JOURNAL OF MODERN DYNAMICS
VOLUME 6, NO. 2, 2012, 139–182
doi: 10.3934/jmd.2012.6.139

ON THE BRIN PRIZE WORK OF ARTUR AVILA IN TEICHMÜLLER DYNAMICS AND INTERVAL-EXCHANGE TRANSFORMATIONS

GIOVANNI FORNI
(Communicated by Anatole Katok)

ABSTRACT. We review the Brin prize work of Artur Avila on Teichmüller dynamics and Interval Exchange Transformations. The paper is a nontechnical self-contained summary that intends to shed some light on Avila's early approach to the subject and on the significance of his achievements.

CONTENTS

1. INTRODUCTION

The Brin prize work of Artur Avila includes two papers related to Teichmüller dynamics and interval-exchange transformations. In the paper [4], Avila and the author solve a long standing open problem in ergodic theory by proving that almost all irreducible interval-exchange transformations (that are not rotations) are weakly mixing. In [8], Avila and M. Viana prove the simplicity of the so-called Kontsevich–Zorich spectrum, that is, the Lyapunov spectrum of the Kontsevich–Zorich cocycle, thereby finishing off the proof of the Kontsevich–Zorich conjecture, partially achieved by the author of this paper [21]. It should be noted that the approach of Avila–Viana's paper [8] is different from that of

2010 *Mathematics Subject Classification:* Primary: 37-02, 37D25, 37A25, 37E05, 37E20, 37E35; Secondary: 32G15, 30F60.
Key words and phrases: Brin prize, Avila.

[21] and provides a completely independent proof of the full Kontsevich–Zorich conjecture (or at least its spectral part). Avila's early work on the subject includes two other papers, namely [6] and [3], which are closely related to the Prize work. In [6], Avila, S. Gouëzel and J.-C. Yoccoz prove that the Teichmüller flow has exponential decay of correlations (for observables in the Ratner class) and the $SL(2,\mathbb{R})$ action on the moduli space of Abelian differentials has a spectral gap (in the sense of analysis or representation theory). This result answers a natural question in Teichmüller dynamics asked by several authors after Veech's proof of the mixing property [52]. In the paper [3], Avila and A. Bufetov prove exponential decay of correlations for a certain return map of the Teichmüller flow to a noncompact section of infinite transverse volume, called the Rauzy–Veech–Zorich induction. This work is a spin-off of [6] and of Bufetov's thesis work (which contains a weaker result for the same system) and will not be examined in this paper.

Among the more recent contributions of Avila on the subject, his joint paper with M. J. Resende generalizes the result of [6] on exponential decay of correlations of the Teichmüller flow to strata of quadratic differentials and his joint paper with S. Gouëzel goes further to prove a precise spectral gap result for all algebraic $SL(2,\mathbb{R})$ measures on the moduli space of Abelian differentials. The latter paper, quite remarkable, represents a departure from previous work of Avila on the subject. In fact, as we will outline in this paper, Avila's approach to Teichmüller flow in his early work is via a well-known symbolic model based on the Rauzy–Veech induction map and the techniques are often combinatorial. The paper [5] does not rely on the aforementioned symbolic model and applies analytic techniques developed in the study of mixing for Anosov flows together with geometric estimates. It is beyond the scope of our paper. Avila's work on Teichmüller dynamics is tied to his work on one-dimensional dynamics and Schrödinger operators by the emphasis on *renormalization* methods in dynamical systems. In fact, the Rauzy–Veech–Zorich induction and the Teichmüller flow are renormalization dynamics for interval exchange transformations and their suspensions, that is, for translation flows. They respectively generalize the Gauss map and the hyperbolic geodesic flow on the modular surface, classical examples of renormalization systems (for rotations of the circle and for linear flows on tori).

The two Brin prize papers [4] and [8] and the related paper [6] will be the focus of our review. We will describe the main results and ideas in these papers emphasizing common threads with the goal of giving an account of Avila's early point of view on the subject. It should be noted that, because of publishing delays, the above three papers were published in reverse chronological order with respect to the order in which they were written between 2004 and 2005.

The plan of this this paper is as follows. In Section 2 we review the fundamental definitions and results of the theory of interval-exchange transformations, of the Rauzy–Veech–Zorich cocycle, of the zippered rectangles and of the Teichmüller flow and state the main results of Avila in this subject. In Section 3 we give a nontechnical outline of the main ideas in Avila's approach to the dynamics of the Rauzy–Veech–Zorich cocycle and of the proofs of his main theorems.

2. MAIN RESULTS

In this section we review basic notions and results on the dynamics interval-exchange transformations and of the associated renormalization schemes, the Rauzy–Veech–Zorich induction and the Teichmüller flow, and state Avila's main results on the subject. There are several excellent surveys on interval-exchange transformations and Teichmüller dynamics (see for instance [62, 53, 55]) and we refer the reader to those for a complete introduction to the subject.

2.1. **Interval Exchange Transformations.** Interval Exchange Transformations can be briefly defined as piecewise continuous isometries (translations) of a finite interval with finitely many discontinuities. They were introduced by Oseledets [44] motivated by a question of Arnold [1]. An important source of motivation for their study also came in the '80s from work of Novikov and its school on Hamiltonians with multivalued first integrals (locally Hamiltonian vector fields), in particular semiclassical models of the motion of an electron on the Fermi surface of an atom and related questions (see [61] and references therein). From the point of view of dynamical systems, interval-exchange transformations and their suspensions are the simplest generalization of circle rotations and linear flows on two-dimensional tori.

Let $d \geq 2$ be a natural number and let π be an *irreducible* permutation of $\{1,\ldots,d\}$, that is, $\pi\{1,\ldots,k\} \neq \{1,\ldots,k\}$, $1 \leq k < d$. Given $\lambda \in \mathbb{R}^d_+$, the *interval-exchange transformation* $f := f(\lambda,\pi)$ is the map defined as follows (see [44, 16, 32]): we consider the interval

$$I := I(\lambda) = \left[0, \sum_{i=1}^{d} \lambda_i\right) ,$$

break it into subintervals

$$I_i := I_i(\lambda,\pi) = \left(\sum_{j<i} \lambda_j, \sum_{j\leq i} \lambda_j\right), \quad 1 \leq i \leq d,$$

and rearrange the I_i according to π (in the sense that the i-th interval is mapped onto the $\pi(i)$-th interval). In other words, $f\colon I \to I$ is given by the formula

$$x \mapsto x + \sum_{\pi(j)<\pi(i)} \lambda_j - \sum_{j<i} \lambda_j , \quad x \in I_i .$$

Interval-exchange transformations on two subintervals ($d = 2$) are isomorphic to circle rotations, interval-exchange transformations on three subintervals can all be obtained as return maps of a circle rotation to a subinterval. Thus the minimum number of subintervals for genuinely new dynamical phenomena is four. Most results on the dynamics of interval-exchange transformations hold for almost all $\lambda \in \mathbb{R}^d_+$. A condition, due to Keane, called the *Keane condition* [32], which holds everywhere but for a countable union of subspaces of the parameter space implies *minimality* of interval-exchange transformations (see also [44]).

One of the reasons interval-exchange transformations are interesting from a dynamical systems point of view is the fact that not all minimal interval-exchange transformations on at least four intervals are uniquely ergodic, in contrast with the case of rotations. Nevertheless, by the *Keane conjecture*, eventually proved independently by H. Masur [41] and W. Veech [50] using different renormalization methods, almost all (irreducible) interval-exchange transformations are *uniquely ergodic*. A. Katok [30] had proved earlier that interval-exchange transformations are *never mixing*. It is thus fair to say that after the proof of the Keane conjecture the question whether almost all interval-exchange transformations are *weakly mixing* became the main open problem on the dynamics of interval-exchange transformations.

2.1.1. *Weak mixing.* We recall that the weak mixing property of a measurable map f on a probability space (X,μ) is intermediate between ergodicity and mixing and has several equivalent formulations. For instance, the map f is weakly mixing if and only if, for all pairs of measurable sets A, $B \subset X$,

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=0}^{N} |\mu(f^{-n}(A)\cap B) - \mu(A)\mu(B)| = 0\,. \tag{1}$$

Another equivalent formulation is the existence of *mixing sequences* of density one, that is, of subsets $I \subset \mathbb{N}$ of upper density one such that

$$\lim_{n\in I} \mu(f^{-n}(A)\cap B) = \mu(A)\mu(B)\,. \tag{2}$$

Weak mixing is an important *spectral property*, equivalent to the statement that the unitary action of f on $L^2(X,d\mu)$ has no nonconstant eigenfunctions (that is, the spectrum is continuous). It is the latter characterization that is applied in most proofs of weak mixing, including Avila's work [4] as we shall see below.

Avila's main result on interval-exchange transformations [4] establishes weak mixing for almost all interval-exchange transformations that are not rotations.

THEOREM 1 ([4])**.** *Let π be any irreducible permutation of $\{1,\dots,d\}$ that is not a rotation. For almost every $\lambda\in\mathbb{R}^d_+$, the map $f(\lambda,\pi)$ is weakly mixing.*

Katok and Stepin [31] had proven much earlier that interval-exchange transformations on three subintervals are indeed almost always weakly mixing, so it was natural to conjecture that the same conclusion should hold for interval-exchange transformations on at least four subintervals that are not rotations. The first important partial result was obtained by Veech [51] who proved weak mixing for almost all parameter values and for infinitely many special permutations. The Veech criterion for weak mixing (see below) will also be the starting point of Avila's work [4] (although the authors did not know Veech's work, it did not take long for Avila to rederive the Veech criterion). Later Nogueira and

Rudolph [43] were able to prove topological weak mixing for general permutations. It is interesting to note that interval-exchange transformations can be topologically mixing as proved recently by Jon Chaika [15].

2.1.2. *The Kontsevich–Zorich conjecture.* After the proof of the Keane conjecture, another crucial development in the theory of interval-exchange transformations came in the '90s with A. Zorich's work (in collaboration with M. Kontsevich). It is interesting to note that the Kontsevich–Zorich work led to advances that also turned out to play a key role in the weak-mixing result stated above. Zorich became interested in deviations of ergodic averages from the mean, which he discovered numerically, after his work on Novikov's problem on the semiclassical motion of an electron in a homogeneous magnetic field [56, 61].

We briefly summarize below the main aspects of the Kontsevich–Zorich work on deviations of ergodic averages. Let π be any irreducible permutation of $\{1,\dots,d\}$, let $\lambda \in \mathbb{R}^d_+$ and let $f := f(\pi,\lambda)$ be the corresponding interval-exchange transformation. Let $I_i = I_i(\lambda)$ denote as above the subintervals of continuity of f on the interval $I := I(\lambda)$. For all $i \in \{1,\dots,d\}$ and for all $n \in \mathbb{N}$, let

$$N_i(x,n) := \#\{k \in \{0,\dots,n-1\} \,|\, f^k(x) \in I_i\}\,. \tag{3}$$

By unique ergodicity (the Keane conjecture, proved by Masur [41] and Veech [50]), it follows that, for almost all $\lambda \in \mathbb{R}^d_+$, for all $i \in \{1,\dots,d\}$ and for all $x \in I$ with infinite orbit

$$\lim_{n\to+\infty} \frac{N_i(x,n)}{n} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{I_i} \circ f^k(x) = \int_I \chi_{I_i}(x)\,dx = \lambda_i\,.$$

Zorich (see [59, 61]) proved the following result on the deviation of ergodic averages of interval-exchange transformations. Let $g := g(\pi) \ge 1$ be the genus of the surface obtained by suspension of an interval-exchange transformation (see Section 2.3.2 below). The genus only depends on the number of subintervals and on the permutation. It is one for interval-exchange transformations on two or three subintervals and higher than one for interval-exchange transformations on at least four intervals for most permutations. For all $(x,n) \in I \times \mathbb{N}$, let $N(x,n) := (N_1(x,n),\dots,N_d(x,n))$. There exist deviation exponents

$$\nu_1 = 1 > \nu_2 \ge \cdots \ge \nu_g \ge \nu_{g+1} = 0 \tag{4}$$

and, for almost all $\lambda \in \mathbb{R}^d_+$, there exists a filtration of subspaces

$$V_1(\lambda) = \mathbb{R}\lambda \subset V_2(\lambda) \subset \cdots \subset V_g(\lambda) \subset V_{g+1} \subset \mathbb{R}^d$$

such that whenever $\Phi \in V_s(\lambda)^\perp \setminus V_{s+1}(\lambda)^\perp$, for all $x \in I$ with infinite orbit,

$$\limsup_{n\to+\infty} \frac{\log|\langle \Phi, N(x,n)\rangle|}{\log n} = \nu_{s+1}\,.$$

In addition, whenever $\Phi \in V(\lambda)^\perp_{g+1}$, then

$$|\langle \Phi, N(x,n)\rangle| \le \text{ Constant}, \quad \text{ for all } n \in \mathbb{N}.$$

Motivated by numerical experiments, Zorich conjectured that

$$\nu_1 = 1 > \nu_2 > \cdots > \nu_g > 0\,. \tag{5}$$

The Zorich conjecture was reformulated by Kontsevich [35] for the equivalent case of flows on surfaces of higher genus. It became known as the Kontsevich–Zorich (or Zorich–Kontsevich) conjecture. The author of this paper was able to prove that the numbers $\nu_1 \geq \cdots \geq \nu_g$ are all nonzero (see [21, 22, 24]). Avila and Viana were later able to give an independent complete proof of the full Kontsevich–Zorich conjecture.

THEOREM 2 ([8, 9])**.** *The Kontsevich–Zorich conjecture on the deviation exponents holds for all irreducible permutations.*

In Zorich's work [58] the numbers in formula (4) arise as Lyapunov exponents of the so-called *Zorich acceleration* of the *Rauzy–Veech cocycle*, which will be introduced in the next section. Kontsevich [35] introduced an essentially equivalent continuous-time cocycle over the Teichmüller geodesic flow, now called the *Kontsevich–Zorich cocycle*. The Kontsevich–Zorich conjecture therefore affirms that the Lyapunov spectrum of the Rauzy–Veech–Zorich cocycle (or, equivalently, of the Kontsevich–Zorich cocycle) is *nonuniformly hyperbolic* and *simple* (that is, all the exponents are nonzero and distinct).

The Kontsevich–Zorich conjecture has several implications for the deviation of ergodic averages of interval-exchange transformations. In fact, if the exponents are strictly positive, then polynomial deviations are really taking place and the Denjoy–Koksma inequality fails; Zorich had proved several conditional results which can be summarized by the statement that the filtration $V_1(\lambda) \subset \cdots \subset V_g(\lambda)$ is a *Lagrangian flag* (in the sense that $V_{i+1}(\lambda)/V_i(\lambda)$ is a line for all $i \in \{1,\ldots,g-1\}$, and $V_g(\lambda) = V_{g+1}(\lambda)$ is Lagrangian with respect to a natural symplectic structure. Although the nonvanishing of exponents is the crucial information for many applications, the additional information provided by the full conjecture is often relevant (see for instance [14]). There are other important consequences of the Kontsevich–Zorich conjecture, in particular for the fine properties of the cohomological equation for interval-exchange transformations and translation flows (see [20, 38, 23]) and for the related local smooth conjugacy problem (see [39]).

2.2. **The Rauzy–Veech–Zorich cocycle.** The Rauzy–Veech–Zorich cocycle is the Rauzy–Veech cocycle over the Zorich acceleration of the Rauzy–Veech map. The Rauzy–Veech map is based on an *induction procedure* for interval-exchange transformations [45, 49, 50]. It is a key elementary fact that the return map of any interval-exchange transformation on d subintervals to any subinterval (not necessarily to a subinterval of continuity) is an interval-exchange transformation on at most $d+1$ subintervals. If the end-points of a subinterval are discontinuities (or end-points) of an interval-exchange transformation on d subintervals, then the induced interval-exchange transformation is again on exactly d subintervals. Inducing therefore defines a map on the space of all interval-exchange transformations on d subintervals, which acts (after rescaling) as a 'renormalization'. For the case of two subintervals the above construction yields the *Farey map*, which appears in the theory of continued-fraction expansions of real numbers.

The combinatorial structure of the Rauzy–Veech induction is easier to analyze by writing the combinatorial data of interval-exchange transformations in a more efficient and flexible way, which avoids computing the inverse of a permutation at each step of the induction. We recall that notation before proceeding to give the definition of the induction.

Let $\mathscr{A}$ be a set with $d \geq 2$ letters, called the *alphabet* and let (π_t, π_b) be a pair of bijections $\pi_t, \pi_b \colon \mathscr{A} \to \{1, \dots, d\}$. For any $\lambda \in \mathbb{R}_+^{\mathscr{A}}$, the interval-exchange transformation $f := f(\lambda, \pi_t, \pi_b)$ is described as follows: arrange d intervals labeled by the letters of the alphabet $\mathscr{A}$ and of corresponding lengths given by the vector $\lambda \in \mathbb{R}_+^{\mathscr{A}}$ according to the map π_t (top arrangement) and according to the map π_b (bottom arrangement), then translate every subinterval of the top arrangement into the subinterval of the bottom arrangement of the same label.

The above setup is related to the more traditional definition as follows. The map $f(\lambda, \pi_t, \pi_b)$ is the interval-exchange transformation with permutation $\pi_b \circ \pi_t^{-1}$ and length vector $\lambda \circ \pi_t^{-1}$. In particular, for each $\alpha \in \mathscr{A}$, we define the subintervals

$$I_\alpha^t(\lambda, \pi_t) = \left(\sum_{\pi^t(\beta) < \pi^t(\alpha)} \lambda_\alpha, \sum_{\pi^t(\beta) \leq \pi^t(\alpha)} \lambda_\alpha \right),$$

$$I_\alpha^b(\lambda, \pi_b) = \left(\sum_{\pi^b(\beta) < \pi^b(\alpha)} \lambda_\alpha, \sum_{\pi^b(\beta) \leq \pi^b(\alpha)} \lambda_\alpha \right).$$

The interval-exchange transformation $f := f(\lambda, \pi_t, \pi_b)$ is defined on the interval

$$I(\lambda) = [0, \sum_{\alpha \in \mathscr{A}} \lambda_\alpha),$$

as the piecewise translation which maps $I_\alpha^t(\lambda, \pi_t)$ onto $I_\alpha^b(\lambda, \pi_t)$, for all $\alpha \in \mathscr{A}$, given by the formula

$$x \mapsto x + \sum_{\pi_b(\beta) < \pi_b(\alpha)} \lambda_\beta - \sum_{\pi_t(\beta) < \pi_t(\alpha)} \lambda_\beta, \quad x \in I_\alpha^t(\lambda, \pi_t).$$

The irreducibility condition on the pair (π_t, π_b) is by definition equivalent to the irreducibility condition on the permutation $\pi_b \circ \pi_t^{-1}$, that is,

$$\pi_t^{-1}\{1, \dots, k\} \neq \pi_b^{-1}\{1, \dots, k\}, \quad \text{for all } k \in \{1, \dots, d\}.$$

2.2.1. *The Rauzy–Veech map.* Let us now introduce the Rauzy induction. Let us consider irreducible combinatorial data and let us denote by α_t, α_b the letters of the alphabet such that $\pi_t(\alpha_t) = d$ and $\pi_b(\alpha_b) = d$, that is, the labels of the last intervals to the right on the top and bottom arrangements. If $\lambda_{\alpha_t} \neq \lambda_{\alpha_b}$ (hence by irreducibility $\alpha_t \neq \alpha_b$), we consider the first-return map to the complement of the shortest of the rightmost interval in the top arrangement and the rightmost interval in the bottom arrangement. If $\lambda_{\alpha_t} = \lambda_{\alpha_b}$, we let the induction be undefined. In case $\lambda_{\alpha_t} > \lambda_{\alpha_b}$ the procedure is called a *top operation*;

in case $\lambda_{\alpha_t} < \lambda_{\alpha_b}$ it is called a *bottom operation*. In case of a top/bottom operation the labeling on the top/bottom is left unchanged. For a top operation all intervals in the top arrangement return after a single iteration except for the subinterval $I^t_{\alpha_b}$, which returns after two iterations to the bottom subinterval $I^b_{\alpha_t}$, which is thus split into two subintervals, labeled by α_b and α_t. Thus, while the map π_t is unchanged, the map π_b is modified by composition with a cycle on $(\pi_b(\alpha_t)+1,\dots,d)$. The letter α_t is called the *winner*, the letter α_b is called the *loser*. Similarly, for a bottom operation all subintervals in the top arrangement return after a single iteration with the exception of the portion of the top interval $I^t_{\alpha_b}$, which is mapped onto the top subinterval $I^t_{\alpha_t}$. Thus while the map π_b is unchanged, the map π_t is modified by composition with a cycle on $(\pi_t(\alpha_b)+1,\dots,d)$. The letter α_b is called the *winner*, the letter α_t is called the *loser*. The bottom operation is equivalent to the top operation for the inverse interval-exchange transformation and vice versa.

According to the above description, the induced map is an interval-exchange transformation $f(\lambda',\pi'_t,\pi'_b)$ that is uniquely determined by the following data (λ',π'_t,π'_b). For a top Rauzy operation, the combinatorial data transform as

$$\pi'_t = \pi_t \quad \text{and} \quad \pi'_b = (\pi_b(\alpha_t)+1,\dots,d)\circ\pi_b\,, \tag{6}$$

while the length data transform according to the formulas

$$\lambda'_\alpha = \begin{cases} \lambda_{\alpha_t} - \lambda_{\alpha_b}\,, & \text{for } \alpha = \alpha_t\,;\\ \lambda_\alpha\,, & \text{for } \alpha \neq \alpha_t\,. \end{cases} \tag{7}$$

For a bottom Rauzy operation, the combinatorial data transform as

$$\pi'_t = (\pi_t(\alpha_b)+1,\dots,d)\circ\pi_t \quad \text{and} \quad \pi'_b = \pi_b\,, \tag{8}$$

while the length data transform according to the formulas

$$\lambda'_\alpha = \begin{cases} \lambda_{\alpha_b} - \lambda_{\alpha_t}\,, & \text{for } \alpha = \alpha_b\,;\\ \lambda_\alpha\,, & \text{for } \alpha \neq \alpha_b\,. \end{cases} \tag{9}$$

The above formulas for the transformation of the length data can be summarized as follows. For all $\alpha \neq \beta \in \mathscr{A}$, let $E_{\alpha\beta}$ denote the elementary matrix with a single nonzero entry, in the position $\alpha\beta$, equal to 1. For any well-defined Rauzy–Veech induction step starting at (λ,π_t,π_b) with winner α and loser β, let $C(\lambda,\pi_t,\pi_b) = I + E_{\alpha\beta}$. The length data transform according to the formula

$$\lambda = C(\lambda,\pi_t,\pi_b)\lambda' = (I+E_{\alpha\beta})\lambda'\,. \tag{10}$$

It is crucial for the theory of the Rauzy–Veech induction that all the matrices $C(\lambda,\pi_t,\pi_b)$ and their products are nonnegative with integer entries.

Let $\Sigma_{\mathscr{A}} = \{(\pi_t,\pi_b) | \pi_t,\pi_b : \mathscr{A} \to \{1,\dots,d\}\}$ be the set of all (irreducible) combinatorial data. The Rauzy–Veech induction yields a map on the space $\mathbb{R}^{\mathscr{A}} \times \Sigma_{\mathscr{A}}$, defined on the complement of finitely many hyperplanes. The *Rauzy–Veech map* is the projectivization R on $\mathbb{P}^{\mathscr{A}}_+ \times \Sigma_{\mathscr{A}}$ of the Rauzy–Veech induction, that is, the map defined as

$$R([\lambda],\pi_t,\pi_b) = ([\lambda'],\pi'_t,\pi'_b)\,, \quad \text{for all } ([\lambda],(\pi_t,\pi_b)) \in \mathbb{P}^{\mathscr{A}}_+ \times \Sigma_{\mathscr{A}}\,.$$

The dynamics of the Rauzy–Veech map is thus defined almost everywhere with respect to the Lebesgue measure. In fact, every interval-exchange transformation satisfying the Keane condition has an infinite forward orbit under the Rauzy–Veech map. It should be noted that the Rauzy–Veech map is *weakly expanding*, in the sense that, since

$$R\{([\lambda],\pi_t,\pi_b)|\lambda_{\alpha_t}>\lambda_{\alpha_b}\} = R\{([\lambda],\pi_t,\pi_b)|\lambda_{\alpha_t}<\lambda_{\alpha_b}\} = \mathbb{P}_+^{\mathscr{A}}\times\{(\pi'_t,\pi'_b)\},$$

the simplex $\mathbb{P}_+^{\mathscr{A}}\times\{(\pi_t,\pi_b)\}$ is mapped 2-to-1 onto the simplex $\mathbb{P}_+^{\mathscr{A}}\times\{(\pi'_t,\pi'_b)\}$.

Under the action of the Rauzy–Veech map the space $\mathbb{P}_+^{\mathscr{A}}\times\Sigma_{\mathscr{A}}$ splits into the union of finitely many invariant components. A *Rauzy class* $\mathscr{C}(\pi_t,\pi_b)$ of a pair $(\pi_t,\pi_b)\in\Sigma_{\mathscr{A}}$ is the subset of all combinatorial data (π'_t,π'_b), which can be obtained from (π_t,π_b) after finitely many top/bottom operations (given in formulas (6) and (8)). By construction, for every Rauzy class $\mathscr{C}\subset\Sigma_{\mathscr{A}}$, the Rauzy–Veech map restricts to a map $R_{\mathscr{C}}$ on $\mathbb{P}_+^{\mathscr{A}}\times\mathscr{C}$. By a theorem of Masur [41] and Veech [50], for every Rauzy class, the Rauzy–Veech map admits an ergodic, conservative, absolutely continuous invariant measure of infinite total mass, unique up to scalar multiples, with a positive analytic density. The refined combinatorial structure of the Rauzy–Veech induction is encoded in the so-called *Rauzy graph*, which contains the information about exactly which paths in the Rauzy classes are admissible, that is, do appear under the action of the Rauzy–Veech map. The vertices of the Rauzy graph are elements of the set $\Sigma_{\mathscr{A}}$. Every vertex has exactly two incoming and two outgoing arrows corresponding to top/bottom Rauzy–Veech operations. Each arrow can also be labeled by the letters of the alphabet that are the winner and the loser of the corresponding Rauzy–Veech operation. A *Rauzy diagram* is a connected component of the Rauzy graph; hence, it corresponds to a unique Rauzy class.

2.2.2. *The Rauzy–Veech–Zorich cocyle.* The Rauzy–Veech cocycle C_R is a cocycle over the Rauzy–Veech map R generated by the transpose C^* of the matrix-valued function $C\colon \mathbb{P}_+^{\mathscr{A}}\times\Sigma_{\mathscr{A}}\to SL(d,\mathbb{Z})$ of formula (10). The cocycle is defined as follows: for all $([\lambda],\pi_t,\pi_b,v)\in\mathbb{P}_+^{\mathscr{A}}\times\Sigma_{\mathscr{A}}\times\mathbb{R}^{\mathscr{A}}$, let

$$C_R([\lambda],\pi_t,\pi_b,v) = \big(R([\lambda],\pi_t,\pi_b), C^*(\lambda,\pi_t,\pi_b)v\big)\,. \tag{11}$$

The significance of this cocycle for the dynamics of interval-exchange transformations is as follows. For all $n\in\mathbb{N}$, let $([\lambda^{(n)}],\pi_t^{(n)},\pi_b^{(n)}) = R^n([\lambda],\pi_t,\pi_b)$ and let

$$C_n^*(\lambda,\pi_t,\pi_b) = \prod_{j=1}^{n-1} C^*(\lambda^{(j)},\pi_t^{(j)},\pi_b^{(j)})\,.$$

Let $f := f(\lambda,\pi_t,\pi_b)$ and, for all $n\in\mathbb{N}$, let $f^{(n)} = f(\lambda^{(n)},\pi_t^{(n)},\pi_b^{(n)})$. For all $\alpha\in\mathscr{A}$ and all $x\in I_\alpha^t(\lambda^{(n)},\pi_t^{(n)})$, let $N_{\alpha\beta}^{(n)}(\lambda,\pi_t,\pi_b)$ be the number of visits of the orbit of x under f to the subinterval $I_\beta^t(\lambda,\pi_t)$ up to the first return time to $I(\lambda^{(n)})$. For all $n\in\mathbb{N}$ and for every α, $\beta\in\mathscr{A}$, we have

$$C_n^*(\lambda,\pi_t,\pi_b)_{\alpha\beta} = N_{\alpha\beta}^{(n)}(\lambda,\pi_t,\pi_b)\,. \tag{12}$$

This property is easily verified for $n=1$ since by construction the top interval $I^t_\beta(\lambda',\pi'_t,\pi'_b)$ labeled by the loser $\beta\in\mathscr{A}$ visits the top subinterval $I^t_\alpha(\lambda,\pi_t,\pi_b)$ labeled by the winner $\alpha\in\mathscr{A}$ exactly once (at the first iteration) before returning (at the second iteration), while all other subintervals return at the first iteration. By the cocycle property the identity is preserved under iteration of the Rauzy–Veech induction procedure.

In view of the above discussion, it becomes clear that the Rauzy–Veech cocycle is designed exactly to keep track of the frequency of the visits of orbits of an interval-exchange transformation to the subintervals. The Lyapunov structure of the Rauzy–Veech cocycle is therefore relevant to the question of deviations of ergodic averages for interval-exchange transformations. Unfortunately, Oseledets Theorem cannot be applied directly to the Rauzy–Veech cocycle to prove existence of Lyapunov exponents.

Zorich [58] solved this problem by considering a suitable acceleration of the Rauzy–Veech induction step. For every $([\lambda],\pi_t,\pi_b)\in\mathbb{P}^{\mathscr{A}}_+\times\Sigma_{\mathscr{A}}$, let $n(\lambda,\pi_t,\pi_b)$ be the maximal number of successive Rauzy–Veech induction steps starting at $([\lambda],\pi_t,\pi_b)$ that are of the same top/bottom type. The *Zorich acceleration* is the map Z on $\mathbb{P}^{\mathscr{A}}_+\times\Sigma_{\mathscr{A}}$ defined as follows: for almost all $([\lambda],\pi_t,\pi_b)\in\mathbb{P}^{\mathscr{A}}_+\times\Sigma_{\mathscr{A}}$,

$$Z([\lambda],\pi_t,\pi_b)=R^{n(\lambda,\pi_t,\pi_b)}([\lambda],\pi_t,\pi_b)\,. \tag{13}$$

The Rauzy–Veech cocycle induces a cocycle Z_B, called the *Rauzy–Veech–Zorich cocycle*, over the Zorich acceleration map, generated by the matrix-valued function $C_Z:\mathbb{P}^{\mathscr{A}}_+\times\Sigma_{\mathscr{A}}\to GL(d,\mathbb{Z})$ defined by formula: for all $([\lambda],\pi_t,\pi_b)\in\mathbb{P}^{\mathscr{A}}_+\times\Sigma_{\mathscr{A}}$,

$$C_Z(\lambda,\pi_t,\pi_b)=\prod_{n=0}^{n(\lambda,\pi_t,\pi_b)-1}(C^*\circ R^n)(\lambda,\pi_t,\pi_b)\,. \tag{14}$$

Zorich's key contribution was to prove that the Zorich acceleration map has, for every Rauzy class $\mathscr{R}$, a *finite* absolutely continuous ergodic invariant measure $\mu_{\mathscr{R}}$ (unique up to normalization) and that the Rauzy–Veech–Zorich cocycle is log integrable, that is, it satisfies the hypotheses of the Oseledets Theorem [58]. The Rauzy–Veech and the Zorich cocycle preserve the antisymmetric bilinear form $\Omega(\pi_t,\pi_b)$ on $\mathbb{R}^{\mathscr{A}}$ given by the matrix

$$\Omega_{\alpha\beta}(\pi_t,\pi_b)=\begin{cases}1 & \text{if } \pi_b(\beta)<\pi_b(\alpha)\,,\pi_t(\beta)>\pi_t(\alpha)\,;\\ -1 & \text{if } \pi_b(\beta)>\pi_b(\alpha)\,,\pi_t(\beta)<\pi_t(\alpha)\,;\\ 0 & \text{otherwise}\,.\end{cases}$$

The bilinear form $\Omega(\pi_t,\pi_b)$ is nondegenerate, hence symplectic, on the range $H(\pi_t,\pi_b)\subset\mathbb{R}^{\mathscr{A}}$ of the matrix $\Omega(\pi_t,\pi_b)$. For any Rauzy class $\mathscr{R}$, the invariant symplectic bundle

$$H_{\mathscr{R}}:=\bigcup\{H(\pi_t,\pi_b)\,|\,(\lambda,\pi_t,\pi_b)\in\mathbb{P}^{\mathscr{A}}_+\times\Sigma_{\mathscr{A}}\}$$

has even dimension, which only depends on the Rauzy class of the combinatorial data. We write $\dim H_{\mathscr{R}}=2g(\mathscr{R})$ and the integer $g(\mathscr{R})$ is the genus of the surfaces that arise in the suspension of interval-exchange transformations with combinatorial data in the given Rauzy class (see Section 2.3.2 below). Zorich proved that all the Lyapunov exponents on the quotient $\mathbb{R}^{\mathscr{A}}/H_{\mathscr{R}}$ are zero and that the restriction of the cocycle to $H_{\mathscr{R}}$ (called the *restricted Rauzy–Veech–Zorich cocycle*) has a Lyapunov spectrum of the form

$$\theta_1 > \theta_2 \geq \ldots \theta_g \geq -\theta_g \geq \cdots \geq -\theta_2 > -\theta_1\,. \tag{15}$$

(In fact, Zorich derived the strict 'spectral gap' inequality $\theta_1 > \theta_2$ from Veech's theorem on the nonuniform hyperbolicity of the Teichmüller flow [52]). By the above interpretation (12) of the Rauzy–Veech cocycle, it is not hard to derive the relation between the deviation exponents in formula (4) and the Lyapunov exponents (15) of the Rauzy–Veech–Zorich cocycle:

$$\nu_1 = \theta_1/\theta_1 > \nu_2 = \theta_2/\theta_1 \geq \cdots \geq \nu_g = \theta_g/\theta_1\,.$$

Kontsevich–Zorich conjecture states that the Rauzy–Veech–Zorich cocycle has simple Lyapunov spectrum, that is, the exponents in formula (15) are all distinct (hence they are all nonzero). This conjecture implies the Zorich picture on deviations of ergodic averages for interval-exchange transformations and the related conjecture discussed above. The author of this paper proved in [21] that the cocycle is nonuniformly hyperbolic, that is, the exponents are all nonzero (in fact, he proved the corresponding equivalent statement for the continuous-time version of the Rauzy–Veech–Zorich cocycle, which is known as the Kontsevich–Zorich cocycle [35]). Avila and Viana proved the full conjecture:

THEOREM 3 ([8, 9])**.** *For any Rauzy class $\mathscr{R}$, the restricted Rauzy–Veech–Zorich cocycle on the bundle $H_{\mathscr{R}}$ is simple. Hence, in particular,*

$$\theta_1 > \theta_2 > \cdots > \theta_g > 0\,.$$

As we have outlined above, by Zorich's work, Theorem 3 implies Theorem 2 on deviations of ergodic averages (see [59] and for smooth functions [21, 14]). In fact, Avila and Viana prove Theorem 3.

It should be noted that Avila–Viana's approach is completely different, and more general, than the method developed in [21, 24] by the author of this paper. In fact, their proof of the Kontsevich–Zorich conjecture is based on a criterion for the simplicity of a class of cocycles over strongly expanding transformations, which we will outline below in Section 3.3. This criterion is an extension of previous work by Bonatti, Gomez-Mont and Viana [11, 12] on simplicity of the Lyapunov spectrum of deterministic cocycles of results quite well-known for random cocycles (see [27, 26, 37, 29, 28]).

2.3. **The Teichmüller flow.** In this section, we briefly introduce the Teichmüller geodesic flow, review its main ergodic properties, and state the theorem of Avila, Gouëzel and Yoccoz [6] that the Teichmüller geodesic flow is exponentially mixing. We briefly introduce Veech's 'zippered rectangles' flow, an almost everywhere finite-to-one cover of the Teichmüller flow, related to the Rauzy–Veech induction by a suspension construction, which provides a symbolic model crucial for the approach of Avila, Gouëzel and Yoccoz to the dynamics of the Teichmüller flow.

2.3.1. *Exponential Mixing.* The Teichmüller flow can be defined as the geodesic flow on the cotangent bundle of the moduli space of Riemann surfaces endowed with the *Teichmüller metric*, a Finsler metric that measures the minimal conformal distortion of a quasiconformal map between two conformal structures. The cotangent bundle of the moduli space of surfaces can be canonically identified with the bundle of holomorphic quadratic differentials. A quadratic differential on a Riemann surface is a section of the square of its cotangent bundle. The squares of holomorphic 1-forms, called *Abelian differentials*, are a particular case. The Teichmüller geodesic flow, therefore, induces a flow on the moduli space of Abelian differentials. In this section we will be interested in the dynamical properties of this flow. Thanks to the uniformization theorem, the *moduli space of Riemann surfaces* is often understood as a moduli space of Riemannian metrics of constant negative curvature. A fruitful point of view on the *moduli space of Abelian differentials* comes from a natural identification between the notions of an Abelian differential and a translation structure on a Riemann surface. A *translation structure* is given by an atlas whose change of coordinate maps are translations. In other terms, a translation structure is equivalent to a flat metric with trivial holonomy (that is, the parallel transport of unit vector along any closed loop is equal to the identity). From this point of view, a translation surface is the union of finitely many planar polygons with edges glued in pairs by translations. It becomes clear that the group $GL(2,\mathbb{R})$ acts on the moduli space of Abelian differentials. The action can be defined on translation atlases by postcomposition of elements of $GL(2,\mathbb{R})$ with coordinate maps. In other terms, the group $GL(2,\mathbb{R})$ acts on the complex plane. Hence, it acts on polygons and respects the gluing rules for translation surfaces (since the subgroup of all translations is a normal subgroup of the linear group).

It is natural to restrict the Teichmüller flow to the unit cotangent bundle, that is, to the moduli space of Abelian differentials of unit total area. The group $SL(2,\mathbb{R})$ is the largest subgroup of $GL(2,\mathbb{R})$ that preserves the area of translation surfaces, hence its acts on the level sets of the area function. The Teichmüller flow is given by the action of the *diagonal subgroup* of $SL(2,\mathbb{R})$. For instance, in the case of elliptic curves, the moduli space of Abelian differentials of unit total area can be identified with the homogeneous space $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ and the Teichmüller flow, which is just the geodesic flow for the Poincaré metric on the modular surface, can be identified to the flow given by the action of diagonal subgroup of $SL(2,\mathbb{R})$ on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ by multiplication on the right. It is well-known that the modular geodesic flow is mixing with exponential decay of correlations for observables in the Ratner's class and, equivalently, the action of $SL(2,\mathbb{R})$ on the space of square integrable functions on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ has a spectral gap in the sense of harmonic analysis (or representation theory). It is has been a central goal in the theory to generalize these results to higher genera. We recall that a flow $\{g_t\}$ on a probability space (X,μ) has *exponential decay of correlations* for a class $\mathscr{F}$ of square-integrable functions if there exists a constant $\alpha > 0$ and for any pair functions ϕ, $\psi \in \mathscr{F}$ there exists a constant

$C(\phi,\psi) > 0$ such that the following bound on correlations holds for all $t > 0$:

$$\left|\langle \phi \circ g_t, \psi \rangle - \Big(\int \phi d\mu\Big)\Big(\int \psi d\mu\Big)\right| \le C(\phi,\psi)e^{-\alpha t}. \tag{16}$$

In the case of higher-genus surfaces, the moduli space of Abelian differentials is *stratified* according to the pattern of multiplicities of zeros of the differentials. The strata are not connected (connected components are completely classified in [36]). Veech [52] was able to prove that the Teichmüller geodesic flow is nonuniformly hyperbolic and ergodic, hence mixing by Moore's theorem (any ergodic diagonal flow coming from an action of the group $SL(2,\mathbb{R})$ is mixing) with respect to canonical absolutely continuous $SL(2,\mathbb{R})$-invariant measures on every connected component of every stratum of the moduli space of quadratic differentials. Veech's result greatly extended the ergodicity result of Masur [41] for the so-called principal stratum of the space of quadratic differentials. For many strata, Masur [41] and Veech [50] had proved the foundational result that all canonical measures have finite mass. Veech [52] extends this finiteness result to all strata. It was conjectured by Veech (among others) that, for all canonical measures, the $SL(2,\mathbb{R})$ action has a spectral gap and the Teichmüller flow has exponential decay of correlations. The first step in this direction was accomplished by A. Bufetov, who proved in this thesis [13] that the Rauzy–Veech–Zorich acceleration has a stretched exponential decay of correlations (note that Avila and Bufetov proved later that the decay is in fact exponential [3]). Avila, Gouëzel and Yoccoz [6] gave a complete solution to this problem for canonical measures on strata of the moduli space of Abelian differentials.

THEOREM 4 ([6])**.** *The $SL(2,\mathbb{R})$ action has a spectral gap and the Teichmüller geodesic flow has exponential decay of correlations for observables in the Ratner's class with respect to the canonical measures on all strata of the moduli space of Abelian differentials.*

Avila and Resende [7] have later extended the theorem above to all strata of the moduli space of quadratic differentials.

The approach of Avila to the Teichmüller flow (at least in his early work on the subject) is not direct. In fact, Veech had constructed a symbolic model for the Teichmüller flow on the space of Abelian differentials, based on the Rauzy–Veech induction, often called *'zippered rectangles' flow*. Such a flow acts on a moduli space of 'zippered rectangles', which is finite-to-one cover of the moduli space of Abelian differentials outside of a set of Lebesgue measure zero (in fact, outside of a countable union of codimension-1 subspaces). All properties of the measurable dynamics of the Teichmüller flow with respect to the Lebesgue measure class are equivalent to the corresponding properties of the Veech 'zippered rectangles' flow. In the joint work with Resende, the authors introduce a version of the Rauzy induction for nonorientable interval-exchange transformations and prove the conditions required to apply the main abstract theorem established in [6] (stated in below in Section 3.4).

In the higher-genus case, there are many $SL(2,\mathbb{R})$-invariant probability measures on every stratum. It has been conjectured that such measures are all *algebraic*. Eskin and Mirzakhani [19] have recently announced a proof of this conjecture. For example, by a well-known (orienting double cover) construction, it is possible to embed canonical measures on strata of quadratic differentials as singular algebraic measures (supported on suborbifolds) on strata of Abelian differentials. A recent result of Avila and Gouëzel [5] considerably refines the spectral gap theorem stated above and extends the result to cover all algebraic $SL(2,\mathbb{R})$-invariant probability measures (hence, by the Eskin–Mirzakhani result [19], all $SL(2,\mathbb{R})$-invariant probability measures). As we have mentioned above, the approach of this (remarkable) paper is different from that of Avila's earlier work, in particular it is not based on a symbolic model for the Teichmüller flow, hence it is beyond the scope of this paper.

2.3.2. *The 'zippered rectangles' flow.* The zippered rectangles construction is a suspension of interval-exchange transformations that produces a translation surface with a preferred horizontal interval such that the return map of the vertical foliation to the preferred interval is the given interval-exchange transformation. We describe below the suspension construction.

Let $f := f(\lambda, \pi_t, \pi_b)$ be an interval-exchange transformation with length data $\lambda \in \mathbb{R}_+^d$ and combinatorial data $\pi_t, \pi_b \colon \mathscr{A} \to \{1,\dots,d\}$. Let $\mathscr{T}(\pi_t,\pi_b)$ be the set of vectors $\tau \in \mathbb{R}^{\mathscr{A}}$ such that, for all $i \in \{1,\dots,d-1\}$,

$$\sum_{\pi_t(\alpha)\le i} \tau_\alpha > 0 \quad \text{and} \quad \sum_{\pi_b(\alpha)\le i} \tau_\alpha < 0\,. \tag{17}$$

For all $\alpha \in \mathscr{A}$, let $\zeta_\alpha := (\lambda_\alpha, \tau_\alpha) \in \mathbb{R}^2$ and let $\partial P(\lambda, \tau, \pi_t, \pi_b)$ be the closed polygonal curve formed by the concatenation of vectors

$$\zeta_{\pi_t^{-1}(1)}, \zeta_{\pi_t^{-1}(2)}, \dots, \zeta_{\pi_t^{-1}(d)}, -\zeta_{\pi_b^{-1}(d)}, -\zeta_{\pi_b^{-1}(d-1)}, \dots, -\zeta_{\pi_b^{-1}(1)}\,.$$

In case the above concatenation starts at the origin, the above condition (17) means that the end-points of sums $\zeta_{\pi_t^{-1}(1)} + \cdots + \zeta_{\pi_t^{-1}(i)}$ are on the upper half-plane, while the end-points of sums $\zeta_{\pi_b^{-1}(1)} + \cdots + \zeta_{\pi_b^{-1}(i)}$ are in the lower half-plane, for all $i \in \{1,\dots,d-1\}$. By gluing by a translation each pair of edges labeled by the same letter (which, by definition, are parallel and have the same length), we construct a compact translation surface $M := M(\lambda, \tau, \pi_t, \pi_b)$, called the *suspension surface* of the interval-exchange transformation $f := f(\lambda, \pi_t, \pi_b)$. The interval-exchange transformation f on the interval $I(\lambda)$ coincides with return map of the vertical flow to the interval $I \subset M$ be the interval $I(\lambda) \times \{0\}$. The flat metric on the surface M (induced by the flat metric on the plane) can have conical singularities of angle an integer multiple of 2π at any of the points coming from vertices of the polygon, depending on the permutation. Equivalently, the corresponding Abelian differential, induced by the differential dz on $\mathbb{C} \equiv \mathbb{R}^2$, can have zeros at those points. It can be proved that the number and the total angle of the cone points (or, equivalently, the multiplicities of the zeros), hence the genus $g := g(\pi_t, \pi_b)$ of M, only depend on the Rauzy class of the combinatorial data. Let us then give the *'zippered rectangles'* representation of the surface $M(\lambda, \tau, \pi_t, \pi_b)$. For all $\alpha \in \mathscr{A}$, we define the height

$$h_\alpha := \sum_{\pi_t(\beta) < \pi_t(\alpha)} \tau_\beta - \sum_{\pi_b(\beta) < \pi_b(\alpha)} \tau_\beta = -\sum_{\beta \in \mathscr{A}} \Omega_{\alpha\beta}(\pi_t, \pi_b)\tau_\beta .$$

It follows from the definition of the cone $\mathscr{T} := \mathscr{T}(\pi_t, \pi_b)$ that $h_\alpha > 0$ for all $\alpha \in \mathscr{A}$. Let $H^+(\pi_t, \pi_b) \subset \mathbb{R}_+^{\mathscr{A}}$ denote the cone of heights, that is,

$$H^+(\pi_t, \pi_b) = \Omega(\pi_t, \pi_b)(\mathscr{T}) \subset H(\pi_t, \pi_b) = \Omega(\pi_t, \pi_b)(\mathbb{R}^{\mathscr{A}}) .$$

The translation surface $M(\lambda, \tau, \pi_t, \pi_b)$ can be decomposed up to a finite set of vertical segments as a union of open vertical rectangles $\{R^t_\alpha | \alpha \in \mathscr{A}\}$ or $\{R^b_\alpha | \alpha \in \mathscr{A}\}$ defined as follows:

$$R^t_\alpha = I^t_\alpha(\lambda, \pi_t) \times [0, h_\alpha] \quad \text{and} \quad R^b_\alpha = I^b_\alpha(\lambda, \pi_b) \times [-h_\alpha, 0] .$$

The surface can be recovered from the above rectangles by "zipping" adjacent rectangles along vertical segments up to certain heights and by identifying, for all $\alpha \in \mathscr{A}$, the top rectangle R^t_α to the corresponding bottom rectangle R^b_α by the appropriate translation (note that R^t_α and R^b_α are congruent). Adjacent top and bottom rectangles are not zippered all the way along their common boundaries, but only along the vertical segments given as the intersections of the vertical components of the boundaries ∂R^t_α and ∂R^b_α, for all $\alpha \in \mathscr{A}$, with the polygon $P(\lambda, \tau, \pi_t, \pi_b)$ constructed above. It should be clear from the above construction that the vertical flow on M has a piecewise constant return time function to the horizontal interval $I \subset M$ and that the return time is exactly equal to h_α on the subinterval $I^t_\alpha(\lambda, \pi_t)$, for all $\alpha \in \mathscr{A}$.

It can be also be proved that the space $H(\pi_t, \pi_b)$ is in canonical correspondence with the the cohomology $H^1(M, \mathbb{R})$ with real coefficients of the surface M and that the restriction of the matrix $\Omega(\pi_t, \pi_b)$ to $H(\pi_t, \pi_b)$ represents the symplectic intersection form on $H^1(M, \mathbb{R})$ with respect to an appropriate basis.

The formalism of 'zippered rectangles' is especially adapted to the description of the *invertible Rauzy–Veech map*, a natural extension of the Rauzy–Veech map introduced above. Let us recall that the idea of the Rauzy–Veech induction is to remove the rightmost top or bottom interval (whichever is shorter) and consider the return map to the remaining interval. The removed interval is called "the loser" and the other rightmost interval '"the winner". In terms of the suspension surface this is equivalent to considering a different 'zippered rectangles' decomposition of the same translation surface: for a top/bottom operation, remove the rightmost subrectangle of width equal to the loser from the rectangle of the same name as the winner and stack it on the top/bottom of the rectangle of the same name as the loser; stack the rectangle of same name as the loser to the bottom/top of the rectangle of the same name as the winner.

The new heights are given by the following formulas: for a top Rauzy operation,

$$h'_\alpha = \begin{cases} h_{\alpha_t} + h_{\alpha_b}, & \text{for } \alpha = \alpha_b; \\ h_\alpha, & \text{for } \alpha \neq \alpha_b, \end{cases} \tag{18}$$

and for a bottom Rauzy operation,

$$h'_\alpha = \begin{cases} h_{\alpha_t} + h_{\alpha_b}, & \text{for } \alpha = \alpha_t; \\ h_\alpha, & \text{for } \alpha \neq \alpha_t. \end{cases} \tag{19}$$

The above formulas can be summarized as follows: for all $(\lambda, \pi_t, \pi_b) \in \mathbb{R}_+^{\mathscr{A}} \times \Sigma_{\mathscr{A}}$ and for all $h \in H^+(\pi_t, \pi_b)$, we have

$$h' = C^*(\lambda, \pi_t, \pi_b) h. \tag{20}$$

That is, the action of the invertible Rauzy–Veech induction on the heights at a point (λ, π_t, π_b) is given by the matrix of the Rauzy–Veech cocycle (restricted to the cone $H^+(\pi_t, \pi_b) \subset H(\pi_t, \pi_b)$). Let invertible Rauzy–Veech induction generate an equivalence relation $\mathscr{V}$ on the space of all zippered rectangles, that is, on the space $\hat{\Omega}_{\mathscr{A}} = \{(\lambda, h, \pi_t, \pi_b) \in \mathbb{R}_+^{\mathscr{A}} \times H^+(\pi_t, \pi_b) \times \Sigma_{\mathscr{A}}\}$. The *moduli space of zippered rectangles* is the quotient $\Omega_{\mathscr{A}} := \hat{\Omega}_{\mathscr{A}} / \mathscr{V}$ of the space of all zippered rectangles with respect to the equivalence relation generated by the invertible Rauzy–Veech induction. The *'zippered rectangles' flow* is the flow $\{\Phi_t\}$ on the moduli space $\Omega_{\mathscr{A}}$ of zippered rectangles given as follows: for all $t \in \mathbb{R}$,

$$\Phi_t[\lambda, h, \pi_t, \pi_b]_{\mathscr{V}} := [e^t \lambda, e^{-t} h, \pi_t, \pi_b]_{\mathscr{V}}, \quad \text{for all } [\lambda, h, \pi_t, \pi_b]_{\mathscr{V}} \in \Omega_{\mathscr{A}}. \tag{21}$$

The 'zippered rectangles' flow introduced above projects onto the Teichmüller flow on a subset of full Lebesgue measure of a union of strata of the moduli space of Abelian differentials. In fact, for any Rauzy class $\mathscr{R} \subset \Sigma_{\mathscr{A}}$, there is a flow invariant component $\Omega_{\mathscr{A}}(\mathscr{R})$ of the moduli space of zippered rectangles that projects onto a full measure subset of a connected component of a stratum of the moduli space of Abelian differentials. Avila, Gouëzel and Yoccoz [6] prove exponential decay of correlations for the zippered rectangles flow and for sufficiently smooth observables, and then deduce Theorem 4 from this result.

The 'zippered rectangles' flow is a suspension of the *invertible Rauzy–Veech map*, a natural extension of the Rauzy–Veech map described in Section 2.2.1, related to the invertible Rauzy–Veech induction described above. Let $\hat{\mathscr{Y}}_{\mathscr{A}} \subset \hat{\Omega}_{\mathscr{A}}$ be the subset defined as follows:

$$\hat{\mathscr{Y}}_{\mathscr{A}} := \{(\lambda, h, \pi_t, \pi_b) \in \hat{\Omega}_{\mathscr{A}} | \sum_{\alpha \in \mathscr{A}} \lambda_\alpha = 1\}.$$

The projection $\mathscr{Y}_{\mathscr{A}} := \hat{\mathscr{Y}}_{\mathscr{A}} / \mathscr{V} \subset \Omega_{\mathscr{A}}$ is a transverse section for the 'zippered rectangles' flow. The invertible Rauzy–Veech map $\hat{R} \colon \mathscr{Y}_{\mathscr{A}} \to \mathscr{Y}_{\mathscr{A}}$ is the first-return map of the zippered rectangle flow. By construction, the invertible Rauzy–Veech map is an extension of the Rauzy–Veech map as we shall see below.

For any $[\lambda, h, \pi_t, \pi_b]_{\mathcal{V}} \in \mathcal{Y}_{\mathcal{A}}$, the first return time $r(\lambda, h, \pi_t, \pi_b)$ of the 'zippered rectangles' flow $\{\Phi_t\}$ to the transverse section $\mathcal{Y}_{\mathcal{A}}$ is the positive real number

$$r(\lambda, h, \pi_t, \pi_b) = -\log\left(\sum_{\alpha\in\mathcal{A}} \lambda'_\alpha\right). \tag{22}$$

Hence, the invertible Rauzy–Veech map is given by the following formula:

$$\hat{R}(\lambda, h, \pi_t, \pi_b) = (e^{r(\lambda,h,\pi_t,\pi_b)}\lambda', e^{-r(\lambda,h,\pi_t,\pi_b)}h', \pi'_t, \pi'_b).$$

Let $p_{\mathcal{A}}: \mathcal{Y}_{\mathcal{A}} \to \mathbb{P}^{\mathcal{A}}_+ \times \Sigma_{\mathcal{A}}$ be the map that projectivizes the lengths vector and forgets the heights vector, that is,

$$p_{\mathcal{A}}[\lambda, h, \pi_t, \pi_b]_{\mathcal{V}} = ([\lambda], \pi_t, \pi_b) \in \mathbb{P}^{\mathcal{A}}_+ \times \Sigma_{\mathcal{A}}.$$

By construction, the map $p_{\mathcal{A}}$ is a fibration such that

$$p^{-1}_{\mathcal{A}}([\lambda], \pi_t, \pi_b) = H^+(\pi_t, \pi_b), \quad \text{for any}\,([\lambda], \pi_t, \pi_b) \in \mathbb{P}^{\mathcal{A}}_+ \times \Sigma_{\mathcal{A}}$$

and the Rauzy–Veech map is the projection of the invertible Rauzy–Veech map:

$$R \circ p_{\mathcal{A}} = p_{\mathcal{A}} \circ \hat{R} \quad \text{on } \mathcal{Y}_{\mathcal{A}}.$$

In conclusion of the section, it should be clear enough that the Rauzy–Veech map is a crucial tool in the study of interval-exchange transformations, of translation flows and of the Teichmüller flow. With respect to interval-exchange transformations, it plays the role of a renormalization dynamics, while with respect to the Teichmüller flow, it provides a concrete combinatorial framework to analyze the dynamics. There are two important insights in Avila's work that we will try to outline in the next sections. The first concerns the hyperbolic properties of the Rauzy–Veech map. As we have observed, the map is weakly expanding but not strongly expanding since the hyperbolicity degenerates near the boundary of the simplex in its domain of definition. In Avila's work the strong hyperbolicity properties of the Rauzy–Veech map as well as crucial distortion estimates are captured by considering appropriate return maps. The second concerns the dynamics of the map near the boundary. Avila is able to define a boundary dynamics in terms of the Rauzy–Veech map for interval-exchange transformations on fewer intervals and to derive results on the dynamics of the map and of the related cocycle by an induction argument on the number of intervals. The core of Avila's approach is thus based on combinatorial and probabilistic arguments.

3. OUTLINE OF PROOFS

In this section, we will outline some of the main ideas underlying Avila's contributions to Teichmüller dynamics and applications. We will stress once more that Avila's work is based on an analysis of the dynamics of an appropriate acceleration of the Rauzy–Veech induction and cocycle that fully captures its chaotic, strongly mixing properties. This analysis will be outlined in Section 3.1 and lies at the heart of all the early work of Avila on the subject. Section 3.2 will be devoted to the proof by Avila and the author [4] of weak mixing for interval-exchange transformations and translation flow and Section 3.3 to the proof by

Avila and Viana [8] of the Kontsevich–Zorich conjecture on the simplicity of the Kontsevich–Zorich spectrum. These two papers are mentioned in the Brin Prize motivation. Finally, in Section 3.4 we will explain how Avila's ideas are also crucial in the proof by Avila, Gouëzel and Yoccoz [6] of the exponential mixing of the Teichmüller geodesic flow (in fact, of Veech 'zippered rectangles' flow).

3.1. **The Avila acceleration.** We introduce below Avila's notions of a *strongly expanding map* and of a *locally constant, uniform cocycle* over a strongly expanding map [4]. We then explain the proof that appropriate return maps of the Rauzy–Veech map are strongly expanding and that the Rauzy–Veech cocycle over such return maps is locally constant and uniform. In this paper we will call such a procedure of passing to strongly expanding return maps the *Avila acceleration* of the Rauzy–Veech induction and of the Rauzy–Veech–Zorich cocycle. It should be noted however that the idea of considering accelerations of the Rauzy map corresponding to projective contractions is a cornerstone of Veech's proof of the Keane conjecture [49, 50] and appears in most of Veech's work on interval-exchange transformations and on the Teichmüller flow (see for instance [51] and [52]). This idea also appears in the work of S. P. Kerckhoff [34] and A. Bufetov [13]. In fact, Veech's unique ergodicity condition on matrix products of the Rauzy cocycle already appears in the general context of random matrix products in Furstenberg's thesis (see [25], formula (16.13)). However, in our opinion, the Markovian structure of the acceleration, its strongly chaotic, mixing features and the combinatorial structure of Rauzy diagrams, were not fully grasped and not systematically exploited prior to the work on the subject of Avila and his collaborators. In this work, abstract definitions of dynamical systems and cocycles that capture the essential properties have been introduced and all results are derived from general theorems on the dynamics of such general abstract systems.

3.1.1. *Expanding maps.* Let (Δ,μ_0) be a probability space. A measurable map $T\colon \Delta\to\Delta$ that preserves the measure class of the measure μ_0 on Δ, is said to be *weakly expanding* if there exists a finite or countable measurable partition (modulo sets of measure zero)

$$\Delta := \bigcup_{l\in\Lambda} \Delta^{(l)}$$

into sets of positive μ-measure such that, for every $l\in\Lambda$, $T\colon \Delta^{(l)}\to\Delta$ is invertible and the push-forward $T_*(\mu_0|\Delta^{(l)})$ is equivalent to μ_0 on Δ. Let Ω be the set of all finite sequences of elements of Λ. For any $\underline{l} := (l_1,\dots,l_n)\in\Omega$, let us denote

$$\Delta^{\underline{l}} := \{x\in\Delta \,|\, T^{k-1}(x)\in\Delta^{(l_k)} \text{ for } k=1,\dots,n\},$$

and let $T^{\underline{l}} := T^m\colon \Delta^{\underline{l}}\to\Delta$. A weakly expanding map T is said to be *strongly expanding* if the following *bounded distortion* condition holds: there exists a constant $C>0$ such that

$$\frac{1}{C} \le \frac{1}{\mu_0(\Delta^{\underline{l}})} \frac{dT_*^{\underline{l}}(\mu_0|\Delta^{\underline{l}})}{d\mu_0} \le C\,, \quad \text{for all } \underline{l} \in \Omega. \tag{23}$$

3.1.2. *Cocycles.* We are interested in cocycles over strongly expanding maps. A *cocycle* is a pair (T,A) such that $T:(\Delta,\mu)\to(\Delta,\mu)$ is a measure-preserving transformation on a probability space (Δ,μ) and $A:\Delta\to GL(d,\mathbb{R})$ is a measurable map. There is a linear skew-product associated to the pair (T,A), defined as follows:

$$(T,A)(x,v) := (T(x),A(x)v)\,, \qquad \text{for all } (x,v)\in\Delta\times\mathbb{R}^d.$$

The iterates $\{(T,A)^n\}$ of the above linear skew-product are associated to the pair (T^n,A_n) such that $A_n:\Delta\to GL(d,\mathbb{R})$ is the measurable map defined as

$$A_n(x) = A(T^{n-1}(x))\cdots A(x)\,, \qquad \text{for all } (x,n)\in\Delta\times\mathbb{N}.$$

Let $\|\cdot\|$ denote a norm on $GL(d,\mathbb{R})$ and, for any matrix $A\in GL(d,\mathbb{R})$ let

$$\|A\|_0 := \max\{\|A\|,\|A^{-1}\|\}.$$

Let μ be any T-invariant, ergodic probability measure. The cocycle (T,A) is said to be *measurable* if

$$\int_\Delta \log\|A(x)\|\,d\mu(x) < +\infty.$$

and it is said to be *uniform* if the following stronger condition holds:

$$\int_\Delta \log\|A(x)\|_0\,d\mu(x) < +\infty. \tag{24}$$

A cocycle is said to be *integral* if the map A takes its values in the space $GL(d,\mathbb{Z})$ of matrices with integer coefficients. The linear skew-shift associated to an integral cocycle projects onto the skew-product of the space $\Delta\times\mathbb{R}^d/\mathbb{Z}^d$, which is a toral bundle over Δ. All the above properties of cocycles are well-defined for arbitrary measurable maps T on the base space.

Finally, a cocycle (T,A) over a weakly expanding map $T:(\Delta,\mu)\to(\Delta,\mu)$ is said to be *locally constant* if, for all $l\in\Lambda$,

$$A|\Delta^{(l)} = A^{(l)} \in GL(d,\mathbb{R}) \text{ is constant.} \tag{25}$$

Note that for any locally constant cocycle and for any $\underline{l}=(l_1,\dots,l_n)\in\Omega$, the map $A_n|\Delta^{\underline{l}}$ is constant:

$$A_n|\Delta^{\underline{l}} = A^{\underline{l}} := A^{(l_n)}\cdots A^{(l_1)}. \tag{26}$$

The *supporting monoid* of a locally constant cocycle is the submonoid of the linear group $GL(d,\mathbb{R})$ generated by the set $\{A^{(l)}|l\in\Lambda\}$. It is clear from the definitions that, for any $\underline{l}=(l_1,\dots,l_n)\in\Omega$, the matrix $A_n|\Delta^{\underline{l}}=A^{\underline{l}}$ defined in formula (26) belongs to the supporting monoid of the cocycle.

3.1.3. *Return maps.* The Rauzy–Veech map is weakly expanding with respect to the Lebesgue measure on a partition with two elements. In fact, for every Rauzy class $\mathscr{R}$, its domain of definition is a full measure subset of $\mathbb{P}_+^{d-1} \times \mathscr{R}$ that can be partitioned into finitely many disjoint connected components. For any combinatorial data $(\pi_t, \pi_b) \in \mathscr{R}$, let $\Delta_t(\pi_t, \pi_b) \subset \mathbb{P}_+^{d-1}$ be the simplex of length data that correspond to a top operation ($\lambda_{\alpha_t} > \lambda_{\alpha_b}$) and let $\Delta_b(\pi_t, \pi_b) \subset \mathbb{P}_+^{d-1}$ be the simplex of length data that correspond to a bottom operation ($\lambda_{\alpha_t} < \lambda_{\alpha_b}$). Let $\{\Delta_t(\mathscr{R}), \Delta_b(\mathscr{R})\}$ be the partition given by the sets

$$\Delta_t(\mathscr{R}) := \bigcup_{(\pi_t,\pi_b)\in\mathscr{R}} \Delta_t(\pi_t, \pi_b) \quad \text{and} \quad \Delta_b(\mathscr{R}) := \bigcup_{(\pi_t,\pi_b)\in\mathscr{R}} \Delta_b(\pi_t, \pi_b).$$

It is clear from the explicit formulas defining the Rauzy–Veech map (7), (9) that the restrictions of the Rauzy–Veech map R to $\Delta_t(\mathscr{R})$ and $\Delta_b(\mathscr{R})$ are invertible maps onto $\mathbb{P}_+^{d-1} \times \mathscr{R}$. The inverse branches of the maps $R|\mathbb{P}_+^{d-1} \times \{(\pi_t, \pi_b)\}$ are given by the projectivization of matrices with nonnegative integer coefficients, hence the push-forward of the restrictions of the Lebesgue measure to partition elements $\Delta_t(\mathscr{R})$ and $\Delta_b(\mathscr{R})$ are absolutely continuous with respect to Lebesgue. However, the Rauzy–Veech map is not strongly expanding since its distortion is unbounded (near the boundary of the simplex). In particular, it was proved by Veech (see [50] and also [41]) that the Rauzy–Veech map has a conservative absolutely continuous invariant measure; however, such a measure has infinite total mass. Avila's idea is to consider appropriate return maps.

For all $n \in \mathbb{N}$ and for almost all $([\lambda], \pi_t, \pi_b) \in \mathbb{P}_+^{d-1} \times \mathscr{R}$, let $\Delta_n(\lambda, \pi_t, \pi_b)$ denote the connected component of the domain of definition of the n-th iterate Z^n of the Rauzy–Veech map such that $([\lambda], \pi_t, \pi_b) \in \Delta_n(\lambda, \pi_t, \pi_b)$. It is a fundamental result by Veech [50], a unique ergodicity criterion in his proof of the Keane conjecture, that for every Rauzy class $\mathscr{R}$ and for Lebesgue almost all $([\lambda], \pi_t, \pi_b) \in \mathbb{P}_+^{d-1} \times \mathscr{R}$,

$$\bigcap_{n\in\mathbb{N}} \Delta_n(\lambda, \pi_t, \pi_b) = \{([\lambda], \pi_t, \pi_b)\}.$$

It follows that, for n large enough, the connected component $\Delta_n(\lambda, \pi_t, \pi_b)$ is a relatively compact subset of $\mathbb{P}_+^{d-1} \times (\pi_t, \pi_b)$. Let then Δ denote any of such relatively compact connected components of the domain of definition of any iterate of the Rauzy–Veech map. Let $T\colon \Delta \to \Delta$ be the first-return map of the Rauzy–Veech map to Δ. As we shall explain, any such map is strongly expanding with respect to an absolutely continuous, invariant ergodic probability measure. The key point is that by construction there exists a countable partition $\{\Delta^{(l)} | l \in \Lambda\}$ such that $T|\Delta^{(l)}$ is a bijection onto Δ with inverse map given by a matrix with positive coefficients. Such a partition can be described in symbolic terms as follows. For every $n \in \mathbb{N}$, every connected component of $Z^n|\mathbb{P}_+^{d-1} \times \{(\pi_t, \pi_b)\}$ is given by a word of length n in the alphabet given by strings of the form $t \cdots t$ and $b \cdots b$. Let γ be the corresponding path in the Rauzy graph starting at (π_t, π_b). The elements of the partition are given in symbolic terms by all paths in the Rauzy graph starting and ending with the path γ. By construction, the restriction $T|\Delta^{(l)}$ is injective and its inverse is the restriction of a projective map given by a matrix with nonnegative integer coefficients. In fact, since Δ is relatively compact, all inverse branches of T are given by matrices with *positive* integer coefficients.

3.1.4. *Projective contractions.* Following [4], a *projective contraction* is a projective transformation taking the *standard simplex* $\mathbb{P}_+^{d-1}$ into itself. A projective contraction is given by the projectivization of some matrix $A \in GL(d,\mathbb{R})$ with strictly positive coefficients. The image of the standard simplex under a projective contraction is called a simplex. The following crucial result (see [4], Lemma 2.1) establishes that the first-return maps of the Rauzy–Veech map introduced above are strongly expanding.

LEMMA 5. *Let Δ be a simplex relatively compact in the standard simplex $\mathbb{P}_+^{d-1}$ and let $\{\Delta^{(l)} | l \in \mathbb{Z}\}$ be a partition of Δ (modulo sets of Lebesgue measure 0) into sets of positive Lebesgue measure. Let $T: \Delta \to \Delta$ be a measurable transformation such that, for all $l \in \mathbb{Z}$, the restriction $T|\Delta^{(l)}$ is an invertible map onto Δ and its inverse is the restriction of a projective contraction. Then T is strongly expanding with respect to an invariant probability measure μ on Δ that is absolutely continuous with respect to Lebesgue measure and has a continuous, positive density in $\overline{\Delta}$.*

The proof of the lemma is based on the simple observation that all pushforwards of the restriction of the Lebesgue measure under iterates of T on Δ belong to the space of measures that are absolutely continuous with respect to Lebesgue and have a density with d-Lipschitz logarithm with respect to the projective distance (Hilbert metric). In fact, with respect to the Hilbert metric the logarithm of the Jacobian of any projective contraction is d-Lipschitz and any relatively compact simplex has finite diameter.

We will refer to the above construction, which plays a crucial role in the work of Avila on Teichmüller dynamics, as the *Avila acceleration* of the Rauzy–Veech map. More generally, an Avila acceleration of the Rauzy–Veech map is defined to be any map induced by the Rauzy–Veech map on any finite union of *relatively compact* connected components of the domain of any of its iterates. Let $T: \Delta \to \Delta$ denote any Avila acceleration and let (T, A) the corresponding Avila acceleration of the Rauzy–Veech–Zorich cocycle (Z, C_Z) defined in formula (14). By definition, the cocycle (T, A) is also an acceleration of the Rauzy–Veech cocycle. It follows immediately from the construction that that the cocycle (T, A) is a locally constant, integral cocycle. It is also uniform as a corollary of a key result by Zorich [58] who proved that the Zorich acceleration of the Rauzy–Veech map has an absolutely continuous invariant ergodic *probability* measure and that the Rauzy–Veech–Zorich cocycle is uniform in the above sense (see formula (24)).

As we shall see, Avila's Brin prize contributions to Teichmüller dynamics, described in the previous section, are based on abstract theorems on locally constant, uniform, integral cocycles.

3.2. **Weak mixing for interval-exchange transformations and translation flows.** By standard ergodic theory the weak mixing property of a dynamical system is a spectral property equivalent to the property of having continuous spectrum in the orthogonal complement of the subspace of constant functions. In other terms, a dynamical system is weakly mixing if and only if the only eigenfunctions are constants. For interval-exchange transformations and translation flows, Veech (see [51], §7) proved a criterion that characterizes nonweakly mixing systems in terms of the Rauzy–Veech–Zorich cocycle.

3.2.1. *The Veech criterion.* Weak mixing for an interval-exchange transformation f is equivalent to the existence of no nonconstant measurable solutions $\phi\colon I \to \mathbb{C}$ of the equation

$$\phi \circ f(x) = e^{2\pi i t}\phi(x)\,, \quad \text{for any } (x,t) \in I \times \mathbb{R}\,.$$

If f is ergodic, the above condition is equivalent to the following: there are no nonzero measurable solutions $\phi\colon I \to \mathbb{C}$ of the equation

$$\phi \circ f(x) = e^{2\pi i t}\phi(x)\,, \quad \text{for any } (x,t) \in I \times \mathbb{R} \setminus \mathbb{Z}\,.$$

By the Keane conjecture (see [41] and [50]), almost all interval-exchange transformations are uniquely ergodic, hence ergodic, and thus the latter condition is sufficient.

It is convenient to consider the question of weak mixing for interval-exchange transformations in the more general context of translations flows. In fact, weak mixing of an interval-exchange transformation f can be reformulated in terms of weak mixing for the special flow over f with constant roof function. Since the action of the invertible Rauzy–Veech map can be viewed simply as the interval-exchange transformation obtained from the same special flow by inducing on the appropriate subinterval of the original interval, we are naturally led to consider weak mixing for the general special flows over interval-exchange transformations with piecewise constant roof functions.

Let $F := F(\lambda, h, \pi_t, \pi_b)$ be the special flow over the interval-exchange transformation $f := f(\lambda, \pi_t, \pi_b)$ with piecewise constant roof function specified by the vector $h \in \mathbb{R}_+^{\mathscr{A}}$, that is, the roof function is constant equal to h_α on the subinterval $I_\alpha^t := I_\alpha^t(\lambda, \pi_t)$, for all $\alpha \in \mathscr{A}$. We remark that, by Veech's 'zippered rectangles' construction (see Section 2.3.2), if F is a translation flow, then necessarily $h \in H(\pi_t, \pi_b)$. The phase space of F is the union of disjoint rectangles $R_\alpha := I_\alpha \times [0, h_\alpha)$, and the flow F is completely determined by the conditions

$$\begin{aligned} F_s(x,0) &= (x,s)\,, \quad \text{for } x \in I_\alpha^t \text{ and } 0 \le s < h_\alpha, \\ F_{h_\alpha}(x,0) &= (f(x),0)\,, \quad \text{for all } \alpha \in \mathscr{A}\,. \end{aligned}$$

Weak mixing for the flow F is equivalent to the existence of no nonconstant measurable solutions $\phi\colon D \to \mathbb{C}$ of the equation

$$F_s \circ \phi(x) = e^{2\pi i t s}\phi(x), \quad \text{for any } (x,t) \in D \times \mathbb{R},$$

or, in terms of the interval-exchange transformation f, it is equivalent to ergodicity and to the existence of no nonconstant measurable solutions $\phi\colon I \to \mathbb{C}$ of the equation

$$\phi \circ f(x) = e^{2\pi i t h_\alpha} \phi(x), \quad \text{for any } (x,t) \in I^t_\alpha \times \mathbb{R}, \text{ for any } \alpha \in \mathscr{A}. \tag{27}$$

From the above discussion, it should be clear that the above condition is invariant under the action of the Rauzy–Veech cocycle; in fact, under the action of the Rauzy–Veech cocycle, the special flow over the interval-exchange transformation f with roof function given by the vector $h \in \mathbb{R}^{\mathscr{A}}_+$ is unchanged, while the transverse interval is shrank appropriately, so that the flow is represented as a special flow over the interval-exchange transformation f' with roof function given by the vector $h' \in \mathbb{R}^{\mathscr{A}}_+$. Under the Rauzy–Veech induction, the size of the transverse interval converges to zero, so that any continuous eigenfunction is approximately constant. For measurable eigenfunctions an argument based on Luzin's Theorem yields the following result.

THEOREM 6 (Veech, [51], §7). *For any Rauzy class $\mathscr{R} \subset \Sigma_{\mathscr{A}}$, there exists an open set $U_{\mathscr{R}} \subset \mathbb{P}^{d-1}_+ \times \mathscr{R}$ with the following property. Assume that, under the Rauzy induction R, the orbit of $([\lambda], \pi_t, \pi_b)) \in \mathbb{P}^{d-1}_+ \times \mathscr{R}$ visits $U_{\mathscr{R}}$ infinitely many times. If there exists a nonconstant measurable solution $\phi\colon I \to \mathbb{C}$ of the above equation* (27)*, then the vector th converges to zero modulo $\mathbb{Z}^d$ under the action of the Rauzy–Veech–Zorich cocycle along any sequence of return times of the orbit of $([\lambda], \pi_t, \pi_b)$ to $U_{\mathscr{R}}$ under the Rauzy–Veech induction; that is,*

$$\lim_{R^n(([\lambda],\pi_t,\pi_b)\in U_{\mathscr{R}}} \| C^n_R([\lambda], \pi_t, \pi_b, th) \|_{\mathbb{R}^d/\mathbb{Z}^d} = 0. \tag{28}$$

Note that for a continuous eigenfunction $\phi\colon I \to \mathbb{C}$, the limit in the above formula (28) can be taken without restrictions (as $n \to +\infty$) and it is equal to zero. In general, for a measurable eigenfunctions, the argument based on Luzin's Theorem requires that the surface decomposes into zippered rectangles of comparable area. Such a restriction on the geometry of the zippered rectangles decomposition is behind the restriction of the limit to appropriate visiting times.

We recall that the quotient of the Rauzy–Veech–Zorich cocycle to the bundle $\mathbb{R}^{\mathscr{A}}/H_{\mathscr{R}}$ has zero Lyapunov exponents and it is, in fact, isometric. Thus, whenever the vector $h \notin H(\pi_t, \pi_b)$ (see the definition of the Veech space $H(\pi_t, \pi_b)$ and of the corresponding bundle $H_{\mathscr{R}}$ in Section 2.2.2), the corresponding special flow is weakly mixing for almost all $\lambda \in \mathbb{R}^+$. The Veech criterion, therefore, implies the generic weak mixing property for an interval-exchange transformation on 3 intervals that are not rotations (a result originally proved by A. Katok and A. Stepin [31]) and for special combinatorial data in every number of intervals (namely, the combinatorial data (π_t, π_b) such that the vector $(1, \dots, 1) \notin H(\pi_t, \pi_b)$). Note also that translation flows always correspond to the case $h \in H(\pi_t, \pi_b)$, hence for this case, as well as for the case of interval-exchange transformations with general combinatorial data, the analysis of the Lyapunov structure of the restricted Rauzy–Veech–Zorich cocycle becomes relevant. We recall that such a cocycle is nonuniformly hyperbolic (with respect to the canonical measures) as proved in [21] and later by Avila and Viana in [8]. The work of Avila on the weak mixing for interval-exchange transformations and translation flows [4] rests on this important advance in the theory of the Rauzy–Veech–Zorich cocycle [21]; in fact, for Avila–Forni's proof of weak mixing for almost all interval-exchange transformations or translation flows it is enough to know that the cocycle has a second positive Lyapunov exponent.

3.2.2. *The weak stable space.* The Veech criterion motivates the following abstract definition introduced in [4].

DEFINITION 7. Let (T,A) be cocycle on $\Delta\times\mathbb{R}^d$. For any $x\in\Delta$, the *weak stable space* $W^s(x)$ is defined as follows:

$$W^s(x) := \{v\in\mathbb{R}^d \,|\, \|A_n(x)v\|_{\mathbb{R}^d/\mathbb{Z}^d}\to 0\}\,.$$

If the cocycle is integral, $W^s(x)$ has a natural interpretation as the stable space at $(x,0)$ of the zero section in $\Delta\times\mathbb{R}^d/\mathbb{Z}^d$.

It is immediate to see that for almost all $x\in\Delta$, the space $W^s(x)$ is a union of translates of the stable Oseledets space $E^s(x)$ of the cocycle. If the cocycle is bounded, that is, if the function $A\colon\Delta\to GL(d,\mathbb{R})$ is essentially bounded, then it is easy to see that the weak stable space is a countable union of translates of the stable space; that is,

$$W^s(x) = \bigcup_{c\in\mathbb{Z}^d} E^s(x)+c\,.$$

In general, the weak stable space $W^s(x)$ may be the union of uncountably many translates of the stable space $E^s(x)$. Thus, the weak stable space is described by its transverse structure, that is, by the its intersection $W^s(x)\cap E^{cu}$ with the central unstable Oseledets subspace $E^{cu}(x)$ of the cocycle. If the cocycle is (nonuniformly) hyperbolic, it is not hard to bound the transverse Hausdorff dimension, that is, the Hausdorff dimension of $W^s(x)\cap E^{cu}$, for almost all $x\in\Delta$. In fact, the following result holds:

THEOREM 8 (see [4, Theorem A.1]). *Let (T,A) be any measurable cocycle on the space $\Delta\times\mathbb{R}^d$. For almost every $x\in\Delta$, if $G\subset\mathbb{R}^d$ is any affine subspace parallel to a linear subspace $G_0\subset\mathbb{R}^d$ transverse to the central stable space $E^{cs}(x)$, then the Hausdorff dimension of $W^s(x)\cap G$ is equal to 0.*

The proof of the above theorem is based on quite straightforward estimates based on the assumption that, by the Oseledets Theorem, the cocycle is expansive on the affine subspace $G\subset\mathbb{R}^d$ and it has bounded exponential growth rate, by the Birkhoff Ergodic Theorem.

By the Veech criterion, by the above theorem, and by nonuniform hyperbolicity of the Rauzy–Veech–Zorich cocycle, we derive an estimate on the Hausdorff dimension of the set of nonweakly mixing translation flows.

THEOREM 9 (see [4, Theorem A.2]). *Let $(\pi_t,\pi_b)\in\Sigma_{\mathscr{A}}$ be any given combinatorial data. Then for almost every $\lambda\in\mathbb{R}_+^{\mathscr{A}}$, the set of $h\in H(\pi_t,\pi_b)\subset\mathbb{R}_+^d$ such that the vertical flow of the zippered rectangle (λ,h,π_t,π_b) is not weakly mixing has Hausdorff dimension at most $g(\pi_t,\pi_b)+1$. In particular, if $g(\pi_t,\pi_b)\geq 2$, then for* *almost every $\lambda \in \mathbb{R}^d_+$ and almost every $h \in H(\pi_t, \pi_b) \subset \mathbb{R}^d_+$, the vertical flow of the zippered rectangle $(\lambda, h, \pi_t, \pi_b)$ is weakly mixing.*

Theorem 9 above is proved as follows. Let $\mathscr{R} \subset \Sigma_{\mathscr{A}}$ be any Rauzy class and let $U_{\mathscr{R}} \subset \mathbb{P}^{d-1}_+ \times \mathscr{R}$ be the open set given in the statement of the Veech criterion. We choose any relatively compact connected component Δ of the domain of definition of a sufficiently high iterate of the Rauzy–Veech map such that $\Delta \subset U_{\mathscr{R}}$ and let (T, A) denote the Avila acceleration of the Rauzy–Veech–Zorich cocycle given by the first-return map to $\Delta \subset \mathbb{P}^{d-1}_+ \times \mathscr{R}$. This construction is possible since, as we have recalled by the Keane conjecture, such components generically shrink to points. Then, by the Veech criterion, if the vertical flow of the zippered rectangle $(\lambda, h, \pi_t, \pi_b)$ is not weakly mixing then the line $\mathbb{R}h$ intersects the weak stable space $W^s(\lambda, h, \pi_t, \pi_b)$ of the Avila acceleration (T, A). However, by the nonuniform hyperbolicity of the restricted Rauzy–Veech–Zorich cocycle (hence of the Avila acceleration), any linear subspace G_0 transverse to the central stable space (equal to the stable space) has codimension equal to $g(\pi_t, \pi_b)$. Hence, by Theorem 8, the weak stable space of the Avila acceleration has Hausdorff dimension exactly equal to $g(\pi_t, \pi_b)$. Finally, the Hausdorff dimension of the set of vector $h \in H(\pi_t, \pi_b)$ such that the line $\mathbb{R}h \subset W^s(\lambda, h, \pi_t, \pi_b)$ is at most $g(\pi_t, \pi_b) + 1$, as stated. A similar argument proves that the set of nonweakly mixing translation flows has measure zero under the weaker hypothesis that the cocycle has at least two strictly positive exponents.

As we have described, the fact that almost all translation flows are weakly mixing is a rather direct consequence of the nonuniform hyperbolicity of the Rauzy–Veech–Zorich cocycle. It is sufficient to estimate (for almost all length data) the dimension of the set of the nonweakly mixing height data that are to be discarded. We call this procedure a *linear parameter elimination* since for almost all length data we eliminate the nonweakly mixing height data from an open cone in a vector space. The proof of weak mixing for interval-exchange transformations, which is much harder, is based on a probabilistic *nonlinear parameter elimination* of the nonweakly mixing length data, since in that case the height data are fixed.

3.2.3. *Weak mixing for interval-exchange transformations.* By the Veech criterion, the weak mixing property for almost all interval-exchange transformations is a consequence of the following statement. For any given irreducible combinatorial data $(\pi_t, \pi_b) \in \Sigma_{\mathscr{A}}$, and for almost all length vectors $\lambda \in \mathscr{R}$, the line $\mathbb{R} \cdot (1, \dots, 1)$, spanned by the height vector $(1, \dots, 1)$, does not intersect the weak stable space $W^s(\lambda, \pi_t, \pi_b) \subset H(\pi_t, \pi_b)$ of the restricted Rauzy–Veech–Zorich cocycle at $([\lambda], \pi_t, \pi_b) \in \mathbb{P}^{d-1}_+ \times \Sigma_{\mathscr{A}}$. Since the weak stable space contains all integer translations of the Oseledets stable space, it is necessary to prove the following preliminary result. For almost all length vectors $\lambda \in \mathscr{R}$, no nontrivial integer translate of the line $\mathbb{R} \cdot (1, \dots, 1)$ intersects the central stable space $E^{cs}(\lambda, \pi_t, \pi_b) \subset H(\pi_t, \pi_b)$ of the Rauzy–Veech–Zorich cocycle under the assumption that

$$\dim E^{cs}(\lambda, \pi_t, \pi_b) < 2g(\pi_t, \pi_b) - 1 .$$

This preliminary step, called the *elimination of the stable space*, is enough to establish topological weak mixing, that is, the absence of continuous eigenfunctions, and was essentially established by Nogueira and Rudolph in [43] in their proof of topological weak mixing. Note that the crucial hypothesis on the dimension of the central stable space holds whenever $g(\pi_t,\pi_b)\geq 2$ by the result of [21] (and later [8]) on the Lyapunov spectrum of the Rauzy–Veech–Zorich cocycle, which established that the cocycle is nonuniformly hyperbolic, hence the central stable space has dimension equal to the genus $g(\pi_t,\pi_b)$ of the 'zippered rectangles' surface.

In [4] weak mixing for almost all interval-exchange transformations is proved as a consequence of the Veech criterion and of an abstract theorem on locally constant integral uniform cocycles that implies that whenever it is possible to eliminate the stable space, then it is possible to eliminate the weak stable space as well.

A compact set $\Theta\subset\mathbb{P}^{d-1}$ is said to be *adapted* to the a locally constant cocycle (T,A) if $A^{(l)}\Theta\subset\Theta$ for all $l\in\Lambda$ and if, for almost every $x\in\Delta$ and every $v\in\mathbb{R}^d\setminus\{0\}$ such that $[v]\in\Theta$, we have

$$\|A(x)\cdot v\|\geq\|v\|\quad\text{ and }\quad\|A_n(x)\cdot v\|\to+\infty.$$

Let $\mathscr{J}:=\mathscr{J}(\Theta)$ denote the set of lines in $\mathbb{R}^d$, parallel to some element of Θ and not passing through 0 (see [4, §3]).

THEOREM 10 ([4, Theorem 3.1]). *Let (T,A) be a locally constant integral uniform cocycle, and let Θ be adapted to (T,A). Assume that for every line $J\in\mathscr{J}:=\mathscr{J}(\Theta)$, we have that $J\cap E^{cs}(x)=\emptyset$ for almost every $x\in\Delta$. Then for any line $L\subset\mathbb{R}^d$ parallel to some element of Θ, we have $L\cap W^s(x)\subset\mathbb{Z}^d$ for almost every $x\in\Delta$.*

The proof of the above theorem is a rather technical probabilistic elimination procedure. We outline some of the main ideas below.

For any $x\in\Delta$ and for any $\delta>0$, let $W^s_{\delta,n}(x)$ be the set of vectors $v\in\mathbb{R}^d$ such that $\|A_k(x)v\|_{\mathbb{R}^d/\mathbb{Z}^d}\leq\delta$ for all $k\leq n$. Let $W^s_\delta(x)$ be the intersection of the spaces $W^s_{\delta,n}(x)$ for all $n\in\mathbb{N}$. It follows from the definition that $W^s(x)\subset W^s_\delta(x)$ for all $\delta>0$. The theorem then follows immediately from the following result.

LEMMA 11 ([4, Lemma 3]). *There exists $\delta>0$ such that for all $J\in\mathscr{J}$ and for almost all $x\in\Delta$ we have $J\cap W^s_\delta(x)=\emptyset$.*

Under the action of the cocycle along any orbit segment, any interval $J\in\mathscr{J}$ is stretched and its intersection with union of all the balls of radius $\delta\in(0,1/10)$, centered at integer points, may have several connected components. The inverse image of all such components "survives" at this stage along the orbit and may still belong to the space $W^s_\delta(x)$, while of course all the other points already do not "survive". It is clear that in first approximation the likelihood that a segment $J\in\mathscr{J}$ "survives" (that is, that it contains "surviving" points) for a long time is inversely proportional to the distance $\|J\|$ of the segment from the nearest lattice point. The goal of the proof is to establish that the "survival" probability of any interval J, weighted by a negative power of its distance $\|J\|$ from the integer lattice, converges to zero as time goes to infinity.

By the bounded distortion assumptions on a strongly expanding map, the argument goes as if the map were a shift on countably many symbols. For any $J \in \mathscr{J}$ such that $\|J\| < \delta$ and for any finite sequence $\underline{l} = (l_1, \dots, l_n) \in \Omega$ of symbols, we let $J_{\underline{l},0} = A^{\underline{l}} \cdot J$ and let

$$J_{\underline{l},1}, \dots, J_{\underline{l},\phi_\delta(\underline{l},J)}$$

be the connected components (taken modulo the action of the integer lattice) of the intersection of $A^{\underline{l}} \cdot (J \cap B_\delta(0))$ with the union of all balls of radius $\delta > 0$ at all lattice points different from the origin, modulo the action of the lattice. The interval $J_{\underline{l},0}$ may be called the "parent" and the segments $J_{\underline{l},1}, \dots, J_{\underline{l},\phi_\delta(\underline{l},J)}$ may be called the "children". Note that for every $\underline{l} \in \Omega$ there are no "children" at all if $\delta > 0$ is sufficiently small and that the number $\phi_\delta(\underline{l}, J)$ of "children" is bounded by the norm $\|A^{\underline{l}}\|$ of the cocycle matrix. In our situation the probability of "survival" of the "children" adds to the probability of "survival" of the "parent". It is therefore crucial to be able to estimate the likelihood of "survival" of the "children". By an elementary geometric estimate

$$\min_{k \geq 1} \|J_{\underline{l},k}\| \geq 2^{-1} \|A^{\underline{l}}\|_0^{-1}. \tag{29}$$

It follows that, on the one hand, the likelihood of "survival" of the parent decreases with high probability, since any interval $J \in \mathscr{J}$ is pushed away from the origin in the direction of the second positive exponent (as by assumption $J \cap E^{cs}(x) = \emptyset$), and, on the other hand, under sufficiently strong integrability conditions on the cocycle, the number of children and their likelihood of "survival" is not very large sufficiently often. Hence, the weighted probability of "survival" of any interval $J \in \mathscr{J}$ converges to zero with time.

The crucial technical step is to prove a kind of supermartingale inequality for a sufficiently small negative power of the weight function of the "parent" interval, that is, roughly speaking, to prove the following claim. For simplicity, let us assume that the T is a shift on countably many symbols and let P denote the invariant probability.

CLAIM 12 ([4, Claim 3.5])**.** *There exists $N_0 \in \mathbb{N}$ such that for any $N > N_0$ there is a finite subset $Z \subset \Omega^N$ of large measure and number $\rho_0 > 0$ such that for every $0 < \rho < \rho_0$, every $J \in \mathscr{J}$ and every $Y \subset \Delta$ with $P(Y) > 0$, we have*

$$\sum_{\underline{l} \in Z} \|J_{\underline{l},0}\|^{-\rho} P(\Delta^{\underline{l}} | \bigcup_{\underline{l}' \in Z} \Delta^{\underline{l}'} \cap T^{-N}(Y)) \leq (1-\rho)\|J\|^{-\rho}.$$

In other words the expectation of the likelihood of "survival" of the "parent" interval decreases by a factor (< 1) after a sufficiently long time. The inequality holds since on a large measure set the interval is pushed away from the origin uniformly in the direction of the unstable space of the cocycle. Sets of sufficiently small measure can be neglected by the integrability of the cocycle.

In fact, it is remarkable that the above claim holds under the quite general assumption of uniformity (log-integrability) of the cocycle. For Avila accelerations of the Rauzy–Veech–Zorich cocycle, it was later proved by Avila, Gouëzel and Yoccoz [6] that appropriate Avila accelerations satisfy a much stronger integrability condition, namely they belong to the space $L^p(\Delta)$ for all $p < 1$ (see Corollary 29). Under such a stronger integrability assumption, the proof of the above claim is much more straightforward than the proof of the general case, given in [4]. There, the above supermartingale inequality for $\|J\|^{-\rho}$ is deduced, for sufficiently small exponents $\rho > 0$, by derivation with respect to the exponent of the analogous supermartingale inequality for the function $\log\|J\|$, which in turn can be proved directly for all log-integrable cocycles (see [4, Claim 3.4 and the proof of Claim 3.5]). This extremely clever argument is just one small example of Avila's exceptional technical abilities. It should be mentioned that a similar argument can be found in earlier work of Eskin and Margulis (see [18, Lemma 4.2]), but it was found independently to tackle a possible lack of integrabillity of (small) powers of the norm of the cocycle in [4], while in [18] the integrability of small powers is a key assumption (see [18, formula (2)]).

The proof of Lemma 11 is completed as follows. Let $N > N_0$ be any fixed integer such that the above Claim holds. For any $J \in \mathscr{J}$, for any $\delta > 0$, and for all $m \in \mathbb{Z}$, let $\Gamma^m_\delta(J) := \{x \in \Delta | J \cap W^s_{\delta,mN}(x) \neq \emptyset\}$ be the set of points such that the interval J "survives" (within distance $\delta > 0$ of the integer lattice) up to time $mN \in \mathbb{N}$. The goal is then to prove that the probability $P(\Gamma^m_\delta(J))$ converges to zero as m diverges to infinity. Let $\hat\Omega^N$ be the subset of all finite words of length multiple of $N \in \mathbb{N}$ and let $\hat\Omega^N_Z$ be the set of all finite sequences in the in the alphabet $(\Omega^N \smallsetminus Z) \cup \{Z\}$. There is a natural map $\Phi_N : \hat\Omega^N \to \hat\Omega^N_Z$ that maps every subsequence $\underline{l} \in \Omega^N \smallsetminus Z$ with the letter $\underline{l} \in \hat\Omega^N_Z$ and replaces every sequence $\underline{l} \in Z$ with the letter $Z \in \hat\Omega^N$. For any $\underline{d} = (d_1, \dots, d_m) \in \hat\Omega^N_Z$, let

$$\hat\Delta^{\underline{d}} = \bigcup_{\Phi_N(\underline{l}) = \underline{d}} \Delta^{\underline{l}}.$$

The above definition means that we do not distinguish trajectories that differ only on strings that belong to the finite set $Z \subset \Omega_N$.

CLAIM 13 ([4, Claim 3.6]). *For any* $\underline{d} = (d_1, \dots, d_m) \in \hat\Omega^N_Z$ *we have*

$$\frac{P(\Gamma^m_\delta(J)|\hat\Delta^{\underline{d}})}{\|J\|^{-\rho}} \le \prod_{d_i = Z} (1-\rho) \prod_{d_i \notin Z} \|A^{\underline{d}_i}\|_0^\rho (1 + (2\delta)^\rho \|A^{\underline{d}_i}\|_0).$$

The proof of the above claim is derived from Claim 12 for the estimate on the weight of the "parent" interval and by straightforward bounds on the number and weight of the "children" (see for instance formula (29)). The argument proceeds by induction on the length $m \in \mathbb{N}$ of the symbolic sequence.

Finally, the result follows from the ergodic theorem and by the uniform property (log-integrability) of the cocycle. In fact, if the finite set $Z \subset \Omega^N$ corresponds to a set of sufficiently large measure and $\delta > 0$ is sufficiently small (so that in particular strings in the finite set Z produce no "children"), the function

$$\gamma(x) := \begin{cases} -\rho\,, & x \in \bigcup_{\underline{l}\in Z} \Delta^{\underline{l}}, \\ \rho \log \|A^{\underline{l}}\|_0 + \log(1+(2\delta)^\rho \|A^{\underline{l}}\|_0)\,, & x \in \bigcup_{\underline{l}\in \Omega^N \setminus Z} \Delta^{\underline{l}}, \end{cases}$$

has strictly negative average. It follows that

$$\begin{aligned} P(\Gamma^m_\delta(J)) &= \sum_{\underline{d}\in\hat{\Omega}^N_Z, |\underline{d}|=m} P(\Gamma^m_\delta(J)|\hat{\Delta}^{\underline{d}})P(\hat{\Delta}^{\underline{d}}) \\ &\le \|J\|^{-\rho} \int_\Delta \exp[\sum_{k=0}^{m-1} \gamma\Big(T^{kN}(x)\Big)]\,dP(x)\,. \end{aligned}$$

Hence, by the ergodic theorem, the "survival" probability $P(\Gamma^m_\delta(J))$ converges to zero (exponentially fast) as $m \in \mathbb{N}$ diverges to infinity. Our outline of the proof of Lemma 11, hence of Avila–Forni's proof of weak mixing for almost all interval-exchange transformations that are not rotations, is thus complete.

3.3. **The Kontsevich–Zorich conjecture.** Avila and Viana's proof of the conjecture is in fact the main application of a general criterion for the simplicity of a locally constant cocycle over a strongly expanding transformation (see [8, 9]). In their work, they formulate fundamental notions of *pinching* and *twisting* cocycles. Cocycles that are both pinching and twisting are said to be *simple*. The main abstract theorem states that any simple cocycle over a strongly expanding transformation has a simple Lyapunov spectrum (all Lyapunov exponents are simple). The proof of this result adapts and refines earlier work of several authors (see [37, 11, 12] for deterministic cocycles and [27, 26, 37, 29, 28] for random cocycles). The reader can consult Viana's lectures on Lyapunov exponents (see [54]) for a comprehensive treatment of the theory of Lyapunov exponents including a refined version of the Avila–Viana simplicity theorem (see [54, Chap. 7]).

The proof that the Rauzy–Veech–Zorich cocycle is simple. Hence, it has a simple Lyapunov spectrum, is based on a combinatorial analysis of the dynamics of the Rauzy–Veech–Zorich map near the boundary of the simplex and on an induction argument. These ideas will be developed and refined in Avila's joint work on the exponential decay of correlation of the Teichmüller flow, which we will examine in the next, final section.

We note that since the cocycle is Rauzy–Veech–Zorich cocycle symplectic, it has a symmetric Lyapunov spectrum, and, hence, it is also nonuniformly hyperbolic (as by symmetry the Lyapunov exponent zero has necessarily even multiplicity).

3.3.1. *Simple cocycles.* Pinching and twisting are actually properties of the linear action of the monoid generated by a cocycle. A linear action of a monoid $\mathscr{M}$ is an action by linear isomorphisms of a finite-dimensional vector space H. It naturally induces actions on the projective space $\mathbb{P}(H)$, on the Grassmannians $G_k(H)$ of k-dimensional subspaces and on the space $\mathscr{F}(H)$ of flags. By endowing the vector space H with a euclidean structure (an inner product), we can associate to every element $m \in \mathscr{M}$ its *singular values*

$$\sigma_1(m) \geq \cdots \geq \sigma_{\dim(H)}(m) > 0,$$

which, by definition, are the eigenvalues of the linear self-adjoint operator $m^* m$ on H, as well as its *Lyapunov exponents*

$$\theta_1(m) \geq \cdots \geq \theta_{\dim(H)}(m).$$

DEFINITION 14 (Pinching)**.** The linear action of a monoid $\mathscr{M}$ on a vector space H is said to be *pinching* if for every $C > 0$ there exists $m \in \mathscr{M}$ such that

$$\sigma_i(m) > C\sigma_{i+1}(m), \quad \text{for all } i = 1, \dots, \dim(H) - 1.$$

DEFINITION 15 (Twisting)**.** The linear action of a monoid $\mathscr{M}$ on a vector space H twists a subspace $F \in G_k(H)$ if for every $F_1, \dots, F_J \in G_{\dim(H)-k}$ there exists $m \in \mathscr{M}$ such that

$$mF \cap F_j = \{0\}, \quad \text{for all } j = 1, \dots, J.$$

The action is said to be *twisting* if it twists all subspaces $F \in G(H) = \cup G_k(H)$.

A linear action of a monoid $\mathscr{M}$ is said to be *simple* if it is pinching and twisting. A cocycle is said to be pinching, twisting, simple if the linear action of the monoid generated by the cocycle is respectively pinching, twisting, simple.

Pinching and twisting can be obtained more concretely as follows [12, 9]: a cocycle (T, A) is *pinching* if there exists a periodic point p of period $N \in \mathbb{N}$ such that all eigenvalues of the matrix $A_N(p) = (A \circ T^{N-1})(p) \cdots A(p)$ are distinct; a cocycle (T, A) is *twisting* if there exists a homoclinic point z in the support of the ergodic invariant measure considered such that z belongs to the local unstable manifold $W^u_{loc}(p)$, $T^m(z)$ belongs to the local stable manifold $W^s_{loc}(p)$ and for all $A_N(p)$-invariant subspaces E, $F \subset H$ of complementary dimension

$$A_m(z)E \oplus F = H.$$

The above twisting condition can be checked by computing that all minors of the matrix of $A_m(z)$ with respect to a basis of eigenvectors of $A_N(p)$ are nonzero.

The following result holds:

THEOREM 16 (see [8, Theorem 2])**.** *Let (T, A) be a locally constant measurable cocycle over a strongly expanding transformation $T\colon \Delta \to \Delta$. If the supporting monoid of the cocycle is simple, then the cocycle has simple Lyapunov spectrum.*

The idea underlying the proof of the above theorem is that a strongly expanding transformation is sufficiently chaotic that methods of earlier works on simplicity of the Lyapunov spectrum in the deterministic situation (see in particular [11, 12]) can be applied. The argument is based on the construction of special invariant measures for the action of cocycles on the Grassmannian bundles, called *u-states*, over strongly expanding transformations (the bounded distortion property of strongly expanding transformations plays a key role). An important step is to prove that by the pinching and twisting conditions u-states are delta measures on the Grassmannian fibers. Thus the support of any invariant u-state is an invariant measurable section of a Grassmannian bundle that is in "general" position in a precise sense.

A key feature of the simplicity theorem is that, thanks to chaotic nature of the underlying dynamics, it is possible to derive the conclusion from information gathered from a set of orbits of measure zero. In other terms, the pinching and twisting properties can be established on the basis of a knowledge of the action of the cocycle on special orbits (such as periodic and homoclinic orbits). In the terminology of [8], those special orbits are able to "persuade" almost all orbits to generate simple Lyapunov exponents. This persuasion mechanism is crucial in particular for the application to the Rauzy–Veech–Zorich cocycle since its dynamics are more accessible near the "boundary" of the simplex (near the boundary of the moduli space of 'zippered rectangles' or translation surfaces), as we shall see in the next subsection.

The proof of simplicity of the supporting monoid of the Rauzy–Veech–Zorich cocycle is somewhat simplified since the the cocycle is *symplectic*. For symplectic cocycles, Avila and Viana devise the following strategy [8]. First of all, there is a notion of *strongly pinching* symplectic action:

DEFINITION 17. (Strong Pinching) The linear symplectic action of a monoid $\mathscr{M}$ on a symplectic vector space H (of dimension $2g$) is said to be *strongly pinching* if for every $C > 0$ there exists $m \in \mathscr{M}$ such that

$$\begin{aligned} &\log \sigma_g(m) > C \quad \text{and} \\ &\log \sigma_i(m) > C\sigma_{i+1}(m), \quad \text{for all } i = 1, \dots, g-1. \end{aligned}$$

In the symplectic case, under the strong pinching condition, it is possible to restrict the twisting condition to isotropic subspaces. We say that the symplectic action of a monoid *twists isotropic subspaces* if the twisting property of Definition 15 holds on the Grassmannians of isotropic subspaces.

LEMMA 18 (see [8, Lemma 4.14]). *Let $\mathscr{M}$ be a monoid that acts symplectically on a symplectic vector space H. If the action of $\mathscr{M}$ on H is strongly pinching and twists isotropic subspaces, then it is simple (pinching and twisting).*

The strong pinching property is established thanks to the following result.

LEMMA 19 (see [8, Lemma 4.7]). *Let $\mathscr{M}$ be a monoid that acts symplectically on a symplectic vector space H of dimension $2g$. Assume that for every $C > 0$ there exists $m \in \mathscr{M}$ for which 1 is an eigenvalue of geometric multiplicity one,*

$$\theta_{g-1}(m) > 0 \quad \text{and} \quad \theta_i(m) > C\theta_{i+1}(m), \quad \text{for all } i = 1, \dots, g-2,$$

then the action of $\mathscr{M}$ on H is strongly pinching.

Finally, for a symplectic action the property of twisting isotropic subspaces follows from the minimality of the action on the space of Lagrangian flags.

LEMMA 20 (see [8, Lemma 4.4]). *Let $\mathscr{M}$ be a monoid that acts symplectically on a symplectic vector space H. If the action of $\mathscr{M}$ on the space $\mathscr{L}(H)$ of Lagrangian flags is minimal, then it twists isotropic subspaces.*

From the above discussion, we conclude that the Kontsevich–Zorich conjecture can be derived as soon as it is proved that the action of the Rauzy–Veech–Zorich monoid on the space of Lagrangian flags is minimal and that the strong pinching conditions of Lemma 19 hold.

3.3.2. *Simple reduction/extension.* In this subsection we outline the main ideas of the proof of the main result of Avila and Viana on the Rauzy–Veech–Zorich cocycle: the proof of the Kontsevich–Zorich conjecture.

THEOREM 21 ([8, Corollary 6.2])**.** *For every Rauzy class $\mathscr{R}$, the action of the Rauzy–Veech–Zorich monoid on the symplectic bundle $H_{\mathscr{R}}$ is simple, hence the Lyapunov spectrum of the Rauzy–Veech–Zorich cocycle is simple.*

As we have outlined in the previous subsection, the argument is reduced to the proof that the action of the monoid is minimal on Lagrangian flags and strongly pinching. These properties are verified by an induction procedure on the number of intervals. The idea is that we can consider the Rauzy–Veech–Zorich induction on interval-exchange transformations with at least one "very short" interval. A very short interval will remain short for a relatively long time and the orbit will 'shadow' an orbit of the Rauzy–Veech–Zorich induction on the space of interval-exchange transformations with fewer subintervals. Assuming the desired properties are verified for the Rauzy–Veech–Zorich induction on interval-exchange transformations with fewer subintervals, it is possible to derive them for the given space of interval-exchange transformations.

We outline below the main steps of the inductive argument. It is based on operations of *simple reduction* and *simple extension*, defined as follows.

DEFINITION 22. Let $\mathscr{A}$ be a finite alphabet with $d \geq 3$ elements, let $\alpha \in \mathscr{A}$ and let $\mathscr{A}' := \mathscr{A} \setminus \{\alpha\}$. The *simple reduction* of any irreducible combinatorial data $(\pi_t, \pi_b) \in \Sigma_{\mathscr{A}}$ is given by the combinatorial data (π'_t, π'_b) obtained by erasing α from the top and bottom rows whenever (π'_t, π'_b) is irreducible. Otherwise the simple reduction is not defined.

DEFINITION 23. Let $\mathscr{A}'$ be a finite alphabet with $d \geq 2$ elements. For any irreducible combinatorial data $(\pi'_t, \pi'_b) \in \Sigma_{\mathscr{A}'}$, let $\alpha/\beta \in \mathscr{A}'$ be first in top/bottom respectively. Let $\gamma \notin \mathscr{A}'$ and let $\mathscr{A} := \mathscr{A}' \cup \{\gamma\}$. Let $\mathscr{R}(\pi'_t, \pi'_b)$ be the Rauzy class of the combinatorial data $(\pi'_t, \pi'_b) \in \Sigma_{\mathscr{A}'}$. Let $(\delta, \epsilon) \neq (\alpha, \beta)$ in $\mathscr{A}'$. The operation of *simple extension* on the Rauzy diagram of the Rauzy class $\mathscr{R}(\pi'_t, \pi'_b)$ is defined as follows: on vertices, the simple extension L is defined by adding the letter γ to the left of δ/ϵ in the top/bottom to every combinatorial data; on arrows, the operation of simple extension L_* is defined as follows: let l' be an arrow in the Rauzy diagram of the Rauzy class $\mathscr{R}(\pi'_t, \pi'_b)$ with starting point $(\pi''_t, \pi''_b) \in \mathscr{R}(\pi'_t, \pi'_b)$,

- if δ/ϵ is last in top/bottom and l' is of type bottom/top, then $L_*(l')$ is a sequence of 2 bottom/top arrows starting at the simple extension $L(\pi''_t, \pi''_b)$;
- otherwise, $L_*(l')$ is a single arrow that starts from the simple extension $L(\pi''_t, \pi''_b)$ and has the same type as the arrow l'.

The operations of simple reduction and extension are inverses of each other in the sense that the appropriate simple reduction of a simple extension yields the original combinatorial data. The operation of simple extension is surjective, that is, for every irreducible pair of combinatorial data $(\pi_t, \pi_b) \in \Sigma_{\mathscr{A}}$ on $d \geq 3$ letters there exists an irreducible pair of combinatorial data $(\pi'_t, \pi'_b) \in \Sigma_{\mathscr{A}'}$ on $d-1$ letters such that (π_t, π_b) is a simple extension of (π'_t, π'_b). It is possible to analyze completely the action of the operations of simple reduction and extension on the Rauzy–Veech–Zorich cocyle. We briefly summarize the main results below.

There are two cases. Either (1) the genus $g(\pi_t, \pi_b) = g(\pi'_t, \pi'_b)$, or (2) the genus $g(\pi_t, \pi_b) = g(\pi'_t, \pi'_b) + 1$. Topologically, this means the following. In case (1), the simple reduction corresponds to collapsing two or more zeros without pinching the associated surface; in case (2), there is exactly one cycle nonhomologous to zero that gets pinched, so that the resulting Abelian differential on the degenerate surface is still holomorphic (in general, the Abelian differential on the degenerate, pinched surface has simple poles at the punctures).

Let $\mathscr{R} \subset \Sigma_{\mathscr{A}}$ and $\mathscr{R}' \subset \Sigma_{\mathscr{A}'}$ be the Rauzy classes of the pairs of combinatorial data (π_t, π_b) and (π'_t, π'_b), respectively. In case (1), there is no loss of dimension and it can be proved that the Rauzy–Veech–Zorich cocycles on the symplectic bundles $H_{\mathscr{R}}$ and $H_{\mathscr{R}'}$ are symplectically conjugate, so that one is simple if and only if the other is. In case (2), when the genus changes, it is necessary to gain information to make the argument work. The strategy goes as follows. By Lemma 18, it is sufficient to prove that the cocycle twists isotropic subspaces and it is strongly pinching. By Lemma 20 the twisting property can be derived from the minimality of the action on Lagrangian flags. Thus, by the induction hypotheses, it is sufficient to establish the minimality of the action on lines. In turn, this is a consequence of the spectral gap property (the top exponent is simple), already proved by Veech [52] (generalized in [21]).

The core of the proof then consists of establishing the strong pinching property. By the induction hypotheses, it is sufficient to look at the action on the Grassmannian of Lagrangian subspaces. The conclusion can then be derived from the nonuniform hyperbolicity of the cocycle proved in [21]. Avila and Viana relied on this argument in a first version of the paper, then found a completely independent argument, based on Lemma 19. Note that the argument ends up turning around the construction of orbits for which the cocycle is not even hyperbolic; rather, it is parabolic. In fact, it is supposed to have a geometrically simple eigenvalue equal to 1 corresponding to a 2×2 Jordan block. Such orbits are obviously rather special (zero measure) and are found by a careful combinatorial construction.

From a topological point of view, the main idea is to construct surfaces by a combinatorial procedure of “bubbling a handle” inspired by Kontsevich and Zorich [36]. Indeed, a zippered rectangle with a single very thin handle would provide a long orbit segment along which the handle stays very thin and the (nonzero) homology class of its waist curve is necessarily mapped into itself by the cocycle. In fact, it is the only very short curve on the surface! The Rauzy–Veech–Zorich cocycle therefore acts on the surface by *Dehn twists* along the waist curves of the thin handle. It is by no means clear by this heuristic description that all the assumptions of Lemma 19 can be achieved.

In conclusion, we note that it is a common procedure in Riemann surface theory of establishing results on all Riemann surfaces by a careful analysis of surfaces near the "boundary" of the Deligne–Mumford compactification (see for instance [40, 42, 21]). However, to the author's best knowledge, in all cases only geometric conclusions are directly drawn from the boundary analysis. For instance, in [21] formulas for partial sums of the Lyapunov exponents in terms of the (nonnegative) eigenvalues of the curvature of the Hodge norm on the Hodge bundle are derived. The nonvanishing of the exponents is then derived from the nonvanishing of the Hodge eigenvalues at some point in the moduli spaces. The boundary analysis is applied to prove that the Hodge eigenvalues are nonzero near certain "boundary" points.

The main novelty in Avila's approach is a machinery to analyze the *dynamics* of the Rauzy–Veech–Zorich induction (hence of the 'zippered rectangles" flow or of the Teichmüller flow) near the boundary of its phase space. The analysis of the dynamics near the boundary is carried out by induction combined with combinatorial considerations. Let us recall that, by the "persuasion mechanism" underlying the abstract simplicity criterion outlined in the previous subsection, the proof of the simplicity of the Rauzy–Veech–Zorich monoid only requires the construction of special orbits. In fact, somewhat paradoxically, according to Lemma 19 the orbits that ensure the strong pinching property have a couple of zero Lyapunov exponents.

A refinement of the simple reduction/extension and of the associated combinatorial analysis was later applied in the joint work of Avila, Gouëzel and Yoccoz [6] to prove sharp bounds on the probability of long excursions of the orbits of the 'zippered rectangles' flow outside a given compact set. As we will outline in the next, final section, such estimates are a key step in the proof of the exponential decay of correlations for the Teichmüller geodesic flow.

3.4. **Exponential Mixing for the Teichmüller flow.** The proof by Avila, Gouëzel and Yoccoz [6] of exponential decay of correlations for the geodesic flow is based on a general abstract theorem for *excellent hyperbolic semiflows*. This theorem adapts Dolgopyat's proof [17] of exponential decay of correlations for certain Anosov flows, following closely the version of V. Baladi and B. Vallée [10]. The verification of the hypotheses of the abstract theorem for Veech 'zippered rectangles' flow (or, equivalently, for the Teichmüller flow) is based on combinatorial techniques and on an induction procedure that refines the methods of Avila and Viana that we have outlined in the previous section. Once exponential decay of correlations is proved for smooth observables with compact support, a reverse Ratner's argument, based on the theory of unitary representations of the group $SL(2,\mathbb{R})$, implies the existence of a *spectral gap* in the action of $SL(2,\mathbb{R})$ with respect to all canonical absolutely continuous measures on connected components of strata of the moduli space of Abelian differentials. By the results of M. Ratner [46], the spectral gap property for the action of $SL(2,\mathbb{R})$ in turn implies exponential decay of correlations for the Teichmüller geodesic flow, as well as a precise polynomial bound on the decay of correlations for the Teichmüller horocycle flow, *for all observables in the Ratner's class.*

3.4.1. *Excellent hyperbolic semiflows.* Roughly speaking an excellent hyperbolic semiflow is a special flow over a *uniformly expanding Markov map* under a good roof function having exponential tails. The notion of a uniformly expanding Markov map generalizes that of strongly expanding map, which we have recalled in Section 3.1. The reader should keep in mind, according to the definitions, the Avila acceleration of the invertible Rauzy–Veech map is uniformly expanding, so that the Veech 'zippered rectangles' flow is an excellent hyperbolic semiflow as soon as it is verified that the corresponding roof function is good with exponential tails.

Uniformly expanding Markov maps are defined on John domains.

A *John domain* (see [6, Definition 2.1]) is a finite-dimensional connected Finsler manifold Δ equipped with a measure Leb satisfying certain conditions. It suffices to say that any relatively compact, open subset Δ of a larger manifold with boundary a finite union of smooth hypersurfaces in general position, equipped with the restriction to Δ of any smooth measure defined on $\overline{\Delta}$ is a John domain. It is not hard to check that the domain of any Avila acceleration of the Rauzy–Veech–Zorich cocycle, a relatively compact simplex contained in a standard simplex, is a John domain once equipped with the restrictions of the Hilbert metric (as a Finsler metric) and of the Lebesgue measure.

Definition 24. Let $\{\Delta^{(l)}\}_{l\in\Lambda}$ be a finite or countable partition of a full measure subset of a John domain Δ into open subsets. A map $T\colon \cup\Delta^{(l)} \to \Delta$ is said to be a *uniformly expanding Markov map* if

1. for each $l\in\Lambda$, the map T is a C^1 diffeomorphism of $\Delta^{(l)}$ onto Δ and there exist constants $\kappa > 0$ (independent of $l\in\Lambda$ and $C_l > 0$) such that, for all $x\in\Delta^{(l)}$ and for all $v\in T_x\Delta$,
$$\kappa\|v\| \le \|DT(x)v\| \le C_l\|v\|\,;$$
2. let J be the inverse of the Jacobian of T with respect to the measure Leb and denote by $\mathscr{H}$ the set of inverse branches of T; the function $\log J$ is C^1 on $\Delta^{(l)}$ for each $l\in\Lambda$ and there exists $C>0$ such that

$$\sup_{h\in\mathscr{H}} \|D\big((\log J)\circ h\big)\|_{C^0(\Delta)} \le C\,. \tag{30}$$

Note that while the Rauzy–Veech map is not a uniformly expanding Markov map, its Avila accelerations, that is, the return maps to suitable relatively compact subsets introduced in Section 3.1, indeed are. In fact, the notion of a uniformly expanding Markov map is a generalization of that of *strongly expanding* *map* (see Section 3.1.1) modeled on the Avila acceleration. For instance, the crucial bounded distortion property given above in formula (30) follows from the fact that all inverse branches of Avila accelerations are *projective contractions*, hence the logarithm of the Jacobian is a Lipschitz function (see Section 3.1.4). We proceed to state the assumption on the roof function.

DEFINITION 25 ([6, Definition 2.3]). Let $T: \cup \Delta^{(l)} \to \Delta$ be a uniformly expanding Markov map. A function $r: \cup \Delta^{(l)} \to \mathbb{R}_+$ is said to be a *good roof function* if the following properties are satisfied:

1. there exists $\epsilon > 0$ such that $r \geq \epsilon$;
2. there exists $C > 0$ such that, for all inverse branches $h \in \mathscr{H}$,
$$\|D(r \circ h)\|\|_{C^0} \leq C;$$
3. the function r is not a coboundary; that is, it is NOT possible to write
$$r = \psi + \phi \circ T - \phi \quad \text{on } \cup \Delta^{(l)}$$
with $\psi: \Delta \to \mathbb{R}$ constant on all sets $\Delta^{(l)}$ and $\phi: \Delta \to \mathbb{R}$ a C^1 function.

In the application of the abstract theorem on exponential decay of correlations, which we will state below, to Veech 'zippered rectangles' flow it is not hard to verify that the roof function is indeed good, according to the above definition. In fact, the roof function for the representation of the 'zippered rectangles' flow as a special flow over an Avila acceleration is given by an explicit formula (which can be derived from formula (22) in Section 2.3.2). Let J denote, as above, the Jacobian of the map T (an Avila acceleration). For all inverse branches $h \in \mathscr{H}$,
$$r \circ h = \frac{1}{d} \log J \circ h.$$
In particular, it can be proved that property (2) in Definition 25 follows from the fact that inverse branches of the Rauzy–Veech map are projective contractions, by exactly the same argument that proves the bounded distortion property (30) of Definition 24. Note that in fact it is proved that the functions $r \circ h$ are 1-Lipschitz. Property (3) in Definition 25 follows quite easily from the minimality of the action of the Rauzy–Veech–Zorich monoid on lines (proved in [8, Corollary 3.6]), a direct consequence of the spectral gap property of the Rauzy–Veech–Zorich cocycle (see also Section 3.3.2).

The assumptions considered so far do not take into account one of the main technical difficulties in applying Dolgopyat's ideas to the Teichmüller flow, that is, the *lack of compactness* of the moduli space. In fact, the Teichmüller flow has smooth globally defined foliations and it is "locally uniformly hyperbolic" (a consequence, for instance, of the variational formulas for the Hodge norm proved in [21]), which means in particular that all of its return maps to compact subsets are uniformly hyperbolic. In representing the 'zippered rectangles' flow as a special flow over an Avila acceleration of the invertible Rauzy–Veech map, the Markovian, hyperbolic properties of the return map become clear, but the roof function is unbounded. Athreya [2] had proved in his thesis (written under the direction of A. Eskin) that exponentially long excursions of the Teichmüller geodesic flow outside of certain ‘large’ compact subsets of the moduli space have exponentially small probabilities. In fact, such a result was intended as a step in the proof of exponential decay of correlations for the Teichmüller flow; unfortunately, the return maps to the ‘large’ compact sets of [2] do not have good geometric properties, and so any approach to decay of correlations based on Markov partitions is problematic. We note, however, that Athreya’s results have the great advantage of holding for an arbitrary $SL(2,\mathbb{R})$ orbit. After a first application to the speed of convergence of ergodic averages for almost all directional flows on an arbitrary translation surface [4], they have recently been applied by Avila and GouÂŚzel [5] to the original question of decay of correlations for arbitrary ‘algebraic’ $SL(2,\mathbb{R})$-invariant ergodic measures. We note that in the latter work, the approach based on Markov partitions is abandoned for an analytic approach based on appropriate norms on distributional spaces.

Let us resume our discussion of the work of Avila, Gouëzel and Yoccoz on the exponential mixing of the Teichmüller flow [6]. Athreya’s work [2] motivated the following definition:

DEFINITION 26. A good roof function $r\colon \cup\Delta^{(l)} \to \mathbb{R}_+$ has *exponential tails* if there exists $\sigma > 0$ such that

$$\int_\Delta e^{\sigma r(x)} d\mathrm{Leb}(x) < +\infty.$$

An *excellent hyperbolic semiflow* is, roughly speaking, a special flow over a (hyperbolic) extension $\hat{T}$ of a uniformly expanding Markov map under a good roof function with exponential tails. The precise notion, that of an *hyperbolic skew-product*, is given in [6, Definition 2.5] and we will not give it here. It suffices to point out that definition is set up so that Avila accelerations of the *invertible* Rauzy–Veech map are hyperbolic skew-products over the corresponding Avila acceleration of the Rauzy–Veech map. In the general setting, the skew-product $\hat{T}$ is defined on an open bounded subset of a connected Finsler manifold $\hat{\Delta}$ and there exists a projection $\pi\colon \hat{\Delta} \to \Delta$ such that $\hat{T}\circ\pi = \pi\circ T$; it is therefore possible to define a special semiflow over $\hat{T}$ under the roof function $r\circ\pi\colon \hat{\Delta} \to \mathbb{R}_+$. We recall that such a semiflow $\{\hat{T}_t\}$ is defined on the set $\hat{\Delta}_r := \{(y,s) | y \in \hat{\Delta}, 0 < s < (r\circ\pi)(y)\}$ and it is defined as follows: for any $(y,s) \in \hat{\Delta}_r$, let n be the largest integer such that $\sum_{k=0}^n (r\circ T^n)(\pi(y)) < t+s$; then

$$\hat{T}_t(y,s) = \left(\hat{T}^n(y), s+t-\sum_{k=0}^n (r\circ T^n)(\pi(y))\right).$$

We are finally ready to state the main abstract theorem on exponential decay of correlations:

THEOREM 27 ([6, Theorem 2.7]). *Let $\{\hat{T}_t\}$ be an excellent hyperbolic semiflow (which preserves the probability measure ν_r on $\hat{\Delta}_r$). There exist constants $C > 0$ and $\delta > 0$ such that for all functions $U, V \in C^1(\hat{\Delta}_r)$ and for all $t > 0$,*

$$\left|\int_{\hat{\Delta}_r} U\cdot V\circ\hat{T}_t d\nu_r - \left(\int_{\hat{\Delta}_r} U d\nu_r\right)\left(\int_{\hat{\Delta}_r} V d\nu_r\right)\right| \le C\|U\|\|V\|_{C^1} e^{-\delta t}.$$

As we have noted, the above theorem is essentially a well-written adaptation of Dolgopyat's ideas from [17] following in the footsteps of Baladi and Vallée [10].

3.4.2. *Exponential tails.* The exponential mixing of the Teichmüller flow is derived for smooth observables from the corresponding result for the 'zippered rectangles' flow, since the latter is a finite cover of the former. As we have outlined in Section 2.3.2, the 'zippered rectangles' flow can be seen as a special flow over the invertible Rauzy–Veech–Zorich induction, which in turn is an extension of the Rauzy–Veech map. By considering appropriate Avila accelerations, the 'zippered rectangles' flow is seen as a special flow over a hyperbolic skew-product (an Avila acceleration of the invertible Rauzy–Veech map) over a uniformly expanding Markov map (an Avila acceleration of the Rauzy–Veech map). We have briefly outlined in the previous section the proof that the roof function, that is, the return time function corresponding to an Avila acceleration, is good. It appears, then, that the main step in order to apply the above abstract exponential mixing theorem to the 'zippered rectangles' flow is to prove that the roof function has exponential tails.

Several comments are in order. It is not necessary to establish *optimal* exponential tail estimates to establish exponential decay of correlations. A proof of the exponential mixing of the Teichmüller flow can in fact be derived from the above abstract theorem by establishing any exponential tail estimates (even weaker than optimal). Such estimates were already contained in the work of A. Bufetov [14], but Avila, Gouëzel and Yoccoz were not aware of all the aspects of Bufetov's work. After completing the proof of the nearly optimal estimates, they independently found a much easier argument that establishes exponential tail estimates that, however far from optimal, are enough to establish exponential mixing (see [6, Appendix A]). We note that suboptimal exponential tail estimates are relatively straightforward to prove and have been extended to quadratic differentials by Avila and M. J. Resende [7] who derived exponential mixing in that context. As a comparison, the simplicity of the Lyapunov spectrum for canonical measures on strata of nonorientable quadratic differentials (that is, a generalization of Avila–Viana's proof of the Kontsevich–Zorich conjecture) is still an open problem. Recently R. Treviño [47] has given a proof, based on the approach of [21, 24], that the cocycle is nonuniformly hyperbolic on all strata of the moduli space of holomorphic quadratic differentials.

In [6] nearly optimal estimates are indeed proved. Such estimates are of independent interest and have been applied to other questions (see, for instance, [48]). Optimal estimates greatly increase the complexity of the argument in two ways: the combinatorial analysis needed to establish them is much more involved and the combinatorial structure of the Avila acceleration for which they can be proved is more complicated. However, in our opinion, the refined combinatorial analysis of the degeneration of Rauzy classes appears to be of fundamental importance for future applications to dynamics and geometry.

We formulate Avila, GouÂŚzel and Yoccoz nearly optimal estimates below:

THEOREM 28 ([6, Theorem 4.10]). *For every $\delta > 0$, the Veech 'zippered rectangles' flow can be represented as a special flow over an Avila acceleration of the invertible Rauzy–Veech map $\hat{T}$ (that is, a return map to a finite union $\hat{Z}$ of relatively compact connected components of iterates of the invertible Rauzy–Veech map) under a roof function $r_{\hat{Z}}\colon \hat{Z} \to \mathbb{R}_+$ such that*

$$\int_{\hat{Z}} e^{(1-\delta) r_{\hat{Z}}}\, d\mathrm{Leb} < +\infty.$$

The key step in the proof of Theorem 28 above is given by bounds on the measure of sets that undergo large distortion after some long Teichmüller time. Nearly optimal distortion bounds are proved by an induction scheme with respect to the number of subintervals carried out by a refined combinatorial analysis of the Rauzy–Veech induction. It should already be clear by the above short description that the argument is based on a refinement of the combinatorial ideas of the earlier paper on the Kontsevich–Zorich conjecture [8] that we have outlined in Section 3.3.2. Such a refinement begins with a generalization of the operation of simple reduction. The resulting operation, called *reduction* allows to carry out a degeneration of a full Rauzy class. Let us a give a taste of some of the notions involved in the combinatorial analysis.

Let $\mathscr{A}$ be an alphabet and let $\mathscr{A}' \subset \mathscr{A}$. Intuitively the idea is to let all lengths labeled by letters in $\mathscr{A}'$ go to zero, that is, to erase all letters from $\mathscr{A}'$ from the top and bottom rows of every irreducible element $(\pi_t, \pi_b) \in \Sigma_{\mathscr{A}'}$. However, this operation may run into difficulties as the resulting pair of combinatorial data may fail to be irreducible and the resulting Rauzy graph may be undefined. Several technical steps are introduced to overcome this difficulty.

A notion of *$\mathscr{A}'$-decorated Rauzy class $\mathscr{R}_* \subset \mathscr{R}$* is defined as a maximal subset of a Rauzy class $\mathscr{R}$ of combinatorial data that can be joined by an $\mathscr{A}'$-colored path. An *$\mathscr{A}'$-colored path* is a concatenation of *$\mathscr{A}'$-colored arrows*, that is, of arrows whose winner belongs to $\mathscr{A}'$. A pair of combinatorial data $(\pi_t, \pi_b) \in \mathscr{R}$ is said to be *$\mathscr{A}'$-essential* if the last letters of both the top row π_t and the bottom row π_b belong to $\mathscr{A}'$. The *essential $\mathscr{A}'$-decorated Rauzy class $\mathscr{R}^{ess}_*$* $\subset \mathscr{R}$ is defined to be the subset of essential elements. A decorated Rauzy class $\mathscr{R}_*$ is said to be essential if the subset $\mathscr{R}^{ess}_*$ of essential elements is nonempty. For any essential decorated Rauzy class $\mathscr{R}^{ess}_*$, the Rauzy graph is defined by considering all arrows starting and ending at elements of $\mathscr{R}^{ess}_*$.

The notion of *reduction* is defined as follows. Given a pair of combinatorial data $(\pi_t, \pi_b) \in \mathscr{R}^{ess}_*$, we delete all the letters that do not belong to $\mathscr{A}'$ from the top row π_t and the bottom row π_b. The resulting pair (π'_t, π'_b) of combinatorial data is not always irreducible, so we consider the *irreducible end* (called the admissible end in [6]) of (π'_t, π'_b), which is obtained by deleting as many letters from the beginning of the top and bottom rows of (π'_t, π'_b) as necessary to obtain an admissible permutation. The resulting pair of combinatorial data $(\pi^{red}_t, \pi^{red}_b)$ belongs to $\Sigma_{\mathscr{A}''}$ (the set of irreducible pairs of combinatorial data) for some alphabet $\mathscr{A}'' \subset \mathscr{A}$. It is called the reduction of $(\pi'_t, \pi'_b) \in \mathscr{R}^{ess}_*$. The operation of reduction can be extended to the whole essential decorated Rauzy class $\mathscr{R}_*$ by introducing an operation that associates to any given nonessential pair of combinatorial data $(\pi_t,\pi_b)\in\mathscr{R}$ an essential pair $(\pi_t^{ess},\pi_b^{ess})\in\mathscr{R}_*^{ess}$. The operation of reduction can be defined on paths in the Rauzy diagram and it is compatible with concatenation. As we mentioned above, the core of the proof is a rather involved induction procedure on the cardinality of the alphabet based on the reduction operation just outlined (see [6, Section 5]).

As a conclusion of the section, we note that Theorem 28 implies a strong integrability condition for Avila accelerations of the Rauzy–Veech–Zorich cocycle.

COROLLARY 29. *For any $\delta>0$ and any open set $U\subset\mathbb{P}_+^{d-1}\times\Sigma_{\mathscr{A}}$, there exists an Avila acceleration (T,A) of the Rauzy–Veech–Zorich cocycle, defined on a domain $\Delta\subset U$, that satisfies the following strong integrability condition with respect to the canonical T-invariant measure μ on Δ:*

$$\int_\Delta \|A(x)\|_0^{1-\delta}\,d\mu(x) < +\infty. \tag{31}$$

The above $L^{1-\delta}$ integrability condition should be compared with the integrability condition (24) that follows immediately from Zorich results [58]. The main abstract theorem of [4], on which the proof of weak mixing for almost all interval-exchange transformations is based, works for all *uniform* cocycles, that is, for all cocycles that satisfy the log-integrability condition (24). However, the proof becomes simpler and more transparent under an $L^{1-\delta}$ integrability assumption (which was not yet available when [4] was written).

3.4.3. *The spectral gap theorem.* From the exponential mixing of the Teichmüller geodesic flow, Avila, Gouëzel and Yoccoz [6] derived, by a reverse Ratner argument, a spectral gap theorem for the natural action of the group $SL(2,\mathbb{R})$ on the moduli space of Abelian differentials with respect to any of the canonical measures on connected components of the strata (see Section 2.3.1).

A unitary representation of the group $SL(2,\mathbb{R})$ has a *spectral gap* if the spectrum of the Laplacian or of the Casimir operator (the generator of the center of the enveloping algebra of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$) associated to the representation has a spectral gap in the sense of the spectral theory of self-adjoint operators. In terms of the theory of unitary representations of $SL(2,\mathbb{R})$, the spectral gap property can be formulated as follows.

The Casimir operator acts as a multiple of the identity on every irreducible unitary representation of $SL(2,\mathbb{R})$, and irreducible unitary representations are classified up to unitary equivalence by the associated Casimir parameter. They belong to three series: the *principal series* given by all irreducible unitary representations with Casimir parameter larger than $1/4$, the *complementary series* given by all irreducible unitary representations with positive Casimir parameter in the interval $(0,1/4)$, and the *discrete series* given by a sequence of negative, integer Casimir parameters.

Any linear unitary representation of $SL(2,\mathbb{R})$ on a Hilbert space can be decomposed as direct sum or, in general, as a direct integral of irreducible representation. The representation is said to have a *spectral gap* if its direct integral decomposition into irreducible unitary representations does not contain representations with nonzero Casimir parameter arbitrarily close to zero. In other terms, the representation is a direct integral of irreducible unitary representations with respect to a (spectral) measure on the real line supported outside of a pointed neighborhood of the origin.

THEOREM 30 ([6, Corollary 1.1]). *For any canonical $SL(2,\mathbb{R})$-invariant ergodic measure $\mu_{\mathscr{C}}$ on a connected component $\mathscr{C}$ of a stratum of the moduli space of Abelian differentials, the unitary representation of $SL(2,\mathbb{R})$ on the Hilbert space $L^2(\mathscr{C},\mu_{\mathscr{C}})$ of square-integrable functions has a spectral gap.*

It turns out that it is not hard to derive the spectral gap theorem from the exponential mixing of the Teichmüller flow for smooth observables with compact support. In fact, it is enough to prove the following abstract result by an explicit computation in the standard unitary model for irreducible unitary representations of the complementary series. The correlations (along the Teichmüller geodesic flow) of an $SO(2,\mathbb{R})$-invariant vector (unique up to multiplication by a scalar), in each irreducible representation of the complementary series, are *nonnegative* and decay at an exponential rate that converges to zero with the Casimir parameter. It follows that if a unitary representation of $SL(2,\mathbb{R})$ does not have a spectral gap, matrix elements (correlations) of $SO(2,\mathbb{R})$-invariant vectors do not converge to zero at a uniform exponential rate long the diagonal subgroup. For the representations on spaces of square-integrable functions on the moduli space of Abelian differentials, it would follow that the Teichmüller flow does not have exponential mixing, a contradiction.

By Ratner's theorems on decay of correlations [46] the following holds.

COROLLARY 31 ([6, Appendix B]). *The Teichmüller geodesic flow has exponential decay of correlations and the Teichmüller horocycle flow has polynomial decay of correlations for all observable in the Ratner's class.*

The Ratner's class is roughly the class of square integrable functions that are Hölder along $SO(2,\mathbb{R})$ orbits. Note the "bootstrap" of regularity in the exponential mixing for the Teichmüller flow with respect to the main theorem.

We conclude our review of Avila's early, prize-winning contributions to Teichmüller dynamics by stating a far reaching generalization of the spectral gap theorem, recently proved in a remarkable paper by Avila and Gouëzel [5].

THEOREM 32 ([5]). *Let μ be any $SL(2,\mathbb{R})$-invariant "algebraic" probability measure in the moduli space of quadratic differentials. For any $\delta > 0$, the spectrum of the associated Laplacian or Casimir operator in the interval $[0,1/4-\delta]$ is made of finitely many eigenvalues of finite multiplicity.*

We recall that Eskin and Mirzakhani [19] have recently announced a proof of the long-standing conjecture that all $SL(2,\mathbb{R})$-invariant measures are "algebraic". By the Avila–Gouëzel Theorem, from a spectral point of view the action of $SL(2,\mathbb{R})$ on the moduli space of quadratic differentials is similar to the action on the unit tangent bundle of a finite volume hyperbolic surface (such as the modular surface). Note, however, that in higher genus $SL(2,\mathbb{R})$ orbits have high codimension. Hence, the foliated Laplacian associated to the $SL(2,\mathbb{R})$ action is not subelliptic.

The Avila–Gouëzel paper represents, in many respects, a departure from the techniques of Avila's earlier work and it is beyond the scope of our paper.

Acknowledgments. We are grateful to Sasha Bufetov and Carlos Matheus Silva Santos who read a preliminary version of this paper and greatly helped to improve it with their comments and suggestions.

REFERENCES

[1] V. I. Arnold, *Small denominators and problems of stability of motion in classical and celestial mechanics*, Uspekhi Mat. Nauk, **18** (1963), 91–192; English translation: Russian Math. Surveys, **18** (1963), 85–191.

[2] J. Athreya, *Quantitative recurrence and large deviations for the Teichmüller geodesic flow*, Geom. Dedicata, **119** (2006), 121–140.

[3] A. Avila and A. Bufetov, *Exponential decay of correlations for the Rauzy-Veech-Zorich induction map*, in "Partially Hyperbolic Dynamics, Laminations, and Teichmüller Flow," Fields Institute Communications, **51**, Ameri. Math. Soc., Providence, RI, (2007), 203–211.

[4] A. Avila and G. Forni, *Weak mixing for interval exchange transformations and translation flows*, Ann. of Math. (2), **165** (2007), 637–664.

[5] A. Avila and S. Gouëzel, *Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmüller flow*, `arXiv:1011.5472`.

[6] A. Avila, S. Gouëzel and J.-C. Yoccoz, *Exponential mixing for the Teichmüller flow*, Publ. Math. Inst. Hautes Études Sci., No. 104 (2006), 143–211.

[7] A. Avila and M. J. Resende, *Exponential mixing for the Teichmüller flow in the space of quadratic differentials*, to appear in Comm. Math. Helv.

[8] A. Avila and M. Viana, *Simplicity of Lyapunov Spectra: Proof of the Zorich-Kontsevich conjecture*, Acta Math., **198** (2007), 1–56.

[9] A. Avila and M. Viana, *Simplicity of Lyapunov Spectra: A sufficient criterion*, Portugaliae Mathematica (N.S.), **64** (2007), 311–376.

[10] V. Baladi and B. Vallée, *Exponential decay of correlations for surface semi-flows without finite Markov partitions*, Proc. Amer. Math. Soc., **133** (2005), 865–874.

[11] C. Bonatti, X. Gómez-Mont and M. Viana, *Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices*, Ann. Inst. H. Poincaré Anal. NonLinéaire, **20** (2003), 579–624.

[12] C. Bonatti and M. Viana, *Lyapunov exponents with multiplicity* 1 *for deterministic products of matrices*, Ergod. Th. Dynam. Sys., **24** (2004), 1295–1330.

[13] A. Bufetov, *Decay of Correlations for the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations and the central limit theorem for the Teichmüller flow on the moduli space of abelian differentials*, J. Amer. Math. Soc., **19** (2006), 579–623.

[14] A. Bufetov, *Limit theorems for translation flows*, `arXiv:0804.3970`.

[15] J. Chaika, *There exists a topologically mixing IET*, `arXiv:0910.3986`.

[16] I. P. Cornfeld, S. V. Fomin and Y. G. Sinaǐ, "Ergodic Theory," Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **245**, Springer-Verlag, New York, 1982.

[17] D. Dolgopyat, *On decay of correlations in Anosov flows*, Ann. of Math. (2), **147** (1998), 357–390.

[18] A. Eskin and G. A. Margulis, *Recurrence properties of random walks on finite volume homogeneous manifolds*, in "Random Walks and Geometry" (ed. V. A. Kaimanovich, in collaboration with K. Schmidt and W. Woess), Walter de Gruyter GmbH & Co. KG, Berlin, (2004), 431–444.

[19] A. Eskin and M. Mirzakhani, *Invariant and stationary measures for the $SL(2,\mathbb{R})$ action on moduli space*, available from: `http://math.uchicago.edu/~eskin/`.

[20] G. Forni, *Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus*, Ann. of Math. (2), **146** (1997), 295–344.

[21] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Annals of Math. (2), **155** (2002), 1–103.

[22] G. Forni, *On the Lyapunov exponents of the Kontsevich-Zorich cocycle*, in "Handbook of Dynamical Systems," Vol. 1B (eds. B. Hasselblatt and A. Katok), Elsevier B. V., Amsterdam, (2006), 549–580.

[23] G. Forni, *Sobolev regularity of solutions of the cohomological equation*, `arXiv:0707.0940`.

[24] G. Forni, *A geometric criterion for the nonuniform hyperbolicity of the Kontsevich-Zorich cocycle*, Journal of Modern Dynamics, **5** (2011), 355–395.

[25] H. Furstenberg, "Stationary Processes and Prediction Theory," Annals of Mathematics Studies, No. 44, Princeton University Press, Princeton, N.J., 1960.

[26] H. Furstenberg, *Noncommuting random products*, Transactions of the American Mathematical Society, **108** (1963), 377–428.

[27] H. Furstenberg and H. Kesten, *Products of random matrices*, Ann. Math. Stat., **31** (1960), 457–469.

[28] I. Y. Gold'sheĭd and G. A. Margulis, *Lyapunov indices of a product of random matrices*, Uspekhi Mat. Nauk, **44** (1989), 13–60; translation in Russian Math. Surveys, **44** (1989), 11–71.

[29] Y. Guivarc'h and A. Raugi, *Product of random matrices: Convergence theorems*, in "Random Matrices and their Applications" (Brunswick, Maine, 1984), Contemp. Math., **50**, Amer. Math. Soc., Providence, RI, (1989), 31–54.

[30] A. B. Katok, *Interval-exchange transformations and some special flows are not mixing*, Israel J. Math., **35** (1980), 301–310.

[31] A. B. Katok and A. M. Stepin, *Approximations in ergodic theory*, Uspehi Mat. Nauk, **22** (1967), 81–106; translation in Russian Math. Survey, **22** (1967), 77–102.

[32] M. S. Keane, *Interval exchange transformations*, Math. Z., **141** (1975), 25–31.

[33] M. S. Keane, *Non-ergodic interval exchange transformations*, Israel J. Math., **26** (1977), 188–196.

[34] S. Kerckhoff, *Simplicial systems for interval exchange maps and measured foliations*, Erg. Theory and. Dynam. Sys., **5** (1985), 257–271.

[35] M. Kontsevich, *Lyapunov exponents and Hodge theory*, in "The Mathematical Beauty of Physics" (Saclay, 1996), Adv. Ser. Math. Phys., **24**, World Scientific Publ., River Edge, NJ, (1997), 318–332.

[36] M. Kontsevich and A. Zorich, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Invent. Math., **153** (2003), 631–678.

[37] F. Ledrappier, *Positivity of the exponent for stationary sequences of matrices*, in "Lyapunov Exponents" (Bremen, 1984), Lect. Notes Math., **1186**, Springer, Berlin, (1986), 56–73.

[38] S. Marmi, P. Moussa and J.-C. Yoccoz, *The cohomological equation for Roth-type interval exchange maps*, J. Amer. Math. Soc., **18** (2005), 823–872.

[39] S. Marmi, P. Moussa and J.-C. Yoccoz, *Linearization of generalized interval-exchange maps*, `arXiv:1003.1191`, to appear on Ann. of Math.

[40] H. Masur, *Extension of the Weyl-Petersson metric to the boundary of the Teichmüller space*, Duke Math. J., **43** (1976), 623–635.

[41] H. Masur, *Interval-exchange transformations and measured foliations*, Ann. of Math. (2), **115**, (1982), 169–200.

[42] H. Masur, *Closed trajectories for quadratic differentials with an application to billiards*, Duke Math. J., **53** (1986), 307–314.

[43] A. Nogueira and D. Rudolph, *Topological weak-mixing of interval-exchange maps*, Ergodic Theory Dynam. Systems, **17** (1997), 1183–1209.

[44] V. I. Oseledec, *The spectrum of ergodic automorphisms*, Dokl. Akad. Nauk SSSR, **168** (1966), 1009–1011; translation in Soviet Math. Dokl., **7** (1966), 776–779.

[45] G. Rauzy, *Échanges d'intervalles et transformations induites*, Acta Arith., **34** (1979), 315–328.

[46] M. Ratner, *The rate of mixing for geodesic and horocycle flows*, Ergodic Theory Dynam. Systems, **7** (1987), 267–288.

[47] R. Treviño, *On the non-uniform hyperbolicity of the Kontsevich-Zorich cocycle for quadratic differentials*, to appear on Geom. Dedicata.

[48] C. Ulcigrai, *Absence of mixing in area-preserving flows on surfaces*, Annals of Mathematics (2), **173** (2011), 1743–1778.

[49] W. Veech, *Projective Swiss cheeses and uniquely ergodic interval exchange transformations*, in "Ergodic Theory Dynam. Systems, I" (ed. A. Katok) (College Park, Md., 1979-80), Progr. Math., **10**, Birkhäuser, Boston, Mass., (1981), 113–193.

[50] W. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. (2), **115** (1982), 201–242.

[51] W. Veech, *The metric theory of interval exchange transformations. I. Generic spectral properties*, Amer. J. Math., **106** (1984), 1331–1359.

[52] W. Veech, *The Teichmüller geodesic flow*, Annals of Mathematics (2), **124** (1986), 441–530.

[53] M. Viana, *Lyapunov exponents of Teichmüller flows*, in "Partially Hyperbolic Dynamics, Laminations, and Teichmüller Flow," Fields Institute Communications, **51**, Amer. Math. Soc., Providence, RI, (2007), 139–201.

[54] M. Viana, *IMPA lectures on Lyapunov exponents*, 2010. Available from: `http://w3.impa.br/~viana`, (see w3.impa.br/ viana/out/lle.pdf).

[55] J.-C. Yoccoz, *Interval-exchange maps and translation surfaces*, in "Homogeneous Flows, Moduli Spaces and Arithmetic," Clay Math. Proc., **10**, Amer. Math. Soc., Providence, RI, (2010), 1–69.

[56] A. Zorich, *The S. P. Novikov problem on the semiclassical motion of an electron in homogeneous Magnetic Field*, Russian Math. Surveys, **39** (1984), 287–288.

[57] A. Zorich, *Asymptotic flag of an orientable measured foliation on a surface*, in "Geometric Study of Foliations" (Tokyo, 1993), World Scientific Publ., River Edge, NJ, (1994), 479–498.

[58] A. Zorich, *Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents*, Ann. Inst. Fourier (Grenoble), **46** (1996), 325–370.

[59] A. Zorich, *Deviation for interval exchange transformations*, Ergod. Th. & Dynam. Sys., **17** (1997), 1477–1499.

[60] A. Zorich, *On hyperplane sections of periodic surfaces*, in "Solitons, Geometry, and Topology: On the Crossroad" (eds. V. M. Buchstaber and S. P. Novikov), AMS Trasl. Ser. 2, **179**, AMS, Providence, RI, (1997), 173–189.

[61] A. Zorich, *How do the leaves of a closed* 1*-form wind around a surface?*, in "Pseudoperiodic Topology" (eds. Vladimir Arnold, Maxim Kontsevich and Anton Zorich), AMS Transl. Ser. 2, **197**, AMS, Providence, RI, (1999), 135–178.

[62] A. Zorich, *Flat surfaces*, in "Frontiers in Number Theory, Physics, and Geometry. 1" (eds. P. Cartier, B. Julia, P. Moussa and P. Vanhove), Springer, Berlin, (2006), 437–583.

GIOVANNI FORNI `<gforni@math.umd.edu>`: Department of Mathematics, University of Maryland, College Park, MD 20742-4015, USA