The Picard group of the moduli space of vector bundles on the quadric surface


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Introduction: Moduli space of vector bundles

\( X = \) smooth projective surface \( /\mathbb{C} \),

\( H = \) ample divisor on \( X \).

For a vector bundle \( \mathcal{V} \), set \( \mu(\mathcal{V}) = \frac{c_1(\mathcal{V}) \cdot H}{r(\mathcal{V})} \).

**Definition**

Vector bundle \( \mathcal{V} \) is slope (semi)stable if for any subbundle \( \mathcal{E} \subset \mathcal{V} \), we have

\[ \mu(\mathcal{E}) \leq \mu(\mathcal{V}). \]

**Key fact**: for \( \mathcal{V}, \mathcal{W} \) semistable with \( \mu(\mathcal{V}) > \mu(\mathcal{W}) \), we have \( \text{Hom}(\mathcal{V}, \mathcal{W}) = 0 \).

Fix numerical invariants \( \mathbf{v} = (r, c_1, c_2) \in K(X) \).

**Theorem** *(Mumford, Gieseker, Maruyama, Simpson, Álvarez-Cónsul, King)*

There is a projective moduli space \( M(\mathbf{v}) \) for semistable bundles on \( X \).

We will be interested in the Picard group of the moduli space:

\[ \text{Pic} \left( M(\mathbf{v}) \right), \]

when \( X = \mathbb{P}^1 \times \mathbb{P}^1 \). Previous work: Yoshioka, Nakashima, Qin.
Constructing line bundles on $\mathcal{M}(\mathbf{v})$

$\mathcal{U}/\mathcal{S} = \text{flat family of bundles of Chern character } \mathbf{v} \text{ on } X$:

$$
\begin{array}{c}
\mathcal{U} \\
\downarrow \\
\mathcal{X} \times \mathcal{S} \\
\downarrow q \quad \downarrow p \\
\mathcal{X} \\
\end{array}
$$

The Donaldson homomorphism is a map $\lambda_{\mathcal{U}} : K(X) \to \text{Pic}(S)$ defined by

$$
K(X) \xrightarrow{q^*} K^0(X \times S) \xrightarrow{\mathcal{U}^*} K^0(X \times S) \xrightarrow{p^!} K^0(S) \xrightarrow{\text{det}} \text{Pic}(S)
$$

Set $\mathbf{v}^\perp = \{ e \in K(X) \mid \chi(e \cdot \mathbf{v}) = 0 \}$.

The Donaldson homomorphism $\mathbf{v}^\perp \xrightarrow{\lambda} \text{Pic}(\mathcal{M}(\mathbf{v}))$ gives a natural way to construct line bundles on $\mathcal{M}(\mathbf{v})$. 
The $X = \mathbb{P}^2$ case

**Definition**

Vector bundle $E$ is exceptional if

\[ \text{Hom}(E, E) = \mathbb{C}, \]
\[ \text{Ext}^i(E, E) = 0 \quad \text{for} \quad i > 0. \]

Exceptional bundles are semistable.

Introduce $\nu = (r, \nu, \Delta)$ with $\nu = \frac{c_1}{r}$, $\Delta = \frac{1}{2} \nu^2 - \frac{c_2}{r}$.

**Theorem (Drézet, Le Potier ’85)**

\[ \dim M(\nu) > 0 \iff \Delta \geq \text{DLP}(\nu) \]

The branches of the DLP curve are constructed using exceptional bundles: if $E$ satisfies $0 \leq \mu(E) - \mu(\nu) < 3$, then

\[ \text{Hom}(E, \nu) = 0 \quad \text{by semistability}, \quad \text{Ext}^2(E, \nu) = 0 \quad \text{by Serre duality and semistability} \implies \chi(E, \nu) \leq 0. \]

Using Riemann-Roch, this is a numerical condition $\Delta \geq \text{DLP}(\nu)$ on $\nu = (r, \nu, \Delta)$. 
The $X = \mathbb{P}^2$ case

**Definition**

If $\nu$ lies on the branch of the DLP curve given by the exceptional bundle $E$, then we say $E$ is associated to $\nu$.

\[ \chi(E, \nu) = 0 \implies \text{Ext}^i(E, \nu) = 0 \]

**Theorem (Drézet ’88)**

Let $\nu = (r, \nu, \Delta)$ be a character with $\dim M(\nu) > 0$.

1. **Above DLP**: if $\Delta > \text{DLP}(\nu)$, then $\lambda$ is an isomorphism and
   \[ \text{Pic}(M(\nu)) \simeq \mathbb{Z}^2. \]

2. **On DLP**: if $\Delta = \text{DLP}(\nu)$, then $\lambda$ is surjective,
   \[ \text{Pic}(M(\nu)) \simeq \mathbb{Z}, \]
   and $\ker(\lambda) = \mathbb{Z}[E]$, where $E$ is associated to $\nu$ and $\overline{E}$ is either $E$ or $E^\vee$. 

Recall the Donaldson homomorphism $\nu^\perp \to \text{Pic}(M(\nu))$.
We have $K(\mathbb{P}^2) \simeq \mathbb{Z}^3$ and $\nu^\perp = \{ e \in K(\mathbb{P}^2) \mid \chi(e \cdot \nu) = 0 \} \simeq \mathbb{Z}^2$. 

The DLP curve: graph of $\Delta = \text{DLP}(\nu)$ for $X = \mathbb{P}^2$. 

$\mathcal{O}_{\mathbb{P}^2}(1)$
The $X = \mathbb{P}^1 \times \mathbb{P}^1$ case

**Theorem (Rudakov ’94)**

$$\dim M(\nu) > 0, \quad H = F_1 + (1 \pm \epsilon)F_2$$

\[
\Delta \geq \text{DLP}(\nu)
\]

$\nu \in \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)_{\mathbb{Q}} = \{aF_1 + bF_2 | a, b \in \mathbb{Q}\}$.

$\nu^\perp = \{e \in K(\mathbb{P}^1 \times \mathbb{P}^1) | \chi(e \cdot \nu) = 0\} \simeq \mathbb{Z}^3$.

The Picard number is no longer controlled only by the exceptional bundles.

**Theorem (P. ’20)**

Let $\nu = (r, \nu, \Delta)$ be a character with $\dim M(\nu) > 0$. Let $\nu^\perp \xrightarrow{\lambda} \text{Pic}(M(\nu))$ be the Donaldson homomorphism.

1. Above DLP: $\lambda$ is an isomorphism and $\text{Pic}(M(\nu)) \simeq \mathbb{Z}^3$.

2. On one branch of DLP: $\lambda$ is surjective, and if $E$ is associated to $\nu$, then either

   a) $\text{Pic}(M(\nu)) \simeq \mathbb{Z}^2$, $\text{ker}(\lambda) = \mathbb{Z}[E]$, or

   (!) b) $\text{Pic}(M(\nu)) \simeq \mathbb{Z}$, $\text{ker}(\lambda) \supset \mathbb{Z}[E]$. (!)

3. On two branches of DLP: $\lambda$ is surjective, and if $E_1, E_2$ are associated to $\nu$, then

   $\text{Pic}(M(\nu)) \simeq \mathbb{Z}$, $\text{ker}(\lambda) \simeq \mathbb{Z}[E_1] + \mathbb{Z}[E_2]$.
Take $w_1 = (r, \nu, \Delta) = (4, -\frac{1}{4} F_1 - \frac{1}{4} F_2, \frac{9}{16})$; line bundle $\mathcal{O}$ is associated to $\nu$. Then

$$M(w_1) \cong \mathbb{P}^3.$$ 

One can iteratively construct an infinite sequence $\{w_k\}_{k \in \mathbb{N}}$ with $\text{Pic}(M(w_k)) \cong \mathbb{Z}$. 

DLP surface: graph of $\Delta = \text{DLP}(\nu)$ for $X = \mathbb{P}^1 \times \mathbb{P}^1$
Idea of the proof

1. Build a family $\mathcal{V}_t/T$ of bundles of character $\nu$ admitting convenient resolutions
2. Show $\text{Pic}(M(\nu)) \simeq \text{Pic}^G(T^{ss})$
3. Compute $\text{Pic}^G(T)$ (easy)
4. Analyze the unstable locus $T^{un} \subset T$ (codimension, irreducibility) to find $\text{Pic}^G(T^{ss})$ (hard)

Good characters

In the good case, $\text{codim}_T(T^{un}) \geq 2$ and $\text{Pic}(M(\nu)) \simeq \text{Pic}^G(T^{ss}) \simeq \text{Pic}^G(T) \simeq \mathbb{Z}^2$.

Bad characters

In the bad case, the locus $T^{un} \subset T$ has an irreducible divisorial component, which gives

$$\mathbb{Z} \rightarrow \text{Pic}^G(T) \xrightarrow{\text{res}} \text{Pic}^G(T^{ss}) \rightarrow 0,$$

and causes the Picard rank to drop:

$$\text{Pic}(M(\nu)) \simeq \mathbb{Z}$$

instead of $\mathbb{Z}^2$. 
Thank you!

**Theorem** *(P. ’20)*

Let $\mathbf{v} = (r, \nu, \Delta)$ be a character with $\dim M(\mathbf{v}) > 0$. Let $\mathbf{v} \perp \overset{\lambda}{\rightarrow} \text{Pic}(M(\mathbf{v}))$ be the Donaldson homomorphism. If $\mathbf{v}$ is

1. **Above DLP:** $\lambda$ is an isomorphism and

   $$\text{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}^3.$$  

2. **On one branch of DLP:** $\lambda$ is surjective, and if $E$ is associated to $\mathbf{v}$, then either

   a) $\text{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}^2$, $\ker(\lambda) = \mathbb{Z}[\overline{E}]$, or

   (i) b) $\text{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}$, $\ker(\lambda) \supseteq \mathbb{Z}[\overline{E}]$. (i)

3. **On two branches of DLP:** $\lambda$ is surjective, and if $E_1, E_2$ are associated to $\mathbf{v}$, then

   $$\text{Pic}(M(\mathbf{v})) \simeq \mathbb{Z}, \quad \ker(\lambda) \simeq \mathbb{Z}[\overline{E_1}] + \mathbb{Z}[\overline{E_2}].$$