The Picard group of the moduli space of vector bundles on the quadric surface


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Introduction: Moduli space of vector bundles

\[ X = \text{smooth projective surface over } \mathbb{C}, \]
\[ H = \text{ample divisor on } X. \]

For a vector bundle \( \mathcal{V} \), set \( \mu(\mathcal{V}) = \frac{c_1(\mathcal{V}) \cdot H}{r(\mathcal{V})}. \)

**Definition**

Vector bundle \( \mathcal{V} \) is slope (semi)stable if for any subbundle \( \mathcal{E} \subset \mathcal{V} \), we have
\[ \mu(\mathcal{E}) \leq \mu(\mathcal{V}). \]

**Key fact:** for \( \mathcal{V}, \mathcal{W} \) semistable with \( \mu(\mathcal{V}) > \mu(\mathcal{W}) \), we have \( \text{Hom}(\mathcal{V}, \mathcal{W}) = 0 \).

Fix numerical invariants \( v = (r, c_1, c_2) \in K(X) \).

**Theorem (Mumford, Gieseker, Maruyama, Simpson, Álvarez-Cónsul, King)**

There is a projective moduli space \( M(v) \) for semistable bundles on \( X \).

We will be interested in the Picard group of the moduli space:
\[ \text{Pic } (M(v)), \]
when \( X = \mathbb{P}^1 \times \mathbb{P}^1 \). Previous work: Yoshioka, Nakashima, Qin.
Constructing line bundles on $M(\mathbf{v})$

$\mathcal{U}/S = $ flat family of bundles of Chern character $\mathbf{v}$ on $X$:

The Donaldson homomorphism is a map $\lambda_{\mathcal{U}} : K(X) \to \text{Pic}(S)$ defined by

$$K(X) \xrightarrow{q^*} K^0(X \times S) \xrightarrow{\mathcal{U}^*} K^0(X \times S) \xrightarrow{p!} K^0(S) \xrightarrow{\det} \text{Pic}(S)$$

$$\mathcal{E} \xrightarrow{q^*} q^*\mathcal{E} \xrightarrow{\mathcal{U} \otimes q^*\mathcal{E}} p!(\mathcal{U} \otimes q^*\mathcal{E}) \xrightarrow{\det} \det(p!(\mathcal{U} \otimes q^*\mathcal{E}))$$

$\mathcal{U}/M(\mathbf{v}) = $ universal family of semistable bundles of character $\mathbf{v}$ on $X$.

Set $\mathbf{v}^\perp = \{ e \in K(X) \mid \chi(e \cdot \mathbf{v}) = 0 \}$.

The Donaldson homomorphism $\mathbf{v}^\perp \xrightarrow{\lambda} \text{Pic}(M(\mathbf{v}))$ gives a natural way to construct line bundles on $M(\mathbf{v})$. 
The $X = \mathbb{P}^2$ case

### Definition

Vector bundle $E$ is **exceptional** if

\[
\text{Hom}(E, E) = \mathbb{C},
\]

\[
\text{Ext}^i(E, E) = 0 \text{ for } i > 0.
\]

Exceptional bundles are semistable.

Introduce $\nu = (r, \nu, \Delta)$ with $\nu = \frac{c_1}{r}$, $\Delta = \frac{1}{2} \nu^2 - \frac{c_2}{r}$.

### Theorem (Drézet, Le Potier ’85)

\[
\dim M(\nu) > 0 \iff \Delta \geq \text{DLP}(\nu)
\]

The DLP curve: graph of $\Delta = \text{DLP}(\nu)$ for $X = \mathbb{P}^2$.

The branches of the DLP curve are **constructed using exceptional bundles**: if $E$ satisfies $0 \leq \mu(E) - \mu(\nu) < 3$, then

\[
\text{Hom}(E, \nu) = 0 \text{ by semistability}, \quad \text{Ext}^2(E, \nu) = 0 \text{ by Serre duality and semistability} \implies \chi(E, \nu) \leq 0.
\]

Using Riemann-Roch, this is a numerical condition $\Delta \geq \text{DLP}(\nu)$ on $\nu = (r, \nu, \Delta)$.
The $X = \mathbb{P}^2$ case

The DLP curve: graph of $\Delta = \text{DLP}(\nu)$ for $X = \mathbb{P}^2$.

**Definition**

If $\nu$ lies on the branch of the DLP curve given by the exceptional bundle $E$, then we say $E$ is associated to $\nu$.

$\chi(E, \nu) = 0 \implies \text{Ext}^i(E, \nu) = 0$

**Theorem (Drézet ’88)**

Let $\nu = (r, \nu, \Delta)$ be a character with $\dim M(\nu) > 0$.

1. **Above DLP**: if $\Delta > \text{DLP}(\nu)$, then $\lambda$ is an isomorphism and

   $\text{Pic}(M(\nu)) \simeq \mathbb{Z}^2$.

2. **On DLP**: if $\Delta = \text{DLP}(\nu)$, then $\lambda$ is surjective,

   $\text{Pic}(M(\nu)) \simeq \mathbb{Z}$,

   and $\ker(\lambda) = \mathbb{Z}[-E]$, where $E$ is associated to $\nu$ and $\overline{E}$ is either $E$ or $E^\vee$.
The $X = \mathbb{P}^1 \times \mathbb{P}^1$ case

**Theorem (Rudakov ’94)**

\[
\dim M(\nu) > 0, \quad H = F_1 + (1 \pm \epsilon)F_2
\]

\[
\Delta \geq \text{DLP}(\nu)
\]

$\nu \in \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)_\mathbb{Q} = \{aF_1 + bF_2 | a, b \in \mathbb{Q}\}$.

\[
\nu \perp = \{e \in K(\mathbb{P}^1 \times \mathbb{P}^1) | \chi(e \cdot \nu) = 0\} \cong \mathbb{Z}^3.
\]

The Picard number is no longer controlled only by the exceptional bundles.

**Theorem (P. ’20)**

Let $\nu = (r, \nu, \Delta)$ be a character with $\dim M(\nu) > 0$. Let $\nu \perp \xrightarrow{\lambda} \text{Pic}(M(\nu))$ be the Donaldson homomorphism.

1. Above DLP: $\lambda$ is an isomorphism and $\text{Pic}(M(\nu)) \cong \mathbb{Z}^3$.

2. On one branch of DLP: $\lambda$ is surjective, and if $E$ is associated to $\nu$, then either
   
   a) $\text{Pic}(M(\nu)) \cong \mathbb{Z}^2$, $\ker(\lambda) = \mathbb{Z}[\overline{E}]$, or
   
   (!) b) $\text{Pic}(M(\nu)) \cong \mathbb{Z}$, $\ker(\lambda) \supset \mathbb{Z}[\overline{E}]$. (!)

3. On two branches of DLP: $\lambda$ is surjective, and if $E_1, E_2$ are associated to $\nu$, then
   
   $\text{Pic}(M(\nu)) \cong \mathbb{Z}$, $\ker(\lambda) \cong \mathbb{Z}[\overline{E_1}] + \mathbb{Z}[\overline{E_2}]$. 
Drop in the Picard rank

Take \( w_1 = (r, \nu, \Delta) = (4, -\frac{1}{4}F_1 - \frac{1}{4}F_2, \frac{9}{16}) \); line bundle \( \mathcal{O} \) is associated to \( w_1 \). Then

\[
M(w_1) \cong \mathbb{P}^3.
\]

The DLP surface: graph of \( \Delta = \text{DLP}(\nu) \) for \( X = \mathbb{P}^1 \times \mathbb{P}^1 \)

One can iteratively construct an infinite sequence \( \{w_k\}_{k \in \mathbb{N}} \) with \( \text{Pic}(M(w_k)) \cong \mathbb{Z} \).
**Idea of the proof**

1. Build a family $\mathcal{V}_t / T$ of bundles of character $\nu$ admitting convenient resolutions
2. Show $\text{Pic}(M(\nu)) \simeq \text{Pic}^G(T^{ss})$
3. Compute $\text{Pic}^G(T)$ (easy)
4. Analyze the unstable locus $T^{un} \subset T$ (codimension, irreducibility) to find $\text{Pic}^G(T^{ss})$ (hard)

**Good characters**

In the good case, $\text{codim}_T(T^{un}) \geq 2$ and $\text{Pic}(M(\nu)) \simeq \text{Pic}^G(T^{ss}) \simeq \text{Pic}^G(T) \simeq \mathbb{Z}^2$.

**Bad characters**

In the bad case, the locus $T^{un} \subset T$ has an irreducible divisorial component, which gives

$$\mathbb{Z} \to \text{Pic}^G(T) \xrightarrow{\text{res}} \text{Pic}^G(T^{ss}) \to 0,$$

and causes the Picard rank to drop:

$$\text{Pic}(M(\nu)) \simeq \mathbb{Z}$$

instead of $\mathbb{Z}^2$. 
Theorem (P. ’20)

Let \( \mathbf{v} = (r, \nu, \Delta) \) be a character with \( \dim M(\mathbf{v}) > 0 \). Let \( \mathbf{v} \perp \xrightarrow{\lambda} \text{Pic}(M(\mathbf{v})) \) be the Donaldson homomorphism. If \( \mathbf{v} \) is

1. **Above DLP:** \( \lambda \) is an isomorphism and

   \[
   \text{Pic}(M(\mathbf{v})) \cong \mathbb{Z}^3.
   \]

2. **On one branch of DLP:** \( \lambda \) is surjective, and if \( E \) is associated to \( \mathbf{v} \), then either
   
   a) \( \text{Pic}(M(\mathbf{v})) \cong \mathbb{Z}^2, \ker(\lambda) = \mathbb{Z}[E], \) or
   
   (!) b) \( \text{Pic}(M(\mathbf{v})) \cong \mathbb{Z}, \ker(\lambda) \supsetneq \mathbb{Z}[E]. \) (!)

3. **On two branches of DLP:** \( \lambda \) is surjective, and if \( E_1, E_2 \) are associated to \( \mathbf{v} \), then

   \[
   \text{Pic}(M(\mathbf{v})) \cong \mathbb{Z}, \quad \ker(\lambda) \cong \mathbb{Z}[E_1] + \mathbb{Z}[E_2].
   \]