Zonal Polynomials and Hypergeometric Functions of Matrix Argument
Some matrix spaces

\( G = GL(n, \mathbb{R}) \): The general linear group, containing all \( n \times n \) nonsingular real matrices

\( S = S(n, \mathbb{R}) \): The space of real symmetric matrices

\( G \) “acts” on \( S \): \( x \in G \) acts on \( s \in S \) by \( x \circ s \equiv xsx' \)

\[
x_1x_2 \circ s = (x_1x_2)s(x_1x_2)' = x_1x_2sx_2'x_1'
\]

\[
= x_1(x_2sx_2')x_1' = x_1(x_2 \circ s)x_1' = x_1 \circ (x_2 \circ s)
\]

\[
x_1x_2 \circ s = x_1 \circ (x_2 \circ s)
\]

This is a group action

right-action vs. left-action
Notation: For \( x = (x_{ij}) \in G \),
\[ \Delta(x) := \text{det}(x) \]
\[ \Delta_j(x) = \Delta \begin{pmatrix} x_{11} & \cdots & x_{1j} \\ \vdots & \ddots & \vdots \\ x_{jj} & \cdots & x_{jj} \end{pmatrix}, \quad j = 1, \ldots, n \]

The standard bitriangular structure of \( G \): Each \( x \in G \) can be expressed as \( x = vcu \) where
- \( c \) is diagonal
- \( u \) is upper triangular with 1’s on the main diagonal
- \( v \) is lower triangular with 1’s on the main diagonal
The map from \((u, c, v) \rightarrow vcu\) is smooth when restricted to the subset of matrices \(x \in G\) such that
\[
\prod_{j=1}^{n} \Delta_j(x) \neq 0
\]

\(O(n)\): the group of \(n \times n\) orthogonal matrices

\(O(n)\) is a maximal compact subgroup of \(G\)

\(P(n, \mathbb{R})\): The cone of positive definite \(n \times n\) matrices

Each \(r \in P(n, \mathbb{R})\) is of the form \(r = xx', x \in G\)
Polynomials on $G$

$\phi$: A polynomial function on $G$

$\phi(x)$ is a polynomial in the entries $x_{ij}$ of $x$

$P(G)$: The space of polynomials on $G$

$P_d(G)$: The space of polynomials homogeneous of degree $d$

$P_d(G)$ is a vector space of dimension $\binom{N+d-1}{N-1}$, where $N \equiv n^2$

$G$ is an open subset of $\mathbb{R}^{n \times n}$

$\phi$ extends uniquely to a polynomial on $\mathbb{R}^{n \times n}$

A polynomial on $G$ restricts uniquely to a polynomial on $P(n, \mathbb{R})$
For $a \in G$, define $R_a : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$ by:

$$R_a \phi(x) = \phi(xa), \quad x \in G$$

$R_a$ is called the right regular representation of $G$ on $\mathcal{P}(G)$.

What is $R_a$ if $a = I_n$?

$$R_{ab} \phi = R_b R_a \phi$$

$\mathcal{P}_d(G)$ is invariant under $R_a$: $R_a \mathcal{P}_d(G) = \mathcal{P}_d(G)$
Regard $\mathbb{R}^{n \times n}$ as $\mathbb{R}^N$

Each polynomial $\phi$ on $\mathbb{R}^{n \times n}$ is a polynomial on $\mathbb{R}^N$

$$\phi(x) = \sum_{\alpha_1, \ldots, \alpha_N} a_{\alpha_1, \ldots, \alpha_N} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}$$

Shorthand notation:

$$\phi(x) = \sum_{\alpha} a_{\alpha} X^\alpha$$

$\alpha! = \alpha_1! \cdots \alpha_N!$

Conjugate of $\phi$: $\tilde{\phi}(x) = \sum_{\alpha} \bar{a}_{\alpha} X^\alpha$
The differentiation inner product on $\mathcal{P}(G)$

For $\phi = \sum_{\alpha} a_\alpha X^\alpha$ and $\psi = \sum_{\alpha} b_\alpha X^\alpha$, define the inner product

$$\langle \phi | \psi \rangle = \sum_{\alpha} \alpha! a_\alpha \bar{b}_\alpha$$

This inner product arises from differentiation of $\psi$ by $\phi$

HW: Prove that for $k \in O(n)$, $R_k$ is a unitary operator on $\mathcal{P}(G)$:

$$\langle R_k \phi | R_k \psi \rangle = \langle \phi | \psi \rangle$$

Hint: Show that $\langle R_a^\prime \phi | R_{a^{-1}} \psi \rangle = \langle \phi | \psi \rangle$, $a \in G$

Under the inner product,

$$\mathcal{P}(G) = \sum_{d=0}^\infty \oplus \mathcal{P}_d(G)$$
Representations of $G$

$G$: $GL(n, \mathbb{R})$

$V$: A space of linear transformations

$\rho : G \to V$ is a representation of $G$ on $V$ if

$$\rho(xy) = \rho(x)\rho(y), \quad x, y \in G$$

The dimension of $\rho$ is the dimension of $V$

Example: The right regular representation

Example: $\rho(x) = \Delta(x)^k \Delta(x)^l$ is a one-dimensional representation of $G$

All one-dimensional representations of $G$ are powers of $\Delta(x)$
Given representations $\rho_1, \rho_2$, form the direct sum

$$(\rho_1 \oplus \rho_2(x) = \begin{pmatrix} \rho_1(x) & 0 \\ 0 & \rho_2(x) \end{pmatrix}$$

This gives us a new representation

Irreducible representations: Those which cannot be written as a direct sum of lower-dimensional representations

The basic problem in representation theory: Describe all irreducible representations of a group

$G = GL(n, \mathbb{R})$: We use the standard bitriangular decomposition to describe some irreducible polynomial representations of $G$
Each irreducible, finite-dimensional, polynomial representation of $GL(n, \mathbb{R})$ is parametrized by an $n$-tuple $(m_1, \ldots, m_n)$ of integers where $m_1 \geq \cdots \geq m_n \geq 0$.

$\pi_m$: The representation corresponding to the signature $m = (m_1, \ldots, m_n)$

Return to the $(U, C, V)$ decomposition: $x = vcu$

For each signature $m$ and $c = \text{diag}(c_1, \ldots, c_n)$, let

$$\mu_m(c) = |c_1|^{2m_1} |c_2|^{2m_2} \cdots |c_n|^{2m_n}$$

$\mu_m$ is a character of $C$

$\mu_m$ is a one-dimensional representation of $C$
\( \mathcal{P}^{2m}(G) \): The collection of all \( \phi \in \mathcal{P}(G) \) such that
\[
\phi(vcx) = \mu_{2m}(c)\phi(x), \quad v \in V, \ c \in C, \ x \in \mathbb{R}^{n \times n}
\]

Each \( \phi \) in \( \mathcal{P}^{2m}(G) \) is homogeneous:
\[
\mathcal{P}^{2m}(G) \subset \mathcal{P}_d(G), \quad d = 2|m|, \ |m| = m_1 + \cdots + m_n
\]

A crucial example of a polynomial in \( \mathcal{P}^{2m}(G) \):
\[
\phi_{2m}(x) = \Delta(x)^{2m_n} \prod_{j=1}^{n-1} \Delta_j(x)^{2(m_j - m_{j+1})}
\]

Calculate \( \Delta_j(vcx) \) to see why \( \phi_{2m} \in \mathcal{P}^{2m}(G) \)

More comments on the representation theory of \( G \)
Orthogonally invariant polynomials

\(\mathcal{I}(G)\): The space of left-invariant polynomials \(\phi\) on \(G\),
\[\phi(kx) = \phi(x), \quad k \in O(n), x \in G\]

If \(\phi\) is homogeneous then it is of even degree, because
\(-I_n \in O(n)\)

\(\mathcal{I}_{2d}(G)\): The class of \(\phi \in \mathcal{I}(G)\) which are homogeneous of degree \(2d\)

\[\mathcal{I}(G) = \sum_{d=0}^{\infty} \oplus \mathcal{I}_{2d}(G)\]

The spherical transform: Given \(\phi \in \mathcal{P}(G)\), construct \(\phi^\# \in \mathcal{I}(G)\): \n\[\phi^\#(x) = \int_{O(n)} \phi(kx) \, dk\]
Apply the spherical transform to each $\phi \in \mathcal{P}^{2m}(G)$ to get the space $\mathcal{I}^{2m}(G)$

The spherical transform decomposes $\mathcal{I}_{2d}(G)$ into irreducible, orthogonal subspaces

$$\mathcal{I}_{2d}(G) = \bigoplus_{|m|=d} \mathcal{I}^{2m}(G)$$

$S = S(n, \mathbb{R})$: The space of real symmetric matrices

Each polynomial $\phi$ on $G$ gives rise to a polynomial $q$ on $S$:

$$q(xx') = \phi(x)$$

The spherical transform converts each left-invariant polynomial $p$ on $G$ into a biinvariant polynomial $q$ on $S$
The zonal polynomials

Apply the spherical transform to $I_{2d}(G)$ and $I^{2m}(G)$

$$\mathcal{P}_d(S) = \sum_{|m|=d} \oplus \mathcal{P}^m(S)$$

This decomposition is *multiplicity free*: Up to constant multiples, there exists only one nontrivial, biinvariant polynomial in $\mathcal{P}^m(S)$

This unique polynomial is called the *zonal polynomial*

Let $\phi \in \mathcal{P}_d(S)$, $\phi(ksk') = \phi(s)$, $k \in O(n)$, $s \in S$. Then there exist unique $\phi_m \in \mathcal{P}^m(S)$ such that

$$\phi = \sum_{|m|=d} \phi_m$$

The $\{\phi_m\}$ are orthogonal w.r.t. the differentiation inner product
Is each subspace $\mathcal{P}^m(S)$ nontrivial? Is there an explicit formula for each zonal polynomial?

Apply the spherical transform to the “crucial example” in $\mathcal{P}^{2m}(G)$

$$q_m(s) = \int_{O(n)} \Delta(ksk'^t)^m \prod_{j=1}^{n-1} \Delta_j(ksk'^t)^{m_j-m_j+1} dk$$

$q_m \in \mathcal{P}^m(S)$ and $q_m > 0$ on the cone of positive definite matrices

Conclude: Each $\mathcal{P}^m(S)$ is nontrivial

$q_m$ is the zonal polynomial of weight $m$
In multivariate statistical analysis, we choose the zonal polynomials $Z_m$ to be such that

$$(\text{tr } s)^d = \sum_{|m|=d} Z_m(s)$$

$$Z_m(s) = Z_m(I_n) \int_{O(n)} \Delta(ksk')^m \prod_{j=1}^{n-1} \Delta_j(ksk')^{m_j-m_{j+1}} dk$$

This requires that we be able to calculate $Z_m(I_n)$

Theorem: $Z_m$ is an eigenfunction of the Laplace-Beltrami operator and, more generally, all $G$-invariant differential operators on $S$

$D_s$: A $G$-invariant differential operator

$D_s = D_{x'sx}, s \in S, x \in G$

If $s = (s_{ij}) \in S$, set

$$\partial_s = \left(\frac{1}{2}(1 + \delta_{ij}) \frac{\partial}{\partial s_{ij}}\right)$$

Examples of invariant differential operators

$$\text{tr} \left(s \partial_s\right)^k, \quad k = 1, 2, \ldots; \quad \Delta(s \partial_s)$$
tr \((s \partial_s)^2\) is the Laplace-Beltrami operator

It can be shown that

\[
q_m(s) = \Delta(s)^m \prod_{j=1}^{n-1} \Delta_j(s)^{m_j - m_{j+1}}
\]

is an eigenfunction of every invariant differential operator \(D_s\)

Since \(D_s\) is \(G\)-invariant then \(D_s = D_{ksk'}\)

D.R. (SIAM J. Math. Analysis, 1985): Applications of invariant differential operators ...
\[ D_s Z_m(s) \propto D_s \int_{O(n)} q_m(ksk') \, dk \]
\[ = \int_{O(n)} D_s q_m(ksk') \, dk \]
\[ = \int_{O(n)} D_{ksk'} q_m(ksk') \, dk \]
\[ \propto \int_{O(n)} q_m(ksk') \, dk \]
\[ = Z_m(s) \]
Theorem: For $s, t \in S$, 
\[
\int_{O(n)} Z_m(ksk'^t) \, dk = \frac{Z_m(s) Z_m(t)}{Z_m(I_n)}
\]

Denote the LHS by $f(s)$

Check that $f$ is an eigenfunction of every invariant $D_s$

Therefore $f(s) \propto Z_m(s)$

Evaluate $f$ at the identity matrix

We can also evaluate Laplace transforms of $Z_m$ and $q_m$
The invariant measure on \( P(n, \mathbb{R}) \): \( d_\ast s = \Delta(s)^{-(p+1)/2} ds \)

For \( \text{Re}(\alpha) > (n - 1)/2 \),

\[
\int_P e^{-\text{tr} r \Delta(r)^\alpha q_m(r)} d_\ast r = [\alpha]_m \Gamma_n(\alpha)
\]

where \((\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1) = \Gamma(\alpha + k)/\Gamma(\alpha)\)

\[
[\alpha]_m = \prod_{j=1}^n (\alpha - \frac{1}{2}(j - 1))_{m_j}
\]

\[
\Gamma_n(\alpha) = \pi^{n(n-1)/4} \prod_{j=1}^n \Gamma(\alpha - \frac{1}{2}(j - 1))
\]

Proof: Apply the standard bitriangular structure (a.k.a. Bartlett decomposition), etc.
For $z \in P(n, \mathbb{R})$, $s \in S$,
\[
\int_P e^{-\text{tr} rz} \Delta(r)^\alpha Z_m(rs) d_*r = [\alpha]_m \Gamma_n(\alpha) \Delta(z)^{-\alpha} Z_m(sz^{-1})
\]

Denote this integral by $F(s, z)$

We already know $F(I_n, I_n)$

Check that $F(s, I_n)$ is $O(n)$ invariant and is in $\mathcal{P}^m(S)$

Therefore $F(s, I_n) \propto Z_m(s)$

Use a change of variables to show that
\[
F(s, z) = \Delta(z)^{-\alpha} F(z^{-1/2}sz^{-1/2}, I_n)
\]
For $z \in S$, 

$$\int_{0 < r < I_n} \Delta(r)^\alpha \Delta(I_n - r)^{\beta - \alpha - \frac{1}{2}(n+1)} Z_m(rz) d_\ast r$$

$$= \frac{\Gamma_n(\alpha)\Gamma_n(\beta - \alpha)}{\Gamma_n(\beta)} \frac{[\alpha]_m}{[\beta]_m} Z_m(z)$$

As in the classical setting, apply the convolution formula for the Laplace transform
Hypergeometric functions of matrix argument

Recall:

\[
[\alpha]_m = \prod_{j=1}^{n} (\alpha - \frac{1}{2}(j - 1))_{m_j}
\]

The generalized hypergeometric function with argument \( s \in S \):

\[
pFq(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; s) = \sum_{d=0}^{\infty} \sum_{|m|=d} \frac{[\alpha_1]_m \cdots [\alpha_p]_m Z_m(s)}{[\beta_1]_m \cdots [\beta_q]_m d!}
\]

When do these series converge?

Example 1: The series for $_0F_0$ converges everywhere on $S$

\[
_{0}F_{0}(s) = \sum_{d=0}^{\infty} \sum_{|m|=d} \frac{Z_m(s)}{d!}
\]

\[
= \sum_{d=0}^{\infty} \frac{1}{d!} \sum_{|m|=d} Z_m(s)
\]

\[
= \sum_{d=0}^{\infty} \frac{1}{d!} (\text{tr } s)^d
\]

\[
= \exp(\text{tr } s)
\]
Example 2: The series for $1F_0$ converges for $||s|| < 1$

$$\Delta(I_n - s)^{-\alpha} = \frac{1}{\Gamma_n(\alpha)} \int_P e^{-\text{tr}(I_n - s)r} \Delta(r)^\alpha d_*r$$

$$= \frac{1}{\Gamma_n(\alpha)} \int_P e^{-\text{tr}r} {}_0F_0(rs) \Delta(r)^\alpha d_*r$$

$$= \frac{1}{\Gamma_n(\alpha)} \sum_{d=0} \frac{1}{d!} \sum_{|m|=d} \int_P e^{-\text{tr}r} \Delta(r)^\alpha Z_m(rs) d_*r$$

$$= \sum_{d=0} \frac{1}{d!} \sum_{|m|=d} \frac{[\alpha]_m Z_m(s)}{d!}$$

$$= {}_1F_0(\alpha; s)$$
Laplace and beta integrals

For \( p \leq q \) and \( \Re(\alpha_{p+1}) > (n - 1)/2 \),

\[
\frac{1}{\Gamma_n(\alpha_{p+1})} \int_{P} e^{-\text{tr} rz} p F_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; r) \Delta(r)^{\alpha_{p+1}} \, d_* r = \Delta(z)^{-\alpha_{p+1}} \, p + 1 F_q(\alpha_1, \ldots, \alpha_p, \alpha_{p+1}; \beta_1, \ldots, \beta_q; z^{-1})
\]

Note: If \( p = q \) then the RHS converges for \( ||z^{-1}|| < 1 \)

\[
\int_{0 < r < I_n} \Delta(r)^{\alpha_{p+1}} \Delta(I_n - r)^{\beta_{q+1} - \alpha_{p+1} - \frac{1}{2}(n+1)} \times p F_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; rs) \, d_* r
\]

\[
= \frac{\Gamma_n(\alpha_{p+1}) \Gamma_n(\beta_{q+1} - \alpha_{p+1})}{\Gamma_n(\beta_{q+1})} \, p + 1 F_{q+1}(\alpha_1, \ldots, \alpha_{p+1}; \beta_1, \ldots, \beta_{q+1}; s)
\]
A few references

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Constantine (1962)

James (1964)

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