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FORTY YEARS OF UNIMODAL DYNAMICS: ON THE OCCASION OF ARTUR AVILA WINNING THE BRIN PRIZE

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1. FIRST 30 YEARS

The field of one-dimensional dynamics, real and complex, emerged from obscurity in the 1970s and has been intensely explored ever since. It combines the depth and complexity of chaotic phenomena with a chance to fully understand it in probabilistic terms: to describe the dynamics of typical orbits for typical maps. It also revealed fascinating universality features that had never been noticed before. The interplay between real and complex worlds illuminated by beautiful pictures of fractal structures adds special charm to the field.

By now, we have reached a full probabilistic understanding of real analytic unimodal dynamics, and Artur Avila has been the key player in the final stage of the story (which roughly started with the new century). To put his work into perspective, we will begin with an overview of the main events in the field from the 1970s up to the end of the last century. Then we will describe Avila's work on unimodal dynamics that effectively closed up the field. We will finish by describing his results in the closely related direction, the geometry of Feigenbaum Julia sets, including a recent construction of a new class of Julia sets of positive area.

1.1. **Object of study.** Let I be a closed interval. A smooth map $f\colon I \to I$ is said to be *unimodal* if it has one critical point c_0, this point belongs to $\operatorname{int} I$, and is an extremum. The main example is given by the quadratic family (also called *logistic family*)

$$f_a\colon [0,1] \to [0,1], \quad f_a\colon x \mapsto ax(1-x), \quad a \in [1,4]. \tag{1.1}$$

We let $c_n = f^n(c_0)$. For technical reasons, we assume that one of the end-points of I is a fixed point with multiplier ≥ 1, and the other one is its preimage.

Such a map is said to be *S-unimodal* if it is of class C^3 and has negative Schwarzian derivative:

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 < 0.$$

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This condition makes the theory cleaner, so we often assume it, though at the end of the day, it can be eliminated, with some minor adjustments.

1.2. **Combinatorics: Milnor–Thurston kneading theory.** To each unimodal map one can associate a combinatorial invariant called the *kneading sequence* $\bar{\epsilon} = (\epsilon_1\epsilon_2\dots)$, where $\epsilon_n \in \{\pm 1, 0\}$. Namely, the critical point c_0 divides I into two subintervals I_- and I_+, where $f|I_+$ is orientation-preserving. Then $\bar{\epsilon}$ is the code of the critical orbit with respect to this partition, i.e., $c_n \in I_{\epsilon_n}$. If $c_n = c_0$ for some $n \geq 1$ then we let $\epsilon_n = 0$ and stop.

The field of unimodal dynamics was inaugurated with the paper by Milnor and Thurston [75], where they introduced this invariant and showed that it essentially determines the dynamics of f (it determines a symbolic model which is nicely semiconjugate to f). The authors went on to characterize admissible kneading sequences and to prove that *the quadratic family (1.1) is full*, i.e., all admissible kneading sequences appear in this family. Moreover, they appear *monotonically* (with respect to a natural twisted lexicographic order on the sequences). This last result remained a conjecture for a while, until it was proven by *complex methods*, which was the beginning of an exciting interplay between real and complex dynamics.

These properties were the first indications of the special role that the quadratic family was going to play in the field.

1.3. **Topological dynamics.** The combinatorial model provided by the kneading sequence has become more precise after the work of Guckenheimer [30] who proved that S-unimodal maps *do not have wandering intervals*. (An interval $J \subset I$ is said to be wandering if $f^n(J) \cap f^m(J) = \emptyset$ for any $n > m \geq 0$, and the $f^n(J)$ do not converge to a cycle.) This, in particular, implies that *any S-unimodal map is topologically conjugate to some quadratic map (1.1)*. Moreover, this led to a good topological picture of the S-unimodal dynamics. Namely, any S-unimodal map is one of the following three types (see Jonker and Rand [38]):

1. *Hyperbolic case:* There is an attracting cycle $\boldsymbol{\alpha}$ whose basin is $I \smallsetminus K$, where K is an expanding Cantor set of zero length. Such maps are also said to be *regular* since almost all orbits behave in a regular fashion: they converge to the attracting cycle.
2. *Mixing case:* there is a cycle of intervals $f^k(J)$, $k = 0, 1, \dots, p-1$, where $\operatorname{int}(f^n(J)) \cap \operatorname{int} f^m(J) = \emptyset$ for $0 \leq m < n < p$ and $f^p(J) \subset J$, such that the return map $f^p \colon J \to J$ is topologically mixing. Moreover, $I \smallsetminus \bigcup_{n\geq 0} f^{-n}(J)$ is an expanding Cantor set of zero length.
3. *Solenoidal case:* There is a nest of periodic intervals $J^1 \supset J^2 \supset \dots$ of periods p_n such that the set
$$\mathcal{O} \equiv \mathcal{O}_f = \bigcap_{n\geq 1} \bigcup_{k=0}^{p_n - 1} f^k(J^n)$$
is an invariant Cantor set and the dynamics $f|\mathcal{O}$ is the *adding machine* (it is translation by 1 on the projective limit of the cyclic groups $\mathbb{Z}/p_n\mathbb{Z}$).

Moreover, almost all orbits in I converge to $\mathcal{O}$, so that $\mathcal{O}$ is a *global attractor*. This case is also referred to as *infinitely renormalizable* for the reason that will become clear momentarily. The ratios p_{n+1}/p_n are called the *relative renormalization periods*.

1.4. **Universality phenomenon.** A remarkable discovery was made in the mid-1970s by Feigenbaum and independently by Coullet and Tresser: some dynamical and parameter objects (within a certain class) have a universal geometry, independent of specific maps and families under consideration. As it had already been known, the quadratic family begins with the cascade of doubling bifurcations a_n: at this moment the attracting periodic cycle of period 2^n bifurcates into the attracting cycle of period 2^{n+1}. Moreover, $a_n \to a_\infty$, where f_{a_∞} is a solenoidal map with periods $p_n = 2^n$. Feigenbaum observed that the parameters a_n converge to a_∞ at exponential rate,

$$a_\infty - a_n \sim C\lambda^{-n}, \quad \lambda > 1,$$

and *the rate λ is independent of the particular family under consideration*, as long as it "looks like the quadratic family" (one can consider $a \sin \pi x$, for instance). Coullet and Tresser observed that *the small-scale geometry of the solenoidal attractor $\mathcal{O}$ is also universal.*

To explain these surprising phenomena, the authors laid down a dynamical *renormalization theory*. A unimodal map f is said to be *renormalizable* if it has a periodic interval $J \ni 0$ of period $p > 1$. Then the *renormalization Rf* of f is defined as the first-return map $f^p : J \to J$ considered up to rescaling. The *combinatorics* of the renormalization is the order of the intervals $f^n(J)$, $n = 0, 1, \dots, p-1$, on the real line. The Feigenbaum–Coullet–Tresser Renormalization Conjecture asserted that *for any given combinatorics, the renormalization operator has a unique fixed point f_* (i.e., it satisfies the Cvitanović-Feigenbaum functional equation $Rf_* = f_*$), and R is hyperbolic at this fixed point with unstable direction of dimension one.*

This conjecture would imply the above universalities: the universal scaling in the parameter plane would be controlled by the unstable eigenvalue of $DR(f_*)$, while the universal geometry of $\mathcal{O}_f$ for all infinitely renormalizable maps would be determined by the solenoidal attractor $\mathcal{O}_*$ of f_*.

1.5. **Abundance of stochastic maps.** A unimodal map is said to be *stochastic* if it has an absolutely continuous invariant measure μ. Such a map should necessarily be of the mixing type (Case (2) in §1.3). Moreover, the measure is unique and ergodic, and $\operatorname{supp}\mu = \bigcup_{n=0}^{p-1} f^n(J)$, where J is the mixing periodic interval. Furthermore, this measure is *nonuniformly hyperbolic* (i.e., it has a positive Lyapunov exponent) and is *asymptotic*, i.e., Lebesgue almost all orbits are equidistributed with respect to μ:

$$\frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k x) \to \int \phi\, d\mu \quad \text{for any } \phi \in C(I) \text{ and a.e. } x \in I.$$

See [17, 60] for all of these results.

Such a measure is also called *SRB measure* (for Sinai, Ruelle and Bowen). It provides us with an excellent picture of the dynamics from the probabilistic point of view.

In the early 1980s, Jakobson proved that *the set of stochastic parameters* $a \in [1,4]$ *in the quadratic family has positive length* [36].

An important sufficient condition for stochasticity was then proposed by Collet and Eckmann [23]: there exist $\lambda > 1$ and $q > 0$ such that

$$|Df^n(c_1)| \geq q\lambda^n, \quad n \in \mathbb{N},$$

i.e., f is exponentially expanding along the critical orbit. Benedicks and Carleson [12] went on to prove that *the set of Collet–Eckmann parameters* $a \in [1,4]$ *has positive length.*

Collet–Eckmann maps exhibit nonuniform hyperbolicity that comes as close as one can get to the uniform one.

1.6. **Methods of holomorphic dynamics.** In the late 1970s, Mandelbrot showed to the mathematical world computer pictures of the complexified logistic family. The object that appeared in these pictures was later called the *Mandelbrot set* M: it is the set of parameters c for which the corresponding Julia set $J(f_c)$ is connected.[1] The set was fascinating and drew great deal of attention in the coming years (all the way up to today). Particularly deep analysis was undertaken in the Orsay Notes by Douady and Hubbard [26]. It led to a precise *topological model* for the Mandelbrot set *as long as the latter is locally connected.* This prompted the most famous *conjecture* in holomorphic dynamics abbreviated as the MLC ("Mandelbrot is Locally Connected"). It was also shown in [26] that the MLC would imply another outstanding conjecture: *density of hyperbolic maps in the complex quadratic family*. (This conjecture is sometimes called *Fatou's Conjecture*.)

One of the prominent features of the Mandelbrot set that is easily observable on its computer images is that it contains all over the place little copies of itself that look exactly like the whole set[2]. To explain this phenomenon, Douady and Hubbard introduced a notion of *complex renormalization* acting in the space of quadratic-like maps.

A *quadratic-like map* is a holomorphic branched covering $f: U \to V$ of degree 2 between two topological disks $U \Subset V$.[3] The set of nonescaping points,

$$K(f) = \{z: \ f^n(z) \in U, \ n = 0, 1, \dots\}$$

is called the *filled Julia set* of f. The *Julia set* of f is the boundary of the filled Julia set, $J(f) = \partial K(f)$. (Note that the Julia set $J(f_c)$ of a quadratic polynomial coincides with the Julia set of its quadratic-like restriction to a sufficiently large disk.)

[1] The standard parametrization of the complex quadratic family is $f_c: z \mapsto z^2 + c$.

[2] Except that the little copy may miss the cusp at its root point—such a copy is called *satellite*; otherwise it is said to be *primitive*.

[3] I.e., U is compactly contained in V.

Roughly speaking, a quadratic-like map is *renormalizable* if there is a topological disk $U' \ni c_0$ and a period $p > 1$ such that the map $f^p : U' \to f^p(U')$ is quadratic-like with connected Julia set. As in the real case, complex renormalization comes together with its *combinatorics*: the way the little Julia sets are located in the big one. The main theorem of [27] asserts that *the parameters c for which the quadratic map f_c is renormalizable with given combinatorics form a topological copy of the Mandelbrot set M.*

The first breakthrough in the MLC Conjecture came in the early 1990s in the work by Yoccoz (see [33, 73]), who proved that *for any quadratic polynomial f_c that is not infinitely renormalizable and does not have neutral cycles*, we have:

- *The Julia set f_c is locally connected* (Dynamical part);
- *the Mandelbrot set is locally connected at c* (Parameter part).

This result revealed a deep connection between the MLC Conjecture and the Renormalization theory. Not less important than the result itself is the technique introduced in this work. It provides us with a nest of dynamically defined tilings of Julia sets and the Mandelbrot set that capture the combinatorics and geometry of the sets in question. These tilings are called *Yoccoz puzzles*.

Using this technique, Shishikura and the author independently proved that *for any quadratic polynomial f_c of the Yoccoz class, the Julia set $J(f_c)$ has zero area* [50, 86]. Shishikura went on to transfer this result to the parameter plane: *the set of nonhyperbolic Yoccoz parameters c has zero area.*

1.7. **Milnor's problem.** Let us say that an invariant compact subset $A \subset I$ is the *global measure-theoretic attractor* (in the sense of Milnor) if $\omega(x) = A$ for almost all $x \in I$.

In [72], John Milnor asked whether it is true that for a unimodal map f, the topological attractor described in §1.3 (which is either the attracting cycle, or the cycle of a mixing interval, or the solenoidal attractor) is the global measure-theoretic attractor. This was proved in [51] for real quadratic-like maps. The main geometric phenomenon behind this result is *decay of geometry for non-renormalizable quadratic-like maps.* (The geometry is controlled by the *scaling factors* between real puzzle pieces in a certain combinatorially defined nest, called the *Principal Nest.*)

This decay of geometry is specific for the quadratic case. For higher-degree unimodal polynomials $f_{c,d} : x \mapsto x^d + c$ it turns out that there may exist *wild attractors* [18]. Namely, for d sufficiently big, there exists a map $f_{c,d}$ (so-called *Fibonacci map*) that has a mixing invariant interval J, but almost all orbits converge to a Cantor attractor $A \subset J$.

1.8. **Density of hyperbolicity.** The real version of the Fatou Conjecture was proved in [52, 31]: *Hyperbolic maps are dense in the real quadratic family.* It was done by combining real and complex methods, and until now no purely real proof for this result was found. The main geometric ingredients of the proof were:

- *Complex a priori bounds* for real infinitely renormalizable maps [46, 58] (which is the main point that needs reality of the maps). This means that the fundamental annuli $V_n \smallsetminus U_n$ of the renormalizations $R^n f\colon U_n \to V_n$ have definite moduli[4]:

$$\operatorname{mod}(V_n \smallsetminus U_n) \geq \epsilon > 0, \quad n = 0, 1, \dots$$

- *Linear growth of the moduli in the Principal Nest of puzzle pieces.* This is the complex counterpart of the decay of geometry property mentioned in §1.7.

These properties led to a proof of Quasiconformal Rigidity for real maps: Two such maps, f_a and $f_{\tilde{a}}$, with the same combinatorics are quasiconformally conjugate. It had been known before (by a soft quasiconformal deformation argument) that this implies that the maps are actually the same (provided they are not hyperbolic), i.e., $a = \tilde{a}$. Then, the Kneading Theory tells us that any such a map can be perturbed to a hyperbolic one.

Carrying the above result further, Kozlovski proved that hyperbolicity is dense in the space of real analytic unimodal maps [43]. Hence, it is also true in the spaces of C^r-unimodal maps, for any $r \geq 1$.

1.9. **Proof of the Renormalization Conjecture.** In his address to the International Congress of Mathematicians in Berkeley [81], Dennis Sullivan proposed a program for approaching the Feigenbaum–Coullet–Tresser Renormalization Conjecture based on Teichmüller theory. The idea was to supply the hybrid class with a natural *Teichmüller metric* that would make the renormalization operator strongly contracting. (Two quadratic-like maps are said to be *hybrid-equivalent* if they are conjugate by a quasiconformal map h such that $\bar{\partial} h = 0$ a.e. on the filled Julia set. A *hybrid class* is the class of quasiconformally conjugate quadratic-like maps.)

This idea was partially realized in [82]: it was proved that the renormalization operator is weakly contracting, which implied existence of a unique renormalization fixed point f_* and convergence to it of all the orbits $\{R^n f\}$ within the hybrid class. It did not imply exponential convergence, though.

McMullen developed another method to give a new proof of the above results, accompanied with exponential convergence [66]. The key of the method was a geometric property of the Feigenbaum Julia sets called *hairiness*: Near the critical point, such a set gets dense exponentially fast. (Here, a *Feigenbaum map* is an infinitely renormalizable quadratic-like map with bounded combinatorics and *a priori* bounds; a *Feigenbaum Julia set* is the Julia set of such a map.)

Finally, in [55] it was proved that the renormalization fixed point f_* is hyperbolic with one-dimensional unstable manifold. This completed the proof of the Renormalization Conjecture for stationary combinatorics in the space of quadratic-like maps.

[4]The *modulus* of a topological annulus $A \subset \mathbb{C}$ can be defined as $\log R$, where $\{1 < |z| < R\}$ is the round annulus that is conformally equivalent to A.

Along these lines, it was demonstrated in the latter work that the hybrid classes form a lamination in the space of quadratic-like maps, with complex codimension-one leaves. Moreover, the quadratic family is a global transversal to this lamination; this further illuminated the special role of this family.

1.10. **"Regular or stochastic" dichotomy.** Finally, it was proved by the author of these notes that *almost every real quadratic map* f_a, $a \in [1,4]$, *is either regular or stochastic*. In the former case, it has an attracting cycle that attracts almost all orbits; in the latter, it has an absolutely continuous invariant measure that governs behavior of almost all orbits. One can also say that *almost every quadratic map is hyperbolic, either uniformly or nonuniformly*. This result gives a clear measure-theoretic picture of the dynamics in the real quadratic family.

There are two main parts of this picture:

- It was proved in [54] that almost every real quadratic map that is not infinitely renormalizable is a stochastic map satisfying the Martens–Nowicki criterion [62]. The proof is based upon geometric analysis of the *parapuzzle* (the Yoccoz puzzle in the parameter plane). The main result was that the moduli of the annuli of the Principal Parameter Nest grow at a linear rate (the parameter counterpart of the geometric result mentioned in §1.8).
- It was proved in [56] that the set of infinitely renormalizable parameters has zero length. To this end, the Renormalization Conjecture was extended to *all* real combinatorics simultaneously. It asserts that in the space of quadratic-like maps there is an invariant subset $\mathscr{A}$ (the *Renormalization Horseshoe*) on which the renormalization operator R acts as the two-sided shift with infinitely many symbols (corresponding to various finite kneading sequences), and this action is uniformly hyperbolic with one-dimensional unstable direction.

The Regular or Stochastic Theorem was the first occasion confirming the general *Palis Conjecture* [78] on attractors that govern the behavior of almost all orbits for typical dissipative dynamical systems.

1.11. **What is next?** So, by the end of the last century the unimodal dynamics reached an advanced stage: combinatorial and topological dynamics were fully understood, while universality and measure-theoretic dynamics were well understood for quadratic-like maps. The outstanding open problems were:

- How to deal with more general families of maps?
- How to deal with higher-degree unimodal maps? (Much in the above techniques depends essentially on the assumption that the maps in question are quadratic-like. This includes the Yoccoz Theorem (§1.6) and the geometric properties of puzzle and parapuzzle from §§1.8, 1.10.)
- To investigate fine stochastic and geometric properties of typical quadratic maps.

This was the moment when Artur Avila came to the scene. He was only 19.

2. AVILA'S WORK ON UNIMODAL DYNAMICS

2.1. **"Regular or stochastic dynamics in real analytic families of unimodal maps," Inventiones Mathematicae 2003** [5]**.** I first met Artur in IMPA in September 1998. At that time he was still an undergraduate in UFRJ, but simultaneously he was a graduate student in IMPA advised by Welington de Melo, and he was already studying the most advanced work in one-dimensional dynamics. During that first meeting, we began to discuss the infinitesimal structure of real hybrid classes. (In the nonhyperbolic case, a *real hybrid class* can be defined as a topological conjugacy class; in the hyperbolic case, it is the class of maps with the same multiplier of the attracting cycle.) These discussions continued the following spring when de Melo and Avila visited Stony Brook, and at that time the project was explicitly framed and put on a firm basis.

Our stated goal was to generalize the Regular or Stochastic Theorem to real analytic families of maps. The strategy was to transfer the result from the quadratic family to other real analytic families of unimodal maps through the holonomy along the real hybrid classes. To this end, we needed to justify that the real hybrid classes are codimension-one real analytic manifolds that form a foliation of the whole (or almost whole) space of real analytic maps. This was far from being obvious since, unlike quadratic-like maps, real analytic maps are lacking a nice external structure. One of the important tools we developed that spring was the *Infinitesimal Pullback Argument*. It helped us to handle many issues, but one of them was left outstanding: we needed to construct a transverse direction to a hybrid class.

Sometime later, Avila came up with an ingenious solution of this problem, by first constructing a special C^1-vector field transverse to the hybrid class, and then approximating it with a polynomial one. The construction is delicate and first looked too good to be true—but it worked, and it completed the argument. The final result is: *In any nontrivial real analytic family of unimodal maps with quadratic critical point, almost every map is either regular or stochastic.*

The method we used is the most definitive confirmation to date of the special place the quadratic family occupies in the space of unimodal maps: it enjoys all the virtues of a globally holomorphic family, which are then inherited, through the holonomy, by other families.

2.2. **"Statistical properties of typical quadratic maps: the quadratic family," Annals of Mathematics 2005** [8]**.** In this paper, Avila and Moreira undertook a thorough statistical analysis of typical stochastic quadratic maps. They proved that almost every stochastic quadratic map satisfies the following properties:

- It is *Collet–Eckmann*, which is, as we know, a strong hyperbolicity property.
- Its geometry decays at a *torrential* rate, like an iterated exponential map. It was known that the decay is exponential, but nobody had noticed that the rate is so fast.

- The recurrence of the critical point is linearly slow:

$$\forall \epsilon > 0\ \exists q > 0 \text{ such that } |f^n c_0 - c_0| \geq \frac{q}{n^{1+\epsilon}}.$$

  Sinai anticipated this in the 1970s, but it was generally believed that exponential recurrence is a more reasonable bet.

The entry point of this study is exponential decay of geometry from [51], but then the authors make further selection of parameters leading to sharp statistical results. It is a true *tour de force* of statistical analysis in dynamics!

Using the methods of [5], the above results can be transferred to arbitrary nontrivial real analytic families of unimodal maps with quadratic critical point.

Let me suggest, in conclusion, that these statistical properties will probably be important in future work on two-dimensional dissipative systems. In particular, it is likely that the Benedicks–Carleson Theorem [13] on the abundance of attractors for Hénon perturbations of *Misiurewicz maps* (i.e., maps with nonrecurrent critical point) extends to perturbations of Collet–Eckmann maps.

2.3. **"Statistical properties of unimodal maps: periodic orbits, physical measures and pathological laminations," Publications Mathématiques de l'IHÉS, 2005** [10]**.** This work of Avila, joint with Carlos Moreira, contains one of most striking results in the field of unimodal dynamics: *If f_λ and g_λ are two essentially different real analytic families of unimodal maps (with quadratic critical point), then their sets of stochastic kneading sequences are disjoint.* Here "essentially different" means that the maps in these families are not analytically conjugate.

Geometrically, this result says that *the real hybrid foliation from §2.1 is totally singular on the set of stochastic maps* (mod analytic conjugacy): the holonomy from one transversal to an essentially different one is singular. This is a kind of *"Fubini foiled"* phenomenon inspired by Katok's example of a foliation in $\mathbb{R}^2$ that admits a set of full measure meeting each leaf at a single point (see [74]).

This property is based on an important and difficult dynamical result proved in [10]: *For a typical stochastic map, the critical orbit is equidistributed with respect to the physical measure μ.* (*Physical measure*, in this context, is yet another name for the absolutely continuous invariant measure.) Moreover, the Lyapunov exponent at the critical value coincides with the one of the physical measure, and in particular, the map is Collet–Eckmann, so this result is sharper than that of [8].

Then the authors went on to carry a delicate analysis of the *typical regularity of the density* of μ that allowed them to control, in terms of μ, the small-scale geometry of hyperbolic Cantor sets (and in particular, the multipliers of periodic points).

Putting the above ingredients together, the authors obtained a *combinatorial formula* for the multipliers of periodic orbits for typical stochastic maps, i.e., typically the multipliers can be read off the kneading sequence. At first glance, it goes against the *Livšic paradigm* that the multipliers play a role of *analytic moduli* and so they cannot be determined by the combinatorial data. (See Martens and de Melo [63] for the precise result in the context of unimodal maps.) However, there is no contradiction here, but rather a new *rigidity phenomenon*: *For a typical unimodal map under consideration, the combinatorics determines the analytic class.* Here *typical* is understood in the *Kolmogorov sense* as a Lebesgue-typical parameter in a nontrivial one-parameter analytic family of maps.

One can wonder how the singularity of the lamination is compatible with the holonomy method described in §2.1. What happens is that in [5] we do not transfer directly the stochastic set of parameters from one family to another, but rather transfer certain geometric properties of the puzzle (like decay of geometry) that imply that typical maps of mixing type are stochastic.

2.4. **Fine geometry of residual parameter sets.** The Regular or Stochastic Theorem describes almost all parameter values in the quadratic family, but does not address the natural problem of how small the residual sets are, in terms of Hausdorff dimension. This problem was thoroughly addressed in the unpublished manuscript by Avila and Moreira [11]. They proved:

- *The Hausdorff dimension of the set of infinitely renormalizable parameters is strictly less than 1.* (On the other hand, it was known that it is at least 1/2 [53].)
- *The Hausdorff dimension of the set of parameters of mixing type that are not stochastic is strictly in between* 0 *and* 1.

These results are based (in a highly nontrivial way) on the hyperbolicity of the full renormalization horseshoe and on further geometric analysis of the parapuzzle.

Overall, the above results provide us with a clear picture of fine stochastic and geometric features of typical real analytic maps with quadratic critical point.

2.5. **Generalization of the Yoccoz theorems to $f_{c,d}\colon z \mapsto z^d + c$.** Now the question arises: how about unimodal maps with a higher-degree criticality? There were several good reasons to address this problem. First of all, clearly our methods are not fully adequate if they cannot handle this case. Second of all, this case can be viewed as an important step towards the general multimodal case.

When Yoccoz proved his results on local connectivity (see §1.6), it was immediately noticed that his method does not work in the higher-degree case. I remember sitting with Adrien Douady in Luxembourg park in the summer of 1990 when he mentioned this problem, saying that this would be an important next breakthrough. The problem remained open for about 15 years.

In the work of Jeremy Kahn and the author [40], a new tool in conformal geometry, the *Covering Lemma*, was developed that allowed us to approach the problem. In particular, it led to *a priori* bounds in the principal nest that imply *local connectivity of the Julia sets* $J(f_{c,d})$, as long as $f_{c,d}$ is not infinitely renormalizable and does not have neutral cycles [41]. This generalizes the dynamical part of the Yoccoz Theorem.

Then Artur Avila and Weixiao Shen joined our team to handle the parameter part [6]. The result was that the *Multibrot set* M_d (the higher-degree analog of the Mandelbrot set) is *locally connected at points c as above*. The final argument was different and arguably simpler than the proofs previously known in the quadratic case.

2.6. **Warwick school.** Before proceeding further with the description of Avila's work, we should briefly stop on a parallel important development in the higher-degree case:

- In [44], Kozlovski, Shen and van Strien proved *density of hyperbolicity* in any family $f_{c,,d}$ (and in fact, in the whole space of real polynomials). In [45], they carried it further to prove density of hyperbolicity in the space of C^r-multimodal maps. (Compare with the discussion in §1.8.) This completed the problem of density of hyperbolicity in one-dimensional dynamics.
- In [19], Bruin, Shen and van Strien established *existence of a unique asymptotic measure for a.e. c*. The typical map $f_{c,d}$ given by this result can be of 4 different types: *regular, stochastic, infinitely renormalizable* (with μ being the unique invariant measure on the solenoidal attractor $\mathcal{O}$), and a map of mixing type with a *wild attractor* (see §1.7). In the two latter cases, μ has zero entropy (note that unlike the SRB measure, the asymptotic measure is allowed to have zero entropy.) This result was an impressive advance towards the Regular or Stochastic Conjecture for higher-degree polynomials.

2.7. **Stochastic maps $f_{c,d}$ are typical: Journal of the European Mathematical Society 2011** [7]**.** The main result of this paper is that *a typical polynomial $f_{c,d}$ of mixing type is stochastic*. Like its quadratic predecessor [54], it is based upon geometric analysis of the parapuzzle. The main result is *a priori bounds* for the principal moduli. However, unlike in the quadratic case, these principal moduli do not, in general, grow. Several new tricks had to be invented in order to handle this problem.

The proof of [8] can now be applied to ensure that a typical map $f_{c,d}$ of mixing type is Collet–Eckmann.

As a corollary of special interest in the higher-degree case, we conclude that the probability of seeing a wild attractor in any family $f_{c,d}$ is zero.

The parapuzzle geometry also implies interesting consequences about the complex parameter plane; in particular, we prove that the set of parameters $c \in \partial M_d$ for which the map $f_{c,d}$ is not infinitely renormalizable has zero area, which is the higher-degree generalization of Shishikura's Theorem (see §1.6).

The higher-degree statement has extra hidden content since it is unknown whether a nonrenormalizable polynomial $f_{c,d}$ without neutral fixed points may or may not have the Julia set of positive area. But if it may, it can happen only with zero probability (as with wild attractors above).

2.8. **Results on a priori beau bounds.** Before going to the final chapter of our story, let us dwell on another parallel development, on the problem of *a priori bounds* for complex infinitely renormalizable maps. To distinguish the complex case from the real one, we call a complex polynomial-like map with a *single* geometric critical point *unicritical* (rather than "unimodal"). The main example is the complex polynomial family $f_{c,d}: z \mapsto z^d + c$.

Let us say that infinitely renormalizable unicritical maps of a certain class have *a priori beau bounds* if there exists an $\epsilon > 0$ such that for any map f in our class there is N with the property that

$$\operatorname{mod}(V_n \smallsetminus U_n) \geq \epsilon > 0, \quad \forall\ n \geq N,$$

where U_n, V_n are the domains and ranges for the renormalizations $R^n f: U_n \to V_n$. Due to the developments of the 1990s, most key problems of the Renormalization Theory were tightly linked to existence of beau bounds.

For real maps, beau bounds were proven in the 1990s in [46, 58] (compare §1.8). In [52], they were also established for a certain class of complex maps, but then there was no further progress in the problem until the breakthrough by Jeremy Kahn [39], who proved beau bounds for complex infinitely renormalizable unicritical maps of bounded primitive type. These two cases were then roughly unified in [42].

From the perspective of our main theme, it is important that this class covers all maps that are hybrid-equivalent to infinitely renormalizable real polynomials with arbitrary periods except 2. (The most classical case of period 2 is still open!)

2.9. **"The full renormalization horseshoe for unimodal maps of higher degree: exponential contraction along hybrid classes," Publications Mathématiques de l' IHÉS 2011** [3]**.** To complete the Regular or Stochastic Theorem for higher-degree unimodal maps, we needed to prove that the set of infinitely renormalizable parameters has zero length. Like in the quadratic case, this would follow from the existence of the full renormalization horseshoe. However, the argument in [56] substantially relied on the decay of the puzzle geometry in the quadratic case. One needed to develop new methods to handle the higher-degree case.

A beautiful idea that Avila introduced was to endow the hybrid classes $\mathcal{H}$ of unicritical maps with the *Carathéodory metric*

$$\operatorname{dist}(f,g) = \sup|\phi(g)|,$$

where the supremum is taken over all holomorphic maps $\phi: (\mathcal{H}, f) \to (\mathbb{D}, 0)$ to the unit disk. The renormalization operator R, being holomorphic, is contracting with respect to this metric. Thus, if we have a relatively compact subset that is mapped "strictly inside itself" under some iterate of R, then we can immediately construct the horseshoe and see that it is exponentially contracting on the hybrid foliation. But this is exactly what *a priori beau bounds* would provide for us. (This argument came straight from the *Erdös Book*!)

This discussion, together with beau bounds of [42], *almost* gives us the full renormalization horseshoe (for all renormalization periods excepts doublings). To handle the doubling renormalization as well, another idea was exploited in [3], *almost-periodicity.*

To fix the idea, let us consider *stationary* real combinatorics, so that the renormalization acts in the corresponding hybrid class, $R\colon \mathscr{H} \to \mathscr{H}$. It is said to be *almost periodic* if all orbits $\{R^n f\}$ are precompact. The real a priori bounds imply almost-periodicity of R. Now, the general theory of almost periodic actions (see [49]) tells us that in the ω-limit set of the iterates of R there is a retraction $P\colon \mathscr{H} \to X$ onto a certain subspace X. Beau bounds for real maps imply that the real slice $X^{\mathbb{R}}$ of X is compact. By the Implicit-Function Theorem, it is a submanifold. Hence $X^{\mathbb{R}}$ is a single point f_*. Since X is the complexification of $X^{\mathbb{R}}$, we conclude that $X = \{f_*\}$ as well. Thus, P retracts the whole hybrid class $\mathscr{H}$ to f_*, which implies that the orbits of R converge to f_* uniformly on compact sets. The Schwarz Lemma makes convergence exponential.

To handle nonstationary combinatorics, we develop a theory of *almost periodic cocycles*. It provides us with the full renormalization horseshoe, which is exponentially contracting in the hybrid lamination. To prove that it is hyperbolic (i.e., expanding in the transverse direction), we apply the method of [56], which works fine in all degrees.

With this, the field of unimodal dynamics has been brought to a remarkably complete form. Artur Avila bears a great deal of responsibility for this to have happened.

3. GEOMETRY OF FEIGENBAUM MAPS: JOURNAL OF THE AMERICAN MATHEMATICAL SOCIETY 2008 [2] AND MANUSCRIPT [4]

3.1. **Geometric trichotomy.** Basic geometric characteristics of a fractal set $J \subset \mathbb{C}$ are its Lebesgue area and Hausdorff dimension. Accordingly, we can roughly classify all the sets into three types:

1. *Lean Case*: $\mathrm{HD}(J) < 2$;
2. *Balanced Case*: $\mathrm{HD}(J) = 2$ but area $J = 0$;
3. *Black Hole Case*: area $J > 0$.

We have explored this geometric trichotomy for *Feigenbaum Julia sets*, i.e., for infinitely renormalizable quadratic-like maps with *a priori* bounds. Our primary motivation came from the renormalization theory that relates tightly this problem to hyperbolicity of the renormalization operator R (see §1.9), but there are many other good reasons to look at it as we will see momentarily.

In [2] we demonstrated that *there exist lean Julia Feigenbaum sets with stationary combinatorics*. We also proved by an Interpolation argument that *if the black hole case is realizable, then so is the balanced case* (albeit, not necessarily with stationary combinatorics).

These results were based on a new method to estimate the *average Poincaré series* by means of a *quadratic recursive equation* that relates the series on different renormalization levels.

The main outstanding problem left after that work was to decide whether the black holes actually exist. Our recent result [4] *resolves this problem affirmatively*.

3.2. **Sullivan's dictionary.** There has been a parallel exciting development in the theory of Kleinian groups. Let Γ be a finitely generated Kleinian group and let Λ be its limit set. Let us assume that $\Lambda \neq \mathbb{C}$. In the mid 1990s, Bishop and Jones proved that $\mathrm{HD}(\Lambda) < 2$ if and only if Γ is geometrically finite. On the other hand, the *Ahlfors Conjecture*, asserting that $\operatorname{area}\Lambda = 0$, has been recently confirmed. (This completed a long story, and many participants were involved, including Ahlfors, Marden, Thurston, Bonahon, Canary, Agol, and Calegari–Gabai.)

Thus, the limit sets of all geometrically infinite Kleinian groups are balanced.

To compare the limit sets with the Julia sets, we need to decide what are geometrically finite rational maps. It should mean strong hyperbolicity, possibly accompanied by parabolic points. So, hyperbolic, parabolic, Misiurewicz, and even Collet–Eckmann maps can be viewed as "geometrically finite". And indeed, for all these maps, the Julia sets are lean (see Przytycki–Rhode [80]).

On the other hand, Shishikura proved in the early 1990s that for a generic quadratic polynomial f_c with $c \in \partial M$, we have $\mathrm{HD}(J(f_c)) = 2$ [85]. Moreover, generically $\operatorname{area}(J(f_c)) = 0$, so these Julia sets are balanced. It seems that these results also fit with the dictionary since generic quadratic maps are obtained by parabolic bifurcation, and hence possess very weak hyperbolicity.

However, Feigenbaum Julia sets are geometrically infinite as well, so it was anticipated that they should be balanced (see [66]). For this reason, existence of lean and black hole examples came as a surprise.

A good question is whether this means that Sullivan's dictionary breaks down when it comes to fine geometric properties of Julia sets. I actually believe it is not necessarily so: indeed, Kleinian groups belong to a special class of *reversible* holomorphic dynamical systems that are conjugate to the inverse by a nice involution (think of the geodesic flow on the hyperbolic 3-manifold). That is why they are balanced, while rational maps can be highly "unbalanced".

3.3. **New features.** Our black holes are not the first examples of Julia sets of positive area. The first ones were constructed by Buff and Cheritat, who carried through the program designed by Douady in the mid 1990s [14]. This strategy is based upon a *Liouvillian mechanism* that produces examples of three types: Cremer, Siegel, and infinitely renormalizable of highly unbounded type. By contrast, the Feigenbaum Julia sets are of Diophantine type, and as such have very different virtues. Here are a few new features of these examples:

Tameness: They are locally connected, and as such admit a precise topological model.

Parameter Visibility: The corresponding set of parameters c has positive Hausdorff dimension (in fact, it is at least $1/2$).

Primitive vs. satellite: They are *primitively* renormalizable (i.e., the renormalization combinatorics corresponds to primitive little Mandelbrot sets). This is a kind of renormalization for which "wild phenomena" were less anticipated.

Measurable Dynamics: The measurable dynamics on the Julia set has a clear nature: it is *ergodic*, and almost all orbits converge to the *Cantor attractor* $\mathcal{O}$ on which f acts as the *adding machine*.

Spectral gap: The *hyperbolic dimension* of $J(f)$ (i.e., the supremum of Hausdorff dimensions over all hyperbolic subsets) is *strictly less than the full Hausdorff dimension*:

$$\mathrm{HD}_{\mathrm{hyp}}(J) < 2 = \mathrm{HD}(J).$$

This is the first known example enjoying this property.

Computer images: It is not so hard to localize black hole parameters c of the type we construct, and to produce pictures of the corresponding Julia sets.

3.4. **Construction of a Feigenbaum Julia set of positive area.** In [2] we gave an efficient criterion to test the black hole case (as well as the lean case) for maps with stationary combinatorics. It depends on two probabilistic parameters:

- the *landing* parameter $\eta > 0$ that controls the probability of going from some renormalization level to a deeper one;
- the *escaping* parameter $\xi > 0$ that controls the probability of escaping from some renormalization level without ever coming back.

If the landing parameter dominates over the escaping one (with a constant depending only on geometric bounds), then the map is black hole. If the opposite domination holds, it is lean.

Now, let us outline the black hole construction. We begin with a *Siegel* quadratic polynomial $P\colon z \mapsto e^{2\pi i\theta} z + z^2$ with a golden mean rotation number θ of high type, and perturb it to a *parabolic* approximand $P_m = P_{p_m/q_m}$. If κ is sufficiently big, then P_m has a distinguished repelling periodic point ζ of period $q_{m-\kappa}$. We perturb P_m further to a *Misiurewicz* map for which ζ is a postcritical point. Finally, this Misiurewicz map can be perturbed to a *superattracting* map that determines the desired renormalization combinatorics (as long as m is sufficiently big).

The renormalization of this map has a large fundamental annulus for big κ, independently of m. This makes the landing parameter η small but definite (as $m \to \infty$). On the other hand, the escaping parameter ξ goes to 0 as $m \to \infty$. This follows from *McMullen's return machinery* [67] which ensures that orbits escaping from the virtual Siegel disk will come back with high probability. Thus, for m sufficiently big, η will dominate ξ.

Three renormalization theories make this construction work:

- *Quadratic-like* renormalization (see §1.9);
- *Siegel* renormalization (see McMullen [67] and Yampolsky [90]);
- *Parabolic* renormalization, particularly the recent advance by Inou and Shishikura [32].

4. FURTHER BIBLIOGRAPHICAL NOTES

The book by Collet and Eckmann [22] can still serve as a good introduction to the early stage of the unimodal dynamics. A thorough overview of the field through the mid 1990s is given in the monograph by de Melo and van Strien [70]. Basic textbooks on holomorphic Dynamics are Milnor [71] and Carleson–Gamelin [20].

4.1. Founders.

Real one-dimensional dynamics originated in the late 19th century in Poincaré's work on circle homeomorphisms. Global holomorphic dynamics was founded in the classical memoirs by Fatou and Julia in the early 20th century.

4.2. Combinatorics and topology.

One cannot talk about one-dimensional dynamics without mentioning the remarkable Sharkovsky Theorem from the 1960s on the forcing relation between periodic orbits. It had appeared ahead of time and had not been noticed in the West for more than a decade (when it was partly rediscovered by Li and Yorke [59]). Besides, Sharkovsky was the first to study in depth topological dynamics on the interval.

The paper by biologist May [64] sparked interest in the dynamics of the logistic maps. The work by Metropolis–Stein–Stein [69] was a predecessor of the kneading theory.

4.3. Stochastic maps.

The first example of a stochastic unimodal map discovered by Ulam and Neumann was the *Chebyshev map* $f_2\colon x \mapsto 4x(1-x)$, which is the "last" map in the quadratic family. Further examples, with nonrecurrent critical point, were provided by Bunimovich, Ruelle, and Misiurewicz, see [68]. The corresponding set of parameters has zero length.

The first example of a nonstochastic map of mixing type was constructed by Johnson [37]. Later, many interesting examples were given by Hofbauer and Keller [35]: for instance, a quadratic map without any asymptotic measure or a quadratic map whose asymptotic measure is supported on a repelling fixed point.

How "stochastic" are stochastic maps? By Ledrappier [48], they are Bernoulli (on their mixing intervals). Collet–Eckmann maps possess an even stronger property of exponential decay of correlations [92].

4.4. Holomorphic dynamics.

Independently of Mandelbrot, the first images (albeit rough) of the bifurcation locus in the complex quadratic family were produced by Brooks and Matelski [16]. Topological models for Julia sets and the Mandelbrot set were built up in [25, 88].

The work of Yoccoz was preceded by the work by Branner and Hubbard [15], who had proved the counterpart of the result for cubic maps with one escaping critical point.

4.5. **Milnor's attractors.** Existence of a unique global measure-theoretic attractor was established in [17]. The article [51] was a continuation of [57] that studied the key example, the quadratic Fibonacci map.

A purely real proof of the decay of geometry (and hence of the absence of wild attractors) for real quadratic maps was found by Shen [83].

4.6. **Renormalization.** The cascade of doubling bifurcations was discovered by Myrberg [76].

The first proof of the Renormalization Conjecture for period doublings was computer-assisted (Lanford [47]). For further developments (that do not use methods of holomorphic dynamics), see in particular, Vul–Sinai–Khanin [89], Epstein [28], and Martens [60].

Complex *a priori* bounds for infinitely renormalizable real maps with bounded combinatorics were originally proved by Sullivan (see [82, 70]).

Local connectivity of Feigenbaum Julia sets was proven in [34, 65]. For ergodicity of the action on these sets, see [79].

An interesting new proof of the Renormalization Conjecture with real stationary combinatorics was given by Daniel Smania [84].

4.7. **The "Fubini foiled" phenomenon.** This also appeared in the work of Shub and Wilkinson on partially hyperbolic dynamics [87], which was another source of inspiration for Avila and Moreira [10].

4.8. **Family $f_{c,d}: x \mapsto x^d + c$.** Combining methods of [52] and [6], Davoud Cheragni [24] gave a new proof of density of hyperbolicity in the families $f_{c,d}$,

Refining methods of [5], Trevor Clark [21] has transferred the Regular or Stochastic Theorem from the families $f_{c,d}$ to arbitrary nontrivial real analytic families of unimodal maps.

4.9. **Smooth dynamics.** We have not discussed much the smooth category. Most of the theory goes through in this case, but there are still some outstanding issues. Let us mention just two important results:

- de Faria, de Melo and Pinto have proved hyperbolicity of the renormalization horseshoes of bounded type in the C^r-spaces [29].
- Avila and Moreira have proved the Regular or Stochastic Theorem for generic smooth families of S-unimodal maps [9].

4.10. **Lean Julia sets.** Lean Julia sets constructed in [2] are Feigenbaum maps with *high combinatorics*, roughly of the same class that was considered in [52] and [91] (it was shown in the latter work that they have zero area). It is shown in [1] that in fact, the Hausdorff dimension of Feigenbaum Julia sets can be arbitrarily close to 1.

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