1. (a) \[ c \int_{-1}^{1} (1 - x^2) dx = 1 \Rightarrow c = \frac{3}{4} \]

(b) \[ F(x) = \frac{3}{4} \int_{-1}^{x} (1 - t^2) dt = \frac{3}{4} \left( x - \frac{x^3}{3} + \frac{2}{3} \right), -1 < x < 1 \]

3. The first function could not be a probability density function because it is negative for \( \sqrt{2} < x < 5/2 \).

The second function could not be a pdf because it is negative for \( 2 < x < 5/2 \).

4. (a) \[ P\{ X > 20 \} = \int_{20}^{\infty} \frac{10}{x^2} dx = - \frac{10}{x}\bigg|_{20}^{\infty} = \frac{1}{2} \]

(b) \[ F(x) = \begin{cases} \int_{-10}^{x} \frac{10}{t^2} dt = 1 - \frac{10}{x}, & x > 10 \\ 0, & x \leq 10 \end{cases} \]

(c) The probability any one of the devices will function for at least 15 hours is \[ P\{ X > 15 \} = 1 - P\{ X < 15 \} = 1 - F(15) = \frac{2}{3} \]

Assuming the devices function independently of each other, we use the binomial distribution, with \( n = 6, p = 2/3 \) to get probability at least 3 of the 6 will function at least 15 hours = \[ \sum_{i=3}^{6} \binom{6}{i} \left( \frac{2}{3} \right)^i \left( \frac{1}{3} \right)^{6-i} = 0.8999 \]

5. We want to choose \( c \) so that
\[ 0.01 = \int_{c}^{1} 5(1-x)^4 dx = \text{probability more than } x \text{ is sold in a week.} \]

\[ 0.01 = (1-c)^5 \]
\[ c = 1 - (0.01)^{1/5} = 0.6019 \]
6. (a) \[ E[X] = \int_0^\infty x \cdot \frac{1}{4} x e^{-x/2} dx \] (let \( y = x / 2 \))

\[ = 2 \int_0^\infty y^2 e^{-y} dy = 2 \Gamma(3) = 4 \]

(b) By symmetry of \( f(x) \) about \( x = 0 \), \( E[X] = 0 \).

(c) \[ E[X] = \int_5^\infty \frac{1}{x} dx = \infty \]

8. \[ E[X] = \int_0^\infty x \cdot x e^{-x} dx = \Gamma(3) = 2. \]

10. (a) \[ P\{ \text{goes to A}\} = P\{5 < X < 15 \text{ or } 20 < X < 30 \text{ or } 35 < X < 45 \text{ or } 50 < X < 60 \} \]

\[ = \frac{2}{3}, \text{ since } X \text{ is uniform on } (0,60) \]

(b) \[ P\{ \text{goes to A}\} = P\{5 < X < 15 \text{ or } 20 < X < 30 \text{ or } 35 < X < 45 \text{ or } 50 < X < 60 \text{ or } 65 < X < 75 \} \]

\[ = \frac{2}{3}, \text{ since } X \text{ is now uniform on } (10,70) \]

16. probability of rainfall of over 50 inches in a given year

\[ = P\{ X > 50 \} = P\left\{ \frac{X - 40}{4} > \frac{50 - 40}{4} \right\} = P\{Z > 2.5\} = 1 - \Phi(2.5) = 1 - 0.9938 \]

We want to find the probability that for the next 10 years, the annual rainfall will be under 50 inches. To find this, we assume that the rainfalls of different years are independent of each other. Then

\[ (P\{ X < 50 \})^{10} = (0.9938)^{10} = 0.9397 \]
T1. Using the integration by parts formula \( \int udv = uv - \int vdu \),

with \( dv = -2bxe^{-bx^2} \), \( u = -x/2b \) yields that

\[
\int_0^\infty x^2 e^{-bx^2} \, dx = \left[ -\frac{xe^{-bx^2}}{2b} \right]_0^\infty + \frac{1}{2b} \int_0^\infty e^{-bx^2} \, dx
\]

\[
= \frac{1}{(2b)^{3/2}} \int_0^\infty e^{-y^2/2} \, dy, \text{ by letting } y = x\sqrt{2b}
\]

\[
= \frac{\sqrt{2\pi}}{2} \cdot \frac{1}{(2b)^{3/2}} = \frac{\sqrt{\pi}}{4b^{3/2}} 
\]

where we are using the fact that \( \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-y^2/2} \, dy = 1/2 \). (Use the normal pdf to see why this is true.) Hence,

\[
a = \frac{4b^{3/2}}{\sqrt{\pi}}
\]

T7. \( SD(aX + b) = \sqrt{Var(aX + b)} = \sqrt{a^2 Var(X)} = |a|\sigma \).