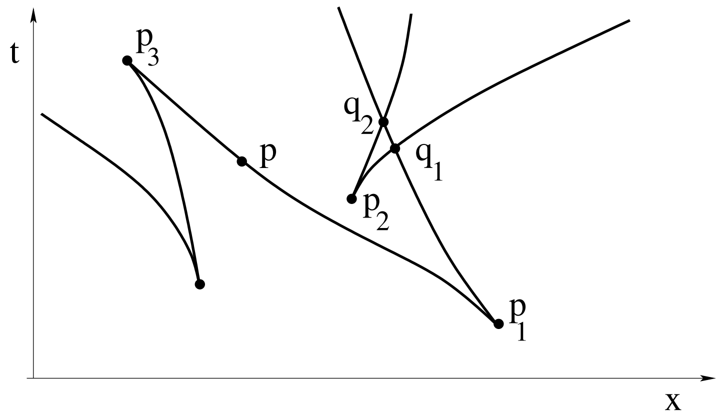

Figure 3: The images of the level sets $\{w = \pi\}$ and $\{z = \pi\}$ in Fig. 2, under the map $\Lambda : (X,Y) \mapsto (x(X,Y), t(X,Y))$. In the x-t plane, these represents the curves where $u = u(x,t)$ is not differentiable. A generic solution of (1.1) with smooth initial data remains smooth outside finitely many singular points and finitely many singular curves, where $u_x \to \pm\infty$. Here p_1, p_2, p_3 are singular points where two new singular curves originate, or two singular curves merge and disappear. At the points q_1, q_2 a forward and a backward singular curve cross each other.

change $\Lambda : (X,Y) \mapsto (x,t)$ is not smoothly invertible. By (2.12)-(2.13), its Jacobian matrix is computed by

$$D\Lambda = \begin{pmatrix} x_X & x_Y \\ t_X & t_Y \end{pmatrix} = \begin{pmatrix} \frac{(1+\cos w)\,p}{4} & -\frac{(1+\cos z)\,q}{4} \\ \frac{(1+\cos w)\,p}{4c(u)} & \frac{(1+\cos z)\,q}{4c(u)} \end{pmatrix}. \tag{3.1}$$

We recall that p, q remain uniformly positive and uniformly bounded on compact subsets of the X-Y plane. By Remark 1, at a point (X_0, Y_0) where $w \neq \pi$ and $z \neq \pi$, this matrix is invertible, having a strictly positive determinant. The function $u = u(x,t)$ considered at (2.20) is thus smooth on a neighborhood of the point

$$(x_0, t_0) = \Big(x(X_0, Y_0)\,,\ t(X_0, Y_0) \Big).$$

To study the set of points x-t plane where u is singular, we thus need to look at points where either $w = \pi$ or $z = \pi$.

If the generic conditions (2.22)–(2.24) are satisfied, then we have the implications

$$\begin{cases} w = \pi \quad \text{and} \quad w_X = 0 \qquad \Longrightarrow \qquad w_Y = \dfrac{c'(u)}{8c^2(u)}(\cos z + 1)q \neq 0\,, \\[2mm] z = \pi \quad \text{and} \quad z_Y = 0 \qquad \Longrightarrow \qquad z_X = \dfrac{c'(u)}{8c^2(u)}(\cos w + 1)p \neq 0\,. \end{cases}$$

Therefore, by the implicit function theorem, the level sets

$$\mathcal{S}^w \doteq \{(X,Y);\ \ w(X,Y) = \pi\}\,, \qquad\qquad \mathcal{S}^z \doteq \{(X,Y);\ \ z(X,Y) = \pi\}\,, \tag{3.2}$$

are the union of a locally finite family of smooth curves. The singularities of u in the x-t plane are contained in the images of S^w and S^z under the map (2.26). Relying on Theorem 2, we shall distinguish three types of singular points $P = (X_0, Y_0)$.

(1) Points where $w = \pi$ but $w_X \neq 0$ and $z \neq \pi$ (or else, where $z = \pi$ but $z_Y \neq 0$ and $w \neq \pi$).

(2) Points where $w = \pi$ and $w_X = 0$, but $w_{XX} \neq 0$ (or else: $z = \pi$ and $z_Y = 0$, but $z_{YY} \neq 0$).

(3) Points where $w = \pi$ and $z = \pi$.

Points of Type 1 form a locally finite family of $\mathcal{C}^2$ curves in the X-Y plane (Fig. 2). Their images $\Lambda(P)$ yield a family of characteristic curves in the x-t plane where the solution $u = u(x,t)$ is singular (i.e., not differentiable).

Points of Type 2 are isolated. Their images in the x-t plane are points where two singular curves initiate or terminate (Fig. 3).

Points of Type 3 are those where two curves $\{w = \pi\}$ and $\{z = \pi\}$ intersect. Their image in the x-t plane are points where two singular curves cross, with speeds $\pm c(u)$.

Our main result provides a detailed description of the solution $u = u(x,t)$ in a neighborhood of each one of these singular points. For simplicity, we shall assume that the initial data (u_0, u_1) in (1.2) are smooth, so we shall not need to count how many derivatives are actually used to derive the Taylor approximations.

Theorem 3. *Let the assumptions* **(A)** *hold, and consider generic initial data* $(u_0, u_1) \in \mathcal{D}$ *as in (2.21), with* $u_0, u_1 \in C^\infty(\mathbb{R})$*. Call* (u, w, z, p, q, x, t) *the corresponding solution of the semilinear system (2.9)–(2.13) and let* $u = u(x,t)$ *be the solution to the original equation (1.1). Consider a singular point* $P = (X_0, Y_0)$ *where* $w = \pi$*, and set* $(x_0, t_0) \doteq (x(X_0, Y_0), t(X_0, Y_0))$*.*

(i) If P *is a point of Type 1, along a curve where* $w = \pi$*, then there exist constants* $a \neq 0$ *and* b_1, b_2 *such that*

$$\begin{aligned} u(x,t) &= u(x_0,t_0) - a \cdot \Big[c(u_0)(t-t_0) + (x-x_0)\Big]^{2/3} \\ &\quad + b_1 \cdot (x-x_0) + b_2 \cdot (t-t_0) + \mathcal{O}(1) \cdot \Big(|t-t_0| + |x-x_0|\Big)^{4/3}. \end{aligned} \tag{3.3}$$

(ii) If P *is a point of Type 2, where* $w = \pi$*,* $w_X = 0$*, and* $w_{XX} > 0$*, then in the* x-t *plane this corresponds to a point* (x_0, t_0) *where two new singular curves* γ^-, γ^+ *originate. In this case, there exists a constant* $a \neq 0$ *such that*

$$u(x,t) = u(x_0,t_0) + a \cdot \Big[c(u_0)(t-t_0) + (x-x_0)\Big]^{3/5} + \mathcal{O}(1) \cdot \Big(|t-t_0| + |x-x_0|\Big)^{4/5}. \tag{3.4}$$

(iii) If P *is a point of Type 3, where* $w = z = \pi$*, then in the* x-t *plane this corresponds to a point* (x_0, t_0) *where two singular curves* $\gamma, \tilde{\gamma}$ *cross each other. In this case, there exist constants* $a_1 \neq 0$ *and* $a_2 \neq 0$ *such that*

$$\begin{aligned} u(x,t) &= u(x_0,t_0) + a_1 \cdot \Big[c(u_0)(t-t_0) + (x-x_0)\Big]^{2/3} \\ &\quad + a_2 \cdot \Big[c(u_0)(t-t_0) - (x-x_0)\Big]^{2/3} + \mathcal{O}(1) \cdot \Big(|t-t_0| + |x-x_0|\Big). \end{aligned} \tag{3.5}$$

Throughout the following, given a point $P = (X_0, Y_0)$ in the X-Y plane where $w = \pi$, we denote by $(u_0, w_0, z_0, p_0, q_0, x_0, t_0)$ the values of (u, w, z, p, q, x, t) at (X_0, Y_0). The three parts of Theorem 3 will be proved separately.

3.1 Singular curves.

Let $P = (X_0, Y_0)$ be a point of Type 1, where

$$w_0 \;=\; \pi, \qquad\qquad z_0 \;\neq\; \pi, \qquad\qquad w_X(X_0, Y_0) \;\neq\; 0. \tag{3.6}$$

By the implicit function theorem, the level set where $w = \pi$ is locally the graph of a smooth function $X = \Phi(Y)$, with $\Phi(Y_0) = X_0$. We claim that, in a neighborhood of the point $(x_0, t_0) = \Lambda(X_0, Y_0)$, the image $\Lambda(\mathcal{S}^w)$ is a smooth curve in the x-t plane, say

$$\gamma \;=\; \big\{(x,t)\,;\; x = \phi(t)\big\}. \tag{3.7}$$

Indeed, the curve γ is the image of the smooth curve $\{X = \Phi(Y)\}$ under the smooth, one-to-one map

$$Y \;\mapsto\; \big(x(\Phi(Y), Y)\,,\; t(\Phi(Y), Y)\big).$$

For future record, we compute the first two derivatives of ϕ at $t = t_0$. Differentiating the identity $w(\Phi(Y), Y) = \pi$ one obtains

$$w_X \Phi' + w_Y \;=\; 0\,,$$

$$w_{XX} \cdot (\Phi')^2 + 2 w_{XY} \Phi' + w_{YY} + w_X \Phi'' \;=\; 0\,.$$

By (2.12)-(2.13), at the point (X_0, Y_0) we have

$$\frac{d}{dY}\big(x(\Phi(Y), Y)\,,\; t(\Phi(Y), Y)\big) \;=\; \left(-\frac{(1 + \cos z_0)\, q_0}{4}\,,\; \frac{(1 + \cos z_0)\, q_0}{4c(u_0)}\right) \;\neq\; (0,0).$$

Observing that

$$\phi'(t(\Phi(Y), Y)) \;=\; \frac{x_X(\Phi(Y), Y) \cdot \Phi'(Y) + x_Y(\Phi(Y), Y)}{t_X(\Phi(Y), Y) \cdot \Phi'(Y) + t_Y(\Phi(Y), Y)}\,,$$

at $t = t_0$ we have

$$\phi'(t_0) \;=\; -\, c(u_0)\,.$$

In a similar way we find

$$\phi''(t_0) \;=\; \frac{x_{YY}(X_0, Y_0) t_Y(X_0, Y_0) - t_{YY}(X_0, Y_0) x_Y(X_0, Y_0)}{t_Y^3(X_0, Y_0)} \;=\; -\,\frac{c'(u_0) \sin z_0}{1 + \cos z_0}\,.$$

Next, by (2.9) one has

$$u_X(X_0, Y_0) \;=\; 0\,, \qquad\qquad u_Y(X_0, Y_0) \;=\; \frac{\sin z_0}{4c(u_0)} q_0 \;\doteq\; \alpha_1\,. \tag{3.8}$$

Differentiating the first equation in (2.9) w.r.t. X and using (2.10)-(2.11) we obtain

$$u_{XX} \;=\; \frac{\cos w}{4c(u)} w_X p - \frac{\sin w}{4c^2(u)} c'(u) \cdot \frac{\sin w}{4c(u)} p^2 + \frac{\sin w}{4c(u)} p_X\,,$$

$$u_{XX}(X_0,Y_0) \;=\; \frac{w_X(X_0,Y_0)}{4c(u_0)}p_0 \;\doteq\; \alpha_2 \;\neq\; 0\,, \tag{3.9}$$

$$u_{XXX}(X_0,Y_0) \;=\; -\frac{1}{4c(u_0)}\Big(w_{XX}(X_0,Y_0)\;p_0 + 2w_X(X_0,Y_0)p_X(X_0,Y_0)\Big) \;\doteq\; \alpha_3\,, \tag{3.10}$$

$$u_{XY}(X_0,Y_0) \;=\; -\frac{p_0}{4c(u_0)}\cdot\frac{c'(u_0)}{8c^2(u_0)}(\cos z_0+1)q_0 \;\doteq\; \alpha_4\,. \tag{3.11}$$

This yields the local Taylor approximation

$$\begin{aligned} u(X,Y) \;&=\; u_0 + \alpha_1\,(Y-Y_0) + \frac{\alpha_2}{2}\,(X-X_0)^2 + \frac{\alpha_3}{6}\,(X-X_0)^3 + \alpha_4\,(X-X_0)(Y-Y_0) \\ &\quad + \mathcal{O}(1)\cdot\Big(|X-X_0|^4 + |Y-Y_0|^2 + |X-X_0|^2\,|Y-Y_0|\Big). \end{aligned} \tag{3.12}$$

Using (2.13), we perform an entirely similar computation for the function t in a neighborhood of (X_0,Y_0).

$$t_X(X_0,Y_0) \;=\; 0, \qquad t_Y(X_0,Y_0) \;=\; \frac{1+\cos z_0}{4c(u_0)}q_0 \;\doteq\; \beta_1 \;>\; 0\,, \tag{3.13}$$

$$t_{XX} \;=\; -\frac{\sin w}{4c(u)}w_X p - \frac{1+\cos w}{4c^2(u)}c'(u)\,u_X p + \frac{1+\cos w}{4c(u)}\,p_X\,,$$

$$t_{XX}(X_0,Y_0) \;=\; t_{XY}(X_0,Y_0) \;=\; 0\,, \tag{3.14}$$

$$t_{XXX}(X_0,Y_0) \;=\; \frac{w_X^2(X_0,Y_0)}{4c(u_0)}\,p_0 \;\doteq\; \beta_3 \;\neq\; 0\,. \tag{3.15}$$

This yields the Taylor approximation

$$\begin{aligned} t(X,Y) \;&=\; t_0 + \beta_1\,(Y-Y_0) + \frac{\beta_3}{6}\,(X-X_0)^3 \\ &\quad + \mathcal{O}(1)\cdot\Big(|X-X_0|^4 + |Y-Y_0|^2 + |X-X_0|^2\,|Y-Y_0|\Big). \end{aligned} \tag{3.16}$$

Finally, for the function x, using (2.12) we find

$$x_X(X_0,Y_0) \;=\; 0\,, \qquad x_Y(X_0,Y_0) \;=\; -\frac{1+\cos z_0}{4}\,q_0 \;\doteq\; -\gamma_1 \;<\; 0\,, \tag{3.17}$$

$$x_{XX} \;=\; -\frac{\sin w\cdot w_X}{4}p + \frac{1+\cos w}{4}\,p_X\,,$$

$$x_{XX}(X_0,Y_0) \;=\; 0, \qquad x_{XY}(X_0,Y_0) \;=\; 0, \tag{3.18}$$

$$x_{XXX}(X_0,Y_0) \;=\; \frac{w_X^2(X_0,Y_0)}{4}\,p_0 \;\doteq\; \gamma_3 \;>\; 0. \tag{3.19}$$

This yields the Taylor approximation

$$\begin{aligned} x(X,Y) \;&=\; x_0 - \gamma_1\,(Y-Y_0) + \gamma_3\,(X-X_0)^3 \\ &\quad + \mathcal{O}(1)\cdot\Big(|X-X_0|^4 + |Y-Y_0|^2 + |X-X_0|^2\,|Y-Y_0|\Big). \end{aligned} \tag{3.20}$$

Observing that the above Taylor coefficients satisfy

$$\gamma_1 \;=\; c(u_0)\,\beta_1\,, \qquad\qquad \gamma_3 \;=\; c(u_0)\,\beta_3\,, \tag{3.21}$$

from (3.16) and (3.20) we deduce

$$\begin{aligned}
&(x-x_0)-c(u_0)(t-t_0)\\
&\qquad = \; -2\gamma_1\,(Y-Y_0)+\mathcal{O}(1)\cdot\Big(|X-X_0|^4+|Y-Y_0|^2+|X-X_0|^2\,|Y-Y_0|\Big),\\
&(x-x_0)+c(u_0)(t-t_0)\\
&\qquad = \; 2\gamma_3\,(X-X_0)^3+\mathcal{O}(1)\cdot\Big(|X-X_0|^4+|Y-Y_0|^2+|X-X_0|^2\,|Y-Y_0|\Big)\,.
\end{aligned}\tag{3.22}$$

Next, using (3.16) and (3.20) we obtain an approximation for X, Y in terms of x, t, namely

$$\frac{1+\cos z_0}{2}q_0(Y-Y_0) \;=\; c(u_0)(t-t_0)-(x-x_0)+\mathcal{O}(1)\cdot\Big(|X-X_0|^4+|Y-Y_0|^2+|X-X_0|^2\,|Y-Y_0|\Big),$$

$$\frac{w_X^2(X_0,Y_0)}{12}p_0(X-X_0)^3 \;=\; c(u_0)(t-t_0)+(x-x_0)+\mathcal{O}(1)\cdot\Big(|X-X_0|^4+|Y-Y_0|^2+|X-X_0|^2\,|Y-Y_0|\Big).$$

Inserting the two above expressions into (3.12), we finally obtain

$$\begin{aligned}
u(t,x) \;=\; & u(t_0,x_0)-\left(\frac{9p_0}{32\,w_X(X_0,Y_0)}\right)^{1/3}\Big[c(u_0)(t-t_0)+(x-x_0)\Big]^{2/3}\\
&+\frac{\sin z_0}{2c(u_0)(1+\cos z_0)}\Big[c(u_0)(t-t_0)-(x-x_0)\Big]\\
&-\frac{w_{XX}(X_0,Y_0)}{2c(u_0)w_X^2(X_0,Y_0)}\Big[c(u_0)(t-t_0)+(x-x_0)\Big]\\
&+\mathcal{O}(1)\cdot\Big(|t-t_0|+|x-x_0|\Big)^{4/3}.
\end{aligned}\tag{3.23}$$

This proves (3.3), with

$$a \;=\; \left(\frac{9p_0}{32\,w_X(X_0,Y_0)}\right)^{1/3} \;\neq\; 0\,. \tag{3.24}$$

The coefficients b_1, b_2 can also be easily computed from (3.23).

Remark 3. By (3.3), the solution u is only Hölder continuous of exponent $2/3$ near the singular curve γ in (3.7). In particular, the Cauchy problem

$$\dot x(t) \;=\; -\,c\big(u(t,x(t))\big), \qquad\qquad x(t_0) \;=\; \phi(t_0)\,,$$

has a solution $t \mapsto x(t)$ which crosses γ at the point (x_0, t_0). Calling $\delta(t) \doteq x(t)-\phi(t)$, to leading order one has

$$\dot\delta \;=\; c'(u_0)\cdot a\,\delta^{2/3}\,.$$

Hence, for $t \approx t_0$ we have

$$\delta(t) \;\approx\; \left(\frac{c'(u_0)\,a}{3}\right)^3 (t-t_0)^3. \tag{3.25}$$

The singular curve γ is thus an envelope of characteristic curves, which cross it tangentially (see Fig. 4).

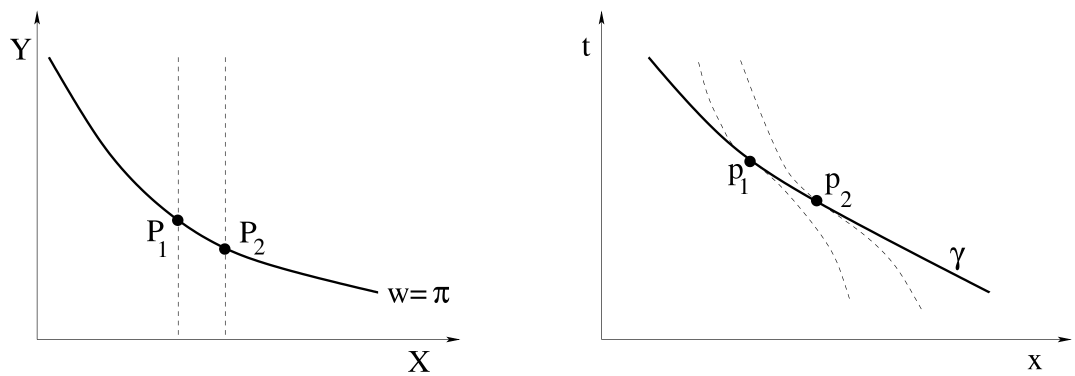

Figure 4: Left: a singular curve where $w = \pi$, in the X-Y plane. Vertical lines where X = constant correspond to characteristic curves of the wave equation (1.1), where $\dot{x} = -c(u)$. Right: the images of these curves in the x-t plane, under the map Λ at (2.26). The singular curve γ is an envelope of characteristic curves, which cross it tangentially.

3.2 Points where two singular curves originate or terminate.

Let $P = (X_0, Y_0)$ be a point of Type 2, where

$$w_0 \;=\; \pi, \qquad z_0 \;\neq\; \pi, \qquad w_X(X_0,Y_0) \;=\; 0, \qquad w_{XX}(X_0,Y_0) \;\neq\; 0\,. \tag{3.26}$$

Recalling (2.15), by (3.26) we have

$$w_Y(X_0,Y_0) \;=\; \frac{c'(u_0)}{8c^2(u_0)}(1+\cos z_0)q_0 \;\neq\; 0.$$

By (2.9), at the point (X_0, Y_0) we have

$$u_X \;=\; u_{XX} \;=\; 0\,, \qquad\qquad u_Y \;=\; \frac{\sin z_0}{4c(u_0)}\,q_0\,,$$

$$u_{XY} \;=\; -\frac{c'(u_0)}{32c^3(u_0)}(1+\cos z_0)p_0q_0\,, \qquad\qquad u_{XXX} \;=\; -\frac{w_{XX}(X_0,Y_0)}{4c(u_0)}p_0\,.$$

In this case, the Taylor approximation for u near the point (X_0, Y_0) takes the form

$$\begin{aligned} u(X,Y) \;&=\; u_0 + \frac{\sin z_0}{4c(u_0)}q_0\,(Y-Y_0) - \frac{w_{XX}(X_0,Y_0)}{24c(u_0)}p_0\,(X-X_0)^3 \\ &\quad + \mathcal{O}(1)\cdot\Big(|X-X_0|^4 + |Y-Y_0|^2 + |X-X_0|\,|Y-Y_0|\Big). \end{aligned} \tag{3.27}$$

Computing the partial derivatives of $x(X,Y)$ and $t(X,Y)$ at the point (X_0, Y_0), by (2.10) and (3.26) we find

$$x_X \;=\; x_{XX} \;=\; x_{XXX} \;=\; x_{XXXX} \;=\; x_{XY} \;=\; x_{XXY} \;=\; 0, \tag{3.28}$$

$$x_{XXXXX} \;=\; \frac{3w_{XX}^2(X_0,Y_0)}{4}\,p_0 \;\neq\; 0\,, \qquad\qquad x_Y \;=\; -\frac{1+\cos z_0}{4}\,q_0 \;\neq\; 0\,, \tag{3.29}$$

$$t_X \;=\; t_{XX} \;=\; t_{XXX} \;=\; t_{XXXX} \;=\; t_{XY} \;=\; t_{XXY} \;=\; 0, \tag{3.30}$$

$$t_{XXXXX} = \frac{3w_{XX}^2(X_0,Y_0)}{4c(u_0)}\,p_0 \neq 0\,, \qquad t_Y = \frac{1+\cos z_0}{4c(u_0)}\,q_0 \neq 0\,. \tag{3.31}$$

This yields the Taylor approximations

$$\begin{aligned} x(X,Y) &= x_0 - \frac{1+\cos z_0}{4} q_0(Y-Y_0) + \frac{3w_{XX}^2(X_0,Y_0)}{5!\,4} p_0(X-X_0)^5 \\ &\quad + \mathcal{O}(1)\cdot\Big(|X-X_0|^6 + |Y-Y_0|^2\Big), \end{aligned} \tag{3.32}$$

$$\begin{aligned} t(X,Y) &= t_0 + \frac{1+\cos z_0}{4c(u_0)} q_0(Y-Y_0) + \frac{3w_{XX}^2(X_0,Y_0)}{5!\,4\,c(u_0)} p_0(X-X_0)^5 \\ &\quad + \mathcal{O}(1)\cdot\Big(|X-X_0|^6 + |Y-Y_0|^2\Big). \end{aligned} \tag{3.33}$$

Combining (3.32) with (3.33) we obtain

$$\begin{aligned} (X-X_0)^5 &= \frac{5!\,2}{3w_{XX}^2(X_0,Y_0)\,p_0}\cdot\big[c(u_0)(t-t_0)+(x-x_0)\big] \\ &\quad + \mathcal{O}(1)\cdot\Big(|X-X_0|^6 + |Y-Y_0|^2\Big), \end{aligned} \tag{3.34}$$

$$Y-Y_0 = \frac{2}{(1+\cos z_0)\,q_0}\cdot\big[c(u_0)(t-t_0)-(x-x_0)\big] + \mathcal{O}(1)\cdot\Big(|X-X_0|^6 + |Y-Y_0|^2\Big). \tag{3.35}$$

Inserting (3.34)-(3.35) into (3.27) we eventually obtain

$$\begin{aligned} u(t,x) &= u(t_0,x_0) - \frac{1}{24c(u_0)}\cdot\left(\frac{80^3\,p_0^2}{w_{XX}(X_0,Y_0)}\right)^{1/5}\cdot\Big[c(u_0)(t-t_0)+(x-x_0)\Big]^{3/5} \\ &\quad + \mathcal{O}(1)\cdot\Big(|t-t_0|+|x-x_0|\Big)^{4/5}. \end{aligned} \tag{3.36}$$

This proves (3.4).

It remains to show that two singular curves originate or terminate at the point (x_0, t_0). To fix the ideas, assume that

$$\kappa \doteq -\frac{w_{XX}(X_0,Y_0)}{2w_Y(X_0,Y_0)} > 0\,. \tag{3.37}$$

By the implicit function theorem, the curve where $w = \pi$ can be approximated as

$$Y - Y_0 = \kappa(X-X_0)^2 + \mathcal{O}(1)\cdot|X-X_0|^3. \tag{3.38}$$

On the other hand, by (3.33) we have

$$Y-Y_0 = \alpha(t-t_0) - \beta(X-X_0)^5 + \mathcal{O}(1)\cdot\Big(|X-X_0|^6 + |t-t_0|^2 + |X-X_0|\,|t-t_0|\Big), \tag{3.39}$$

with

$$\alpha = \frac{4c(u_0)}{(1+\cos z_0)q_0} > 0\,, \qquad \beta = \frac{3w_{XX}^2(X_0,Y_0)\,p_0}{5!\,(1+\cos z_0)q_0} > 0\,.$$

Combining (3.38) with (3.39) we obtain

$$\kappa(X-X_0)^2 = \alpha(t-t_0) + \mathcal{O}(1)\cdot|X-X_0|^3. \tag{3.40}$$

Therefore, as shown in Fig. 5 in a neighborhood of (X_0, Y_0) the following holds:

- The two curves $\{t(X,Y)=t_0\}$ and $\{w(X,Y)=\pi\}$ intersect exactly at the point (X_0,Y_0).
- When $\tau<t_0$, the curves $\{t(X,Y)=\tau\}$ and $\{w(X,Y)=\pi\}$ have no intersection.
- When $\tau>t_0$, the curves $\{t(X,Y)=\tau\}$ and $\{w(X,Y)=\pi\}$ have two intersections, at points $P_1=(X_1,Y_1)$ and $P_2=(X_2,Y_2)$ with

$$\begin{cases} X_1-X_0 &= -\sqrt{\dfrac{\alpha}{\kappa}(\tau-t_0)}+\mathcal{O}(1)\cdot(\tau-t_0), \\ X_2-X_0 &= +\sqrt{\dfrac{\alpha}{\kappa}(\tau-t_0)}+\mathcal{O}(1)\cdot(\tau-t_0). \end{cases} \tag{3.41}$$

$$\begin{cases} Y_1-Y_0 &= \alpha(\tau-t_0)+\mathcal{O}(1)\cdot(\tau-t_0)^{3/2}, \\ Y_2-Y_0 &= \alpha(\tau-t_0)+O(1)\cdot(\tau-t_0)^{3/2}. \end{cases} \tag{3.42}$$

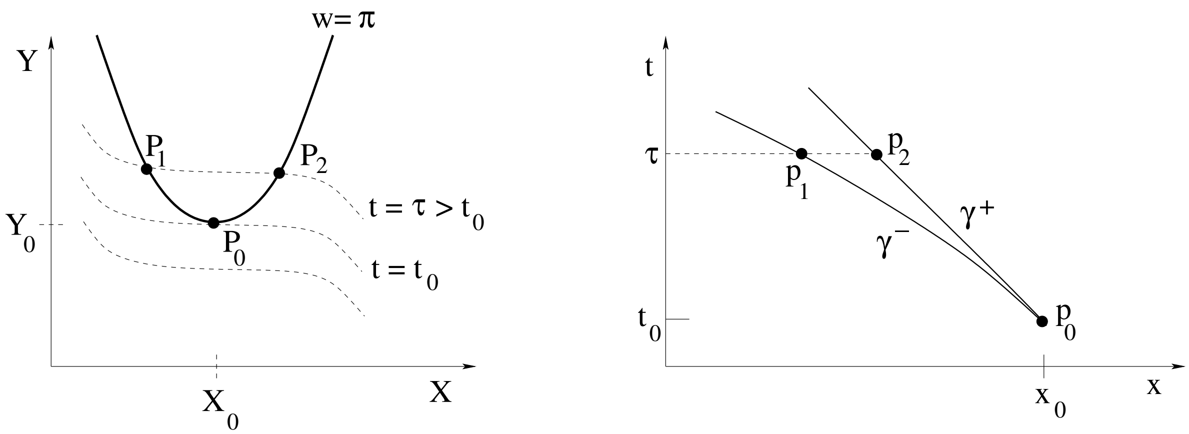

Figure 5: Left: the equation $w(X,Y)=\pi$ implicitly defines a function $Y(X)$ with a strict local minimum at X_0. Under generic conditions, $Y''(X_0)>0$. The dotted curves where $t(X,Y)=\tau$ have 0, 1, or 2 intersections respectively, if $\tau<t_0$, $\tau=t_0$, or $\tau>t_0$. Right: the image of the curve $\{w=\pi\}$ under the map Λ in (2.26) consists of two singular curves γ^-,γ^+ starting at the point $p_0=(x_0,t_0)$. For $\tau>t_0$, the distance between these two curves is $\gamma^+(\tau)-\gamma^-(\tau)=\mathcal{O}(1)\cdot(\tau-t_0)^{5/2}$.

For $t>t_0$, the solution $u=u(t,x)$ is thus singular along two curves γ^-,γ^+ in the x-t plane (see Fig. 5, right). Our next goal is to derive an asymptotic description of these curves in a neighborhood of the point (x_0,t_0), namely

$$\begin{cases} \gamma^-(t) &= x_0-c(u_0)(t-t_0)+\tilde\alpha(t-t_0)^2-\tilde\beta(t-t_0)^{5/2}+\mathcal{O}(1)\cdot(t-t_0)^3, \\ \gamma^+(t) &= x_0-c(u_0)(t-t_0)+\tilde\alpha(t-t_0)^2+\tilde\beta(t-t_0)^{5/2}+\mathcal{O}(1)\cdot(t-t_0)^3, \end{cases} \tag{3.43}$$

for suitable constants $\tilde\alpha,\tilde\beta$.

To prove (3.43), we need to compute more accurate Taylor approximations for t and x near the point (X_0, Y_0).

$$\begin{aligned} t(X,Y) &= t_0 + \frac{1+\cos z_0}{4c(u_0)} q_0 (Y-Y_0) + a(Y-Y_0)^2 + \frac{3w_{XX}^2(X_0,Y_0)}{5!\,4c(u_0)} p_0 (X-X_0)^5 \\ &\quad + \frac{w_Y^2(X_0,Y_0)}{8c(u_0)} p_0 (X-X_0)(Y-Y_0)^2 \\ &\quad + \mathcal{O}(1)\cdot\Big(|X-X_0|^6 + |Y-Y_0|^3 + |X-X_0|^2\,|Y-Y_0|^2\Big), \end{aligned} \tag{3.44}$$

$$\begin{aligned} x(X,Y) &= x_0 - \frac{1+\cos z_0}{4} q_0 (Y-Y_0) + b(Y-Y_0)^2 + \frac{3w_{XX}^2(X_0,Y_0)}{5!\,4} p_0 (X-X_0)^5 \\ &\quad + \frac{w_Y^2(X_0,Y_0)}{8} p_0 (X-X_0)(Y-Y_0)^2 \\ &\quad + \mathcal{O}(1)\cdot\Big(|X-X_0|^6 + |Y-Y_0|^3 + |X-X_0|^2\,|Y-Y_0|^2\Big). \end{aligned} \tag{3.45}$$

The constants a and b are here given by

$$\begin{aligned} a &= -\frac{z_Y \sin z}{8c(u)} q + \frac{1+\cos z}{8c(u)}\left(q_Y - \frac{c'(u)\sin z}{4c^2(u)} q^2\right), \\ b &= \frac{1}{8} z_Y q \sin z - \frac{1}{8}(1+\cos z) q_Y\,, \end{aligned}$$

where the right hand sides are evaluated at the point (X_0, Y_0).

For a fixed $\tau > t_0$, let $P_1 = (X_1, Y_1)$ and $P_2 = (X_2, Y_2)$ be the two points where the curves $\{t(X,Y) = \tau\}$ and $\{w(X,Y) = \pi\}$ intersect. Let $x = \gamma^-(\tau)$ and $x = \gamma^+(\tau)$ describe the corresponding points in the x-t plane (see Fig. 5).

At the intersection point $P_1 = (X_1, Y_1)$, using (3.41) and (3.42) we obtain

$$\begin{aligned} & x(X_1,Y_1) - x_0 + c(u_0)(\tau - t_0) \\ &= (a+b)(Y_1-Y_0)^2 + \frac{3w_{XX}^2(X_0,Y_0)}{5!\,2} p_0 (X_1-X_0)^5 + \frac{w_Y^2(X_0,Y_0)}{4} p_0 (X_1-X_0)(Y_1-Y_0)^2 \\ &\qquad + \mathcal{O}(1)\cdot\Big(|X_1-X_0|^6 + |Y_1-Y_0|^3 + |X_1-X_0|^2\,|Y_1-Y_0|^2\Big) \\ &= \alpha^2(a+b)(\tau-t_0)^2 - \left(\frac{3w_{XX}^2(X_0,Y_0)}{5!\,2\,\kappa^{5/2}} + \frac{w_Y^2(X_0,Y_0)}{4\,\kappa^{1/2}}\right)\alpha^{5/2} p_0 (\tau-t_0)^{5/2} + \mathcal{O}(1)\cdot(\tau-t_0)^3. \end{aligned} \tag{3.46}$$

This yields the equation for γ^- in (3.43), with suitable coefficients $\tilde{\alpha}, \tilde{\beta}$. An entirely similar argument yields the equation for γ^+. In particular, the distance between these two singular curves is

$$\gamma^+(t) - \gamma^-(t) = 2\tilde{\beta}(t-t_0)^{5/2} + \mathcal{O}(1)\cdot|t-t_0|^3. \tag{3.47}$$

3.3 Points where two singular curves cross.

We now consider a point $P = (X_0, Y_0)$ where $w = z = \pi$.

For a generic solution, satisfying the conclusion of Theorem 2, this implies

$$w_X(X_0, Y_0) \neq 0, \qquad z_Y(X_0, Y_0) \neq 0. \tag{3.48}$$

On the other hand, (2.10) yields

$$w_Y(X_0, Y_0) = z_X(X_0, Y_0) = 0.$$

By (2.9) and (2.16), we know that

$$u_X(X_0, Y_0) = u_Y(X_0, Y_0) = u_{XY}(X_0, Y_0) = 0.$$

Hence, in a neighborhood of (X_0, Y_0) the function u can be approximated by

$$u(X,Y) = u_0 - \frac{w_X(X_0,Y_0)}{8c(u_0)} p_0 (X-X_0)^2 - \frac{z_Y(X_0,Y_0)}{8c(u_0)} q_0 (Y-Y_0)^2 + \mathcal{O}(1) \cdot \Big(|X-X_0| + |Y-Y_0|\Big)^3. \tag{3.49}$$

In addition, by (2.12)-(2.13) we have

$$\begin{aligned} t(X,Y) &= t_0 + \frac{w_X^2(X_0,Y_0)}{24c(u_0)} p_0 (X-X_0)^3 + \frac{z_Y^2(X_0,Y_0)}{24c(u_0)} q_0 (Y-Y_0)^3 \\ &\quad + \mathcal{O}(1) \cdot \Big(|X-X_0| + |Y-Y_0|\Big)^4, \end{aligned} \tag{3.50}$$

$$\begin{aligned} x(X,Y) &= x_0 + \frac{w_X^2(X_0,Y_0)}{24} p_0 (X-X_0)^3 - \frac{z_Y^2(X_0,Y_0)}{24} q_0 (Y-Y_0)^3 \\ &\quad + \mathcal{O}(1) \cdot \Big(|X-X_0| + |Y-Y_0|\Big)^4. \end{aligned} \tag{3.51}$$

Using (3.50)-(3.51) in (3.49) we eventually obtain

$$\begin{aligned} u(t,x) &= u(t_0,x_0) - \frac{1}{8c(u_0)} \left(\frac{144\, p_0}{w_X(X_0,Y_0)}\right)^{1/3} \big[c(u_0)(t-t_0) + (x-x_0)\big]^{2/3} \\ &\quad - \frac{1}{8c(u_0)} \left(\frac{144\, q_0}{z_Y(X_0,Y_0)}\right)^{1/3} \big[c(u_0)(t-t_0) - (x-x_0)\big]^{2/3} + \mathcal{O}(1) \cdot \Big(|t-t_0| + |x-x_0|\Big). \end{aligned} \tag{3.52}$$

This proves (3.5). □

4 Dissipative solutions

In this last section we assume $c'(u) > 0$ and study the structure of a dissipative solution in a neighborhood of a point where a new singularity appears. We recall that dissipative solutions can be characterized by the property that R, S in (2.2) are bounded below, on any compact subset of the domain $\{(t,x)\,;\ \ t>0,\ x\in\mathbb{R}\}$. As proved in [3], dissipative solutions can be constructed by the same transformation of variables as in (2.5), (2.7), and (2.8). However, the equations (2.10)-(2.11) should now be replaced by

$$\left\{\begin{array}{rl} w_Y &= \theta\cdot\frac{c'(u)}{8c^2(u)}\,(\cos z-\cos w)\,q\,,\\[2ex] z_X &= \theta\cdot\frac{c'(u)}{8c^2(u)}\,(\cos w-\cos z)\,p\,,\end{array}\right. \tag{4.1}$$

$$\left\{\begin{array}{rl} p_Y &= \theta\cdot\frac{c'(u)}{8c^2(u)}\,\big[\sin z-\sin w\big]\,pq\,,\\[2ex] q_X &= \theta\cdot\frac{c'(u)}{8c^2(u)}\,\big[\sin w-\sin z\big]\,pq\,,\end{array}\right. \tag{4.2}$$

where

$$\theta\ =\ \left\{\begin{array}{ll} 1 & \text{if}\quad \max\{w,z\}<\pi\,,\\ 0 & \text{if}\quad \max\{w,z\}\geq\pi\,.\end{array}\right. \tag{4.3}$$

Notice that, by setting $\theta\equiv 1$, one would again recover the conservative solutions.

It is interesting to compare a conservative and a dissipative solution, with the same initial data. Consider a point $P=(X_0,Y_0)$ of Type 2, where two new singular curves γ^-,γ^+ originate, in the conservative solution. To fix the ideas, assume that the singularity occurs in backward moving waves, so that $R\to+\infty$ but S remains bounded. Moreover, let the conditions (3.26) and (3.37) hold.

Up to the time $t_0=t(X_0,Y_0)$ where the singularity appears, the conservative and the dissipative solution coincide. For $t>t_0$, they still coincide outside the domain

$$\Omega\ \doteq\ \big\{(x,t)\,;\qquad t\geq t_0\,,\quad \gamma^-(t)\leq x\leq\tilde\gamma(t)\big\}, \tag{4.4}$$

where $\tilde\gamma$ is the forward characteristic through the point (x_0,t_0). Figure 6 shows the positions of these singularities in the X-Y plane and in the x-t plane. Figure 7 illustrates the difference in the profiles of the two solutions for $t>t_0$. Our results can be summarized as follows.

Theorem 4. *In the above setting, the conservative solution $u^{cons}(t,\cdot)$ has two strong singularities at $x=\gamma^-(t)$ and $x=\gamma^+(t)$, where $|u_x^{cons}|\to\infty$, and is smooth at all other points.*

On the other hand, the dissipative solution $u^{diss}(t,\cdot)$ has a strong singularity at $x=\gamma^-(t)$, where $|u_x^{diss}|\to\infty$, and a weak singularity along the forward characteristic $x=\tilde\gamma(t)$, where u_x^{diss} is continuous but the second derivative u_{xx}^{diss} does not exist.

The difference between these two solutions can be estimated as

$$\big\|u^{cons}(t,\cdot)-u^{diss}(t,\cdot)\big\|_{\mathcal{C}^0(\mathbb{R})}\ =\ \mathcal{O}(1)\cdot(t-t_0). \tag{4.5}$$

Proof. 1. To fix the ideas, assume that at the point $P=(X_0,Y_0)$ where the singularity is formed one has

$$w_{XX}\ <\ 0,\qquad\qquad w_Y\ >\ 0,\qquad\qquad c'(u)\ >\ 0.$$

In the X-Y coordinates, for smooth initial data the components (x,t,u,w,z,p,q) of the conservative solution remain globally smooth. On the other hand, for a dissipative solution by (4.1)-(4.2) we only know that these components are Lipschitz continuous.

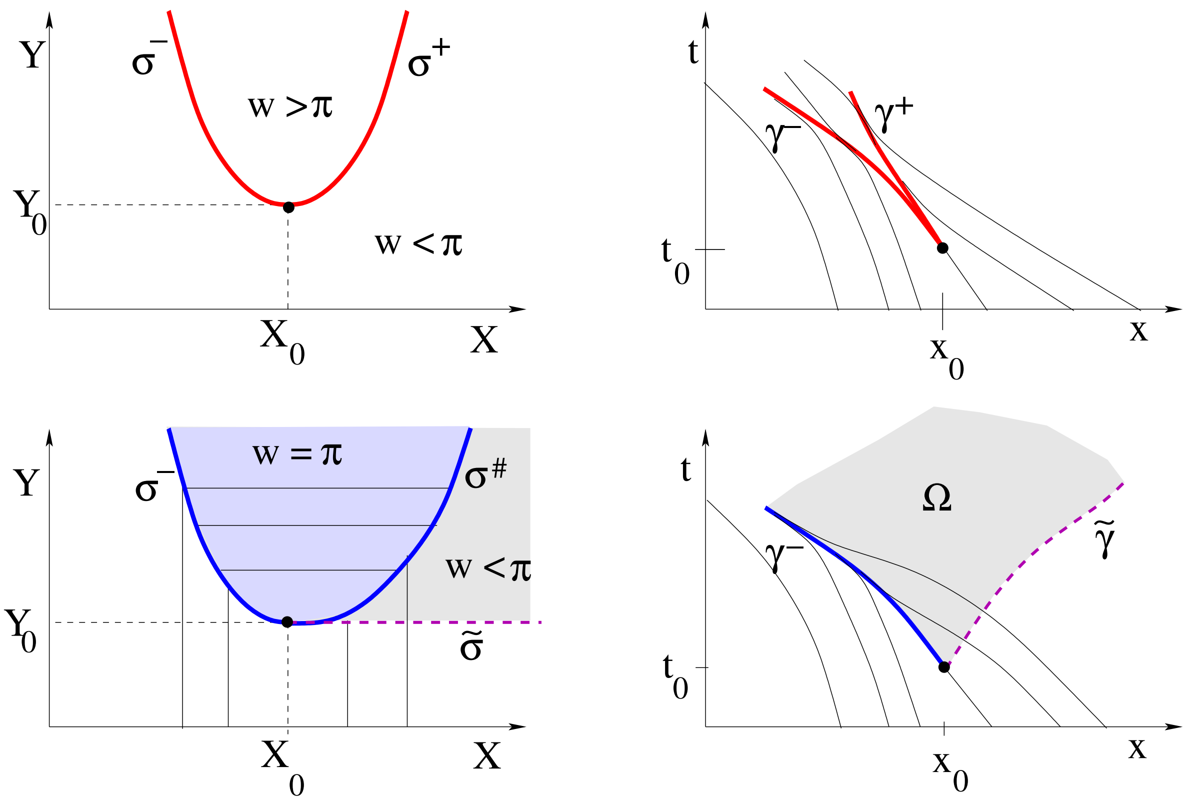

Figure 6: The positions of the singularities in the X-Y plane and in the x-t plane. This refers to a point where a new singularity is formed, in the first family (i.e., for backward moving waves). Above: a conservative solution. Below: a dissipative solution. Notice that the entire region between the curves σ^- and $\sigma^\sharp$ is mapped onto the single curve γ^-. Indeed, horizontal segments in the X-Y plane are mapped into a single point. In the x-t plane, the two solutions differ only on the set Ω, bounded by the characteristic curves γ^- (the image of both σ^- and $\sigma^\sharp$) and $\tilde\gamma$ (the image of the line $\tilde\sigma$).

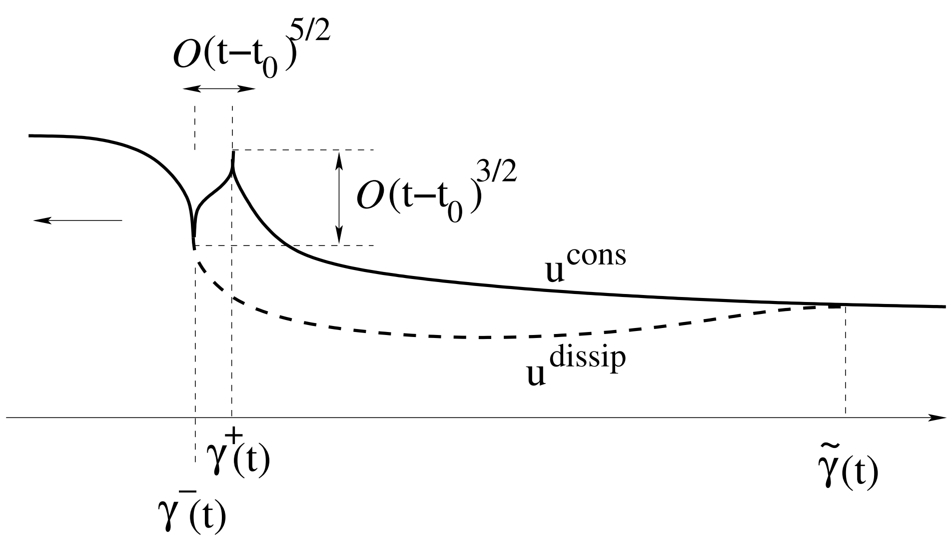

Figure 7: Comparing a conservative and a dissipative solution, at a time $t > t_0$, after a singularity has appeared. The conservative solution has two strong singularities at $\gamma^-(t) < \gamma^+(t)$, while the dissipative solution has a strong singularity at $\gamma^-(t)$ and a weak singularity at $\tilde\gamma(t)$. The two solutions coincide for $x \leq \gamma^-(t)$ and for $x \geq \tilde\gamma(t)$.

2. For $X \geq X_0$ we denote by $Y = \sigma^\sharp(X)$ the curve where $w = \pi$, in the dissipative solution. A Taylor approximation for $\sigma^\sharp$ is derived from the identities

$$w(X, Y_0) \;=\; w_0 + w_{XX}(X_0, Y_0) \cdot \frac{(X - X_0)^2}{2} + \mathcal{O}(1) \cdot (X - X_0)^3,$$

$$w_Y(X, Y) \;=\; w_Y(X_0, Y_0) + \mathcal{O}(1) \cdot \big(|X - X_0| + |Y - Y_0|\big),$$

valid in the region where $w < \pi$. Together, they imply

$$\sigma^\sharp(X) \;=\; Y_0 + \kappa(X - X_0)^2 + \mathcal{O}(1) \cdot (X - X_0)^3, \tag{4.6}$$

where $\kappa > 0$ is the same constant found in (3.37) for the conservative solution.

For $Y' > Y_0$, (3.38) and (4.6) together imply

$$X^\sharp(Y') - X^-(Y') \;=\; 2\left(\frac{Y' - Y_0}{\kappa}\right)^{1/2} + \mathcal{O}(1) \cdot |Y' - Y_0|\,. \tag{4.7}$$

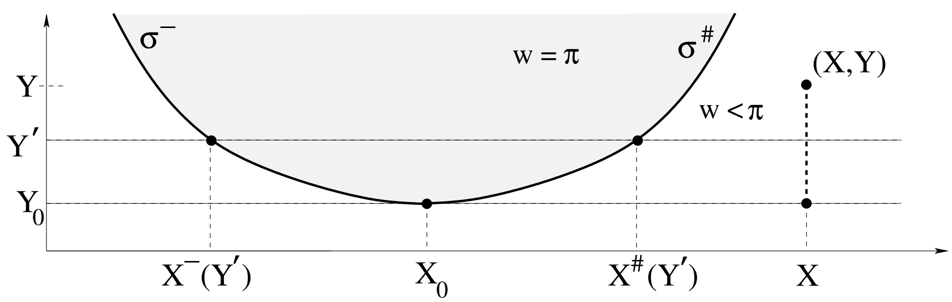

Figure 8: Estimating the values of a dissipative solution near a singularity. Notice that the functions x, t, u are constant on every horizontal segment contained in the shaded region where $w = \pi$.

3. Consider a point (X, Y) with $X > X_0$ and $Y \leq \sigma^\sharp(X)$. By the second equation in (2.9) it follows

$$u(X, Y) \;=\; u(X, Y_0) + \int_{Y_0}^{Y} \left(\frac{\sin z}{4c(u)}\, q\right)(X, Y')\, dY'\,. \tag{4.8}$$

As in Fig. 8, for $Y' \in [Y_0, Y]$, call $X^-(Y')$ and $X^\sharp(Y')$ respectively the points where $\sigma^-(X) = Y'$ and $\sigma^\sharp(X) = Y'$. Since $z_X = q_X = 0$ when $w = \pi$, by the second equations in (4.1) and in (4.2) we have

$$\begin{aligned} z(X, Y') \;&=\; z(X^-(Y'), Y') + \int_{X^-(Y')}^{X} z_X(X', Y')\, dX' \\ &=\; z(X^-(Y'), Y') + \int_{X^\sharp(Y')}^{X} \left(\frac{c'(u)}{8c^2(u)}\,(\cos w - \cos z)\, p\right)(X', Y')\, dX', \end{aligned} \tag{4.9}$$

$$q(X, Y') \;=\; q(X^-(Y'), Y') + \int_{X^\sharp(Y')}^{X} \left(\frac{c'(u)}{8c^2(u)}\,\big[\sin w - \sin z\big]\, pq\right)(X', Y')\, dX'. \tag{4.10}$$

4. For notational convenience, in the following we denote by $(x, t, u, w, z, p, q)(X, Y)$ the components describing a dissipative solution, and by $(\hat{x}, \hat{t}, \hat{u}, \hat{w}, \hat{z}, \hat{p}, \hat{q})(X, Y)$ the corresponding components of the conservative solution. We observe that all these functions are Lipschitz continuous. As shown in Fig. 6, these two solutions can be different only at points (X, Y) in the region bounded by the curves σ^- and $\tilde{\sigma}$, namely

$$\big\{X \le X_0\,, \quad Y > \sigma^-(X)\big\} \,\cup\, \big\{X \ge X_0\,, \quad Y > Y_0\big\}.$$

Consider a point (X, Y') with $X > X_0$, $Y' < \sigma^\sharp(X)$. By (4.1), observing that $z = \hat{z}$ for $Y \le Y_0$ and using (4.7) we find

$$\begin{aligned}
&\hat{z}(X, Y') - z(X, Y') \\
&= \int_{X^-(Y')}^{X^\sharp(Y')} \hat{z}_X(X', Y')\,dX' + \int_{X^\sharp(Y')}^{X} (\hat{z}_X - z_X)\,(X', Y')\,dX' \\
&= -\frac{c'(u_0)(1+\cos z_0)}{8c^2(u_0)}\,p_0 \cdot \big(X^\sharp(Y') - X^-(Y')\big) \\
&\quad +\mathcal{O}(1)\cdot\big(X^\sharp(Y') - X^-(Y')\big)^2 + \mathcal{O}(1)\cdot\big(X - X^\sharp(Y')\big)\,(Y' - Y_0) \\
&= -\frac{c'(u_0)(1+\cos z_0)}{4c^2(u_0)\,\kappa^{1/2}}\,p_0 \cdot (Y' - Y_0)^{1/2} + \mathcal{O}(1)\cdot|Y' - Y_0|\,.
\end{aligned} \tag{4.11}$$

By (4.2), a similar computation yields

$$\begin{aligned}
&\hat{q}(X, Y') - q(X, Y') \\
&= \int_{X^-(Y')}^{X^\sharp(Y')} \hat{q}_X(X', Y')\,dX' + \int_{X^\sharp(Y')}^{X} (\hat{q}_X - q_X)\,(X', Y')\,dX' \\
&= -\frac{c'(u_0)\sin z_0}{8c^2(u_0)}\,p_0 q_0 \cdot \big(X^\sharp(Y') - X^-(Y')\big) \\
&\quad +\mathcal{O}(1)\cdot\big(X^\sharp(Y') - X^-(Y')\big)^2 + \mathcal{O}(1)\cdot\big(X - X^\sharp(Y')\big)\,(Y' - Y_0) \\
&= -\frac{c'(u_0)\sin z_0}{4c^2(u_0)\,\kappa^{1/2}}\,p_0 q_0 \cdot (Y' - Y_0)^{1/2} + \mathcal{O}(1)\cdot|Y' - Y_0|\,.
\end{aligned} \tag{4.12}$$

Next, using the second equation in (2.9) and recalling that $u(X, Y_0) = \hat{u}(X, Y_0)$, for any

$X \in [X_0,\ X_0+1]$ and $Y \in [Y_0, \sigma^\sharp(X)]$ we obtain

$$
\begin{aligned}
\hat u(X,Y) - u(X,Y) \ &= \ \int_{Y_0}^{Y} \left(\frac{\sin \hat z}{4c(\hat u)}\,\hat q - \frac{\sin z}{4c(u)}\,q \right)(X,Y')\,dY' \\
&= \ \int_{Y_0}^{Y} \left(\frac{\cos z_0}{4c(u_0)}\,q_0\cdot(\hat z - z) + \frac{\sin z_0}{4c(u_0)}\,\cdot(\hat q - q) - \frac{c'(u_0)\sin z_0}{4c^2(u_0)}\,q_0\cdot(\hat u - u)\right)(X,Y')\,dY' \\
&\quad +\mathcal{O}(1)\cdot \int_{Y_0}^{Y} \Big(|\hat z - z|^2 + |\hat q - q|^2 + |\hat u - u|^2\Big)(X,Y')\,dY' \\
&\quad +\mathcal{O}(1)\cdot \Big(|X-X_0| + |Y-Y_0|\Big)\cdot \int_{Y_0}^{Y} \Big(|\hat z - z| + |\hat q - q| + |\hat u - u|\Big)(X,Y')\,dY' \\
&= \ \eta_0\cdot (Y-Y_0)^{3/2} + \mathcal{O}(1)\cdot |Y-Y_0|^2,
\end{aligned}
\tag{4.13}
$$

where the constant η_0 is computed by

$$
\eta_0 \ = \ \frac{2}{3}\left[-\frac{c'(u_0)(1+\cos z_0)}{4c^2(u_0)\,\kappa^{1/2}}\,p_0\right]\cdot \frac{\cos z_0}{4c(u_0)}\,q_0 + \frac{2}{3}\left[-\frac{c'(u_0)\,\sin z_0}{4c^2(u_0)\,\kappa^{1/2}}\,p_0 q_0\right]\cdot \frac{\sin z_0}{4c(u_0)}\,. \tag{4.14}
$$

5. Using the second equations in (2.12) and in (2.13), we obtain similar estimates for the variables x, t. Namely,

$$
\begin{aligned}
\hat x(X,Y) - x(X,Y) \ &= \ -\int_{Y_0}^{Y} \left(\frac{1+\cos \hat z}{4}\,\hat q - \frac{1+\cos z}{4}\,q \right)(X,Y')\,dY' \\
&= \ \int_{Y_0}^{Y} \left(\frac{\sin z_0}{4}\,q_0\cdot(\hat z - z) - \frac{1+\cos z_0}{4}\,\cdot(\hat q - q)\right)(X,Y')\,dY' \\
&\quad +\mathcal{O}(1)\cdot \int_{Y_0}^{Y} \Big(|\hat z - z|^2 + |\hat q - q|^2\Big)(X,Y')\,dY' \\
&\quad +\mathcal{O}(1)\cdot \Big(|X-X_0| + |Y-Y_0|\Big)\cdot \int_{Y_0}^{Y} \Big(|\hat z - z| + |\hat q - q|\Big)(X,Y')\,dY' \\
&= \ \mathcal{O}(1)\cdot |Y-Y_0|^2.
\end{aligned}
\tag{4.15}
$$

Indeed, the coefficient of the leading order term $\mathcal{O}(1)\cdot(Y-Y_0)^{3/2}$ vanishes. Similarly,

$$\begin{aligned}
\hat t(X,Y)-t(X,Y) \;&=\; -\int_{Y_0}^{Y}\left(\frac{1+\cos\hat z}{4c(\hat u)}\,\hat q-\frac{1+\cos z}{4c(u)}\,q\right)(X,Y')\,dY' \\
&= \int_{Y_0}^{Y}\left(\frac{\sin z_0}{4c(u_0)}\,q_0\cdot(\hat z-z)-\frac{1+\cos z_0}{4c(u_0)}\cdot(\hat q-q)+\frac{1+\cos z_0}{4c^2(u_0)}c'(u_0)q_0\cdot(\hat u-u)\right)(X,Y')\,dY' \\
&\quad +\mathcal{O}(1)\cdot\int_{Y_0}^{Y}\Big(|\hat z-z|^2+|\hat q-q|^2+|\hat u-u|^2\Big)(X,Y')\,dY' \\
&\quad +\mathcal{O}(1)\cdot\Big(|X-X_0|+|Y-Y_0|\Big)\cdot\int_{Y_0}^{Y}\Big(|\hat z-z|+|\hat q-q|+|\hat u-u|\Big)(X,Y')\,dY' \\
&= \mathcal{O}(1)\cdot|Y-Y_0|^2.
\end{aligned}\tag{4.16}$$

6. The estimate (4.13) provides a bound on the difference $\hat u-u$ between a conservative and a dissipative solution, at a given point (X,Y). However, our main goal is to estimate the difference $\hat u-u$ as functions of the original variables x,t. For this purpose, consider a dissipative solution u and a point

$$P \;=\; (x,t) \;=\; \big(x(X,Y),\, t(X,Y)\big),\tag{4.17}$$

with

$$X \;>\; X_0, \qquad\qquad Y_0 \;<\; Y \;<\; \sigma^\sharp(X).\tag{4.18}$$

Moreover, let $\hat u$ be the conservative solution with the same initial data, and let $(\widehat X,\widehat Y)$ be the point which is mapped to P in the conservative solution, so that

$$P \;=\; (x,t) \;=\; \big(\hat x(\widehat X,\widehat Y),\hat t(\widehat X,\widehat Y)\big).\tag{4.19}$$

Using (4.13), (4.15), (4.16), and recalling that the conservative solution $\hat u=\hat u(x,t)$ is Hölder continuous of exponent $1/2$ w.r.t. both variables x,t, we obtain

$$\begin{aligned}
|\hat u(x,t)-u(x,t)| \;&\le\; \big|\hat u(\widehat X,\widehat Y)-\hat u(X,Y)\big|+\big|\hat u(X,Y)-u(X,Y)\big| \\
&= \mathcal{O}(1)\cdot\Big(\big|\hat x(\widehat X,\widehat Y)-\hat x(X,Y)\big|^{1/2}+\big|\hat t(\widehat X,\widehat Y)-\hat t(X,Y)\big|^{1/2}\Big)+\mathcal{O}(1)\cdot|Y-Y_0|^{3/2} \\
&= \mathcal{O}(1)\cdot\Big(\big|x(X,Y)-\hat x(X,Y)\big|^{1/2}+\big|t(X,Y)-\hat t(X,Y)\big|^{1/2}\Big)+\mathcal{O}(1)\cdot|Y-Y_0|^{3/2} \\
&= \mathcal{O}(1)\cdot|Y-Y_0|.
\end{aligned}\tag{4.20}$$

In a neighborhood of (x_0,t_0) we have

$$t_Y \;=\; \frac{1+\cos z}{4c(u)}\,q \;>\; \frac{1+\cos z_0}{5c(u_0)}q_0 \;>0\,.\tag{4.21}$$

For (x,t) as in (4.17)-(4.18), one has

$$t-t_0 \;=\; \big[t(X,Y)-t(X,Y_0)\big]+\big[t(X,Y_0)-t(X_0,Y_0)\big] \;\ge\; \frac{1+\cos z_0}{5c(u_0)}q_0\cdot|Y-Y_0|\,.$$

Together with (4.20), this proves (4.5).

7. It remains to prove that the solution $u = u(t,x)$ is not twice differentiable along the forward characteristic $\tilde{\gamma}$.

Consider a point $(x_1, t_1) = \big(x(X, Y_0), t(X, Y_0)\big)$ on $\tilde{\gamma}$, with $X > X_0$. Let $x = \gamma(t)$ be the backward characteristic through (x_1, t_1), so that

$$\gamma(t_1) = x_1, \qquad\qquad \dot{\gamma}(t) \;=\; c(u(\gamma(t), t)).$$

Assume that u were twice differentiable at the point (x_1, t_1). Then the map $t \mapsto u(\gamma(t), t)$ would also be twice differentiable at $t = t_1$. Indeed

$$\frac{d}{dt} u\big(\gamma(t), t\big) \;=\; -\,c(u) u_x + u_t\,,$$

$$\frac{d^2}{dt^2} u\big(\gamma(t), t\big) \;=\; -\,c'(u)\,\big(-c(u)u_x + u_t\big)u_x + c^2(u)u_{xx} - 2c(u)u_{xt} + u_{tt}\,. \tag{4.22}$$

To reach a contradiction, consider the map $\tau \mapsto Y(\tau)$ implicitly defined by

$$t(X, Y(\tau)) \;=\; \tau\,.$$

By (4.21) this map is well defined. In particular, $Y(t_1) = Y_0$. We thus have

$$\frac{d}{dt} u\big(\gamma(t), t\big) \;=\; \frac{d}{dt} u(X, Y(t)) \;=\; u_Y \cdot \frac{4c(u)}{(1 + \cos z)q} \;=\; \frac{\sin z}{1 + \cos z}\,. \tag{4.23}$$

We now show that this first derivative cannot be a Lipschitz continuous function of time, for $t \approx t_1$. Indeed, by (4.23) and the mean value theorem we have

$$\begin{aligned} \frac{d}{dt} u(X, Y(t)) - \frac{d}{dt} u(X, Y(t_1)) \;&=\; \frac{\sin z(X, Y(t))}{1 + \cos z(X, Y(t))} - \frac{\sin z(X, Y_0)}{1 + \cos z(X, Y_0)} \\ &=\; \frac{1}{1 + \cos z(X, Y^\sharp)} \cdot \big[z(X, Y(t)) - z(X, Y_0)\big], \end{aligned} \tag{4.24}$$

for some intermediate value $Y^\sharp \in [Y_0, Y(t)]$. Call $\hat{z} = \hat{z}(X, Y)$ the corresponding conservative solution. Observe that $\hat{z}$ is smooth and coincides with z on the horizontal line $\{Y = Y_0\}$. Using (4.11) we obtain

$$\begin{aligned} \big|z(X, Y(t)) - z(X, Y_0)\big| \;&\geq\; \big[z(X, Y(t)) - \hat{z}(X, Y(t))\big] - \big[\hat{z}(X, Y(t)) - \hat{z}(X, Y_0)\big] \\ &\geq\; \frac{c'(u_0)(1 + \cos z_0)\,p_0}{4c^2(u_0)\,\kappa^{1/2}} \cdot (Y(t) - Y_0)^{1/2} - \mathcal{O}(1) \cdot |Y(t) - Y_0|\,. \end{aligned}$$

As a consequence, for $t \approx t_1$, the function $t \mapsto z(X, Y(t))$ is not Lipschitz continuous, and the same applies to the left hand side of (4.24). We thus conclude that the map $t \mapsto u(\gamma(t), t)$ cannot be twice differentiable at $t = t_1$, in contradiction with (4.22). This completes the proof of Theorem 4. □

Acknowledgment. This research was partially supported by NSF, with grant DMS-1411786: "Hyperbolic Conservation Laws and Applications".

References

[1] A. Bressan and G. Chen, Generic regularity of conservative solutions to a nonlinear wave equation, submitted.

[2] A. Bressan, G. Chen, and Q. Zhang, Unique conservative solutions to a nonlinear wave equation. *Arch. Rational Mech. Anal.*, to appear.

[3] A. Bressan and T. Huang, Representation of dissipative solutions to a nonlinear variational wave equation, *Comm. Math. Sci.*, to appear.

[4] A. Bressan and Y. Zheng, Conservative solutions to a nonlinear variational wave equation, *Comm. Math. Phys.* **266** (2006), 471–497.

[5] J. Damon, Generic properties of solutions to partial differential equations. *Arch. Rational Mech. Anal.* **140** (1997) 353–403.

[6] J-G. Dubois and J-P. Dufour, Singularités de solutions d'équations aux dérivées partielles. *J. Differential Equations* **60** (1985), 174–200.

[7] R. T. Glassey, J. K. Hunter and Y. Zheng, Singularities in a nonlinear variational wave equation, *J. Differential Equations*, **129** (1996), 49–78.

[8] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities.* Springer-Verlag, New York, 1973.

[9] J. Guckenheimer, Catastrophes and partial differential equations. *Ann. Inst. Fourier* **23** (1973), 31–59.

[10] H. Holden and X. Raynaud, Global semigroup of conservative solutions of the nonlinear variational wave equation. *Arch. Rational Mech. Anal.* **201** (2011), 871–964.

[11] D. Schaeffer, A regularity theorem for conservation laws, *Adv. in Math.* **11** (1973), 368–386.

[12] P. Zhang and Y. Zheng, Weak solutions to a nonlinear variational wave equation. *Arch. Ration. Mech. Anal.* **166** (2003), 303–319.

[13] P. Zhang and Y. Zheng, Weak solutions to a nonlinear variational wave equation with general data. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005), 207–226.