On the Work and Vision of Dmitry Dolgopyat

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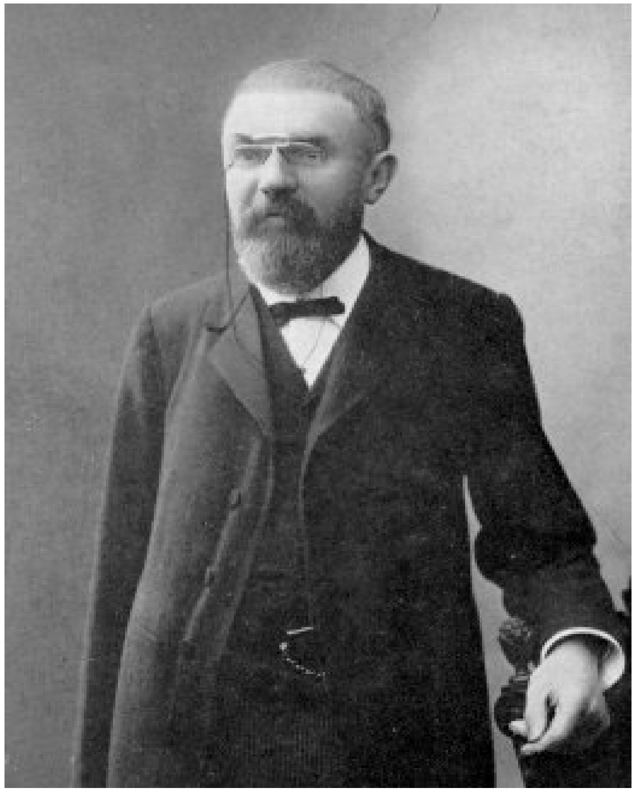
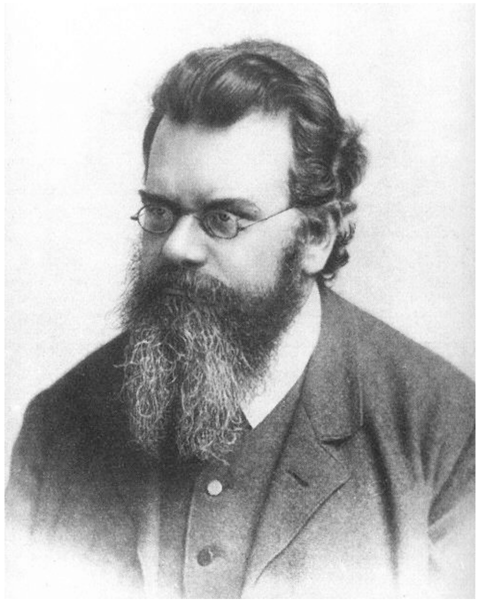
I believe it is not controversial that the roots of Modern Dynamical Systems can be traced back to the work of

Jules Henri Poincaré (1854 – 1912)



Who recognized the phenomenon of instability in systems with few degree of freedoms (Homoclinic tangle)

but (at least to me) equally to the work of



Ludwig Eduard Boltzmann
(1844 –1906)

Founder of Modern Statistical Mechanics

Who introduced the notion of Ergodicity

Ergodicity was put on a solid mathematical basis by the work of Von Neumann and Birkhoff.

Yet it has become clear, at least since the work of Krylov, that for applications to Non-Equilibrium Statistical Mechanics ergodicity is not sufficient: more quantitative properties are required.

In particular, some form of quantitative mixing is necessary.

Such quantitative properties were first obtained for Anosov maps by Sinai-Ruelle-Bowen. Much more recently, thanks to the work of Chernov and, most of all, Dolgopyat, similar results have been obtained for a large class of Anosov flows.

This has opened the doors to a new exciting possibility:

away from low dimensions and back to Statistical Mechanics

Let me explain (beware that the following presentation is quite idiosyncratic)

Some of the most exciting examples of Dynamical Systems are Geodesic flow on manifolds of negative curvature. Their ergodicity has been established by Hopf and then by Anosov. The mixing is due to Sinai.

Important related systems are the various types of Billiards for which the hyperbolicity and ergodicity is understood, starting with the work of Sinai.

What was missing till very recently was an quantitative understanding of the rate of mixing for the above flows.

Let (M, ϕ_t) be a continuos time Dynamical System.

In the footsteps of Andrey Nikolaevich Kolmogorov we consider the evolution of the probability measures:

$$\mathcal{L}_t \mu(f) = \mu(f \circ \phi_t).$$

For many reasonable topologies $\mathcal{L}_t$ is a strongly continuous semigroup, hence it has a generator Z and its resolvent, at least for $\Re(z) = a$ large enough, satisfies

$$R(z) := (z\mathbb{1} - Z)^{-1} = \int_0^\infty e^{-zt} \mathcal{L}_t dt$$

$$\mathcal{L}_t = \lim_{L \to \infty} \int_{-L}^{L} e^{at+ibt} R(a + ib)\, db.$$

Calling μ the SRB measure of the flow (for which $\mathcal{L}_t\mu = \mu$), $R(z)\mu = z^{-1}\mu \otimes 1 + \hat{R}(z)$, where $\hat{R}(z)$ is analytic in a neighborhood of zero.

Thus,

$$\begin{aligned}\mathcal{L}_t &= \mu \otimes 1 + \lim_{L\to\infty} \int_{-L}^{L} e^{at+ibt}\hat{R}(a+ib)\,db \\ &= \mu \otimes 1 + \lim_{L\to\infty} \int_{-L}^{L} \frac{e^{at+ibt}}{(a+ib)^n}\hat{R}(a+ib)Z^n\,db.\end{aligned}$$

We have used the formula $R(z) = \sum_{k=0}^{n} z^{-k-1}Z^k + Z^n R(z)$ and the above analyticity property.

Due to the integral in the flow direction $R(z)$ behaves exactly as the Transfer operator of an Anosov map: for all smooth φ, ψ, setting $dm_\psi = \psi dm$ (m is Lebesgue),

$$|z^{-n}\hat{R}(z)^n m_\psi(\varphi)| \leq C_{\varphi,\psi} e^{-\sigma_z n}$$

Thus,

for each $M > 0$ there is $\omega_M > 0$ such that $\hat{R}(z)$ is analytic in $\{z \in \mathbb{C} \ : \ \Re(z) \geq 0$ or $\Re(z) \geq -\omega_M$ and $|\Im(z)| \leq M\}$.

Accordingly, for each $n \geq 0$,

$$\begin{aligned}\mathcal{L}_t =& \mu \otimes 1 + e^{-\omega_M t} \lim_{L\to\infty} \int_{-M}^{M} e^{ibt} \hat{R}(-\omega_M + ib)\, db \\ &+ \lim_{L\to\infty} \int_{\{M \leq |\Im(z)| \leq L\}} \frac{e^{at+ibt}}{(a+ib)^n} \hat{R}(a+ib) Z^n \, db.\end{aligned}$$

That is, for each $\psi \in \mathcal{C}^n$, and $dm_\psi = \psi dm$,

$$\mathcal{L}_t m_\psi = m(\psi)\mu + \mathcal{O}(M^{-n} + e^{-\omega_M t}).$$

Dolgopyat's inequality

There exists $a, \alpha, \beta > 0$ such that, for each $|b|$ large,

$$\|R(a+ib)^{\beta \ln |b|}\| \leq |b|^{-\alpha}.$$

the above implies, for $0 < \omega < \frac{\alpha a}{\beta}$,

$$\|R(-\omega + ib)\| < |b|^{\beta \ln a}.$$

and the exponential decay of correlations.

The derivations of Dolgopyat's inequality is based on a quantitive version of the joint non-integrability of the strong stable and unstable foliations. The actual proof is rather technical, but it unveils a new (non local) mechanism responsible for mixing. Which has been the basis of many new results in the recent years (e.g. the work of Tsujii).

Thanks to such a strategy Dolgopyat has been able to prove:

- Exponential decay of correlations for mixing Anosov flows with $\mathcal{C}^1$ foliations. (1998)
- Rapid mixing for Axiom A flows with two periodic orbits with period having a Diophantine ratio. (1998)
- Generic exponential mixing for suspension over shifts. (2000)
- Decay of correlation for Group extensions (a quantitative version of Brin theory). (2002)

The above results were technically amazing but still in the path of the traditional approach to the study of statistical properties of Dynamical Systems (Reduction to a symbolic dynamics via Markov partitions).

In the 90's many people deeply felt the need to overcome the traditional approach and develop a strategy independent on Markov partitions.

As a byproduct of a collective effort today there exist several alternative approached.

One of the most powerful and arguably the most flexible is due to Dolgopyat: standard pairs.

Dolgopyat introduced them when he put forward a

- unified approach to the study limit theorems in mixing Dynamical Systems. (2003)

It was then further developed in his work on

- differentiability of the SRB measure for partially hyperbolic system. (2004)

where he elaborated a new version of coupling, pioneered in the field of convergence to equilibrium by Lai-Sang Young.

Another import ingredient developed by Dolgopyat is to adapt Varadhan's martingale problem to the setting of Dynamical Systems. Thanks to the combination all this ideas Dolgopyat has

- created a new general technique to obtain diffusion equations representing the long term behavior of the fluctuations of a Dynamical System even in absence of a natural invariant measure and obtain extremely powerful result for the limiting behavior in systems with slow-fast degree of freedom. (2005)
- Study very refined statistical properties of systems with discontinuities (e.g. Lorentz gases).

Let me describe briefly the idea in a simple setting.

Given a Dynamical Systems (X, f) with a strong unstable foliation, one can consider a class $\mathcal{W}$ of smooth manifolds "close" to the unstable foliation, and the set

$$\Omega_{\alpha,D} = \left\{ (W, \varphi) \ : \ W \in \mathcal{W}, \ \int_W \varphi = 1, \ \|\varphi\|_{\mathcal{C}^\alpha(W,\mathbb{R}_+)} \leq D \right\}.$$

Since for each $\ell = (W, \varphi)$ we can write

$$\mathbb{E}_\ell(A) = \int_W A\varphi$$

$\Omega_{\alpha,D}$ can be naturally viewed as a subset of the probability measures on X.

Let $\overline{\Omega}_{\alpha,D}$ be the convex hull of $\Omega_{\alpha,D}$.

In many relevant cases $f_*\overline{\Omega}_{\alpha,D} \subset \overline{\Omega}_{\alpha,D}$.

Thus any invariant measure obtained by a Krylov-Bogoliubov method starting with a measure in $\overline{\Omega}_{\alpha,D}$ must belong to $\overline{\Omega}_{\alpha,D}$ (Pesin-Sinai, Margulis ...).

More, for each $\ell \in \Omega_{\alpha,D}$ and $n \in \mathbb{N}$ there exist $\ell_i \in \Omega_{\alpha,D}$ and $\alpha_i \geq 1$, $\sum_i \alpha_i = 1$ such that

$$\mathbb{E}_\ell(A \circ f^n) = \sum_i \alpha_i \mathbb{E}_{\ell_i}(A).$$

Since $f^w W$ is a very large manifold it is natural to expect that, if the system is topologically mixing, it will invade all the phase space.

Hence given two standard pairs ℓ, ℓ' we can expect, for n_0 large, to have many of the W_i, W_i' close together.

The basic idea is then to match (couple) the mass in nearby leaves along the weak stable foliation. Then the matched mass will travel together for all the future.

Since a fixed proportion, say δ, of the mass can be matched at any n_0 interval of time, we have that

$$|\mathbb{E}_\ell(A \circ f^{2kn_0}) - \mathbb{E}_{\ell'}(A \circ f^{2kn_0})| \leq \lambda^{-kn_0}|A|_{\mathcal{C}^1} + (1-\delta)^k|A|_{\mathcal{C}^0}$$

In conclusion, Dolgopyat has set the stage for a monumental research program already well underway.

Relevant topics are

- study of an heavy particle interacting with light ones.
- long time behavior of non stationary systems (e.g. particles under the action of an external field).
- systems with weak interaction.

The latter point finally connects to my original remark:

At the moment we can investigate only the case of independent particles, the case of weakly dependent particles is the next, highly non trivial, step to bring the theory of Dynamical Systems at the hart of Non-Equilibrium Statistical Mechanics.

I really look forward to find out what Dolgopyat has in storage for us in the future.