The Picard group of the moduli space of vector bundles on the quadric surface

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Plan of the talk

1. Moduli space of vector bundles on a surface $X$,
2. Case $X = \mathbb{P}^2$,
3. Case $X = \mathbb{P}^1 \times \mathbb{P}^1$. 
Introduction: Moduli space of vector bundles

\( X = \) smooth projective surface \( \mathbb{C} \),

\( H = \) ample divisor on \( X \).

For a vector bundle \( \mathcal{V} \), set \( \mu_H(\mathcal{V}) = \frac{c_1(\mathcal{V}) \cdot H}{r(\mathcal{V})} \).

**Definition**

Vector bundle \( \mathcal{V} \) is slope (semi)stable if for any subbundle \( \mathcal{E} \subset \mathcal{V} \), we have

\[ \mu_H(\mathcal{E}) \leq \mu_H(\mathcal{V}). \]

Fix numerical invariants \( \mathbf{v} = (r, ch_1, ch_2) \in K(X) \).

**Theorem** *(Mumford, Gieseker, Maruyama, Simpson, Álvarez-Cónsul, King)*

There is a projective moduli space \( M(\mathbf{v}) \) for semistable bundles on \( X \).

We will be interested in the Picard group of the moduli space:

\[ \text{Pic} \left( M(\mathbf{v}) \right), \]

when \( X = \mathbb{P}^1 \times \mathbb{P}^1 \).
The $X = \mathbb{P}^2$ case

Introduce $\nu = (r, \nu, \Delta)$ with
$$\nu = \frac{c_1}{r}, \quad \Delta = \frac{1}{2} \nu^2 - \frac{ch_2}{r}.$$

**Theorem (Drézet, Le Potier ’85)**

$$\dim M(\nu) > 0$$

$$\Delta \geq \text{DLP}(\nu)$$

**Theorem (Drézet ’88)**

Let $\nu$ be a character with $\dim M(\nu) > 0$.

1. If $\Delta > \text{DLP}(\nu)$, then $\text{Pic}(M(\nu)) \simeq \mathbb{Z}^2$.
2. If $\Delta = \text{DLP}(\nu)$, then $\text{Pic}(M(\nu)) \simeq \mathbb{Z}$. 

DLP curve: graph of $\Delta = \text{DLP}(\nu)$ for $X = \mathbb{P}^2$. 

$\nu$
The $X = \mathbb{P}^2$ case

DLP curve: graph of $\Delta = \text{DLP}(\nu)$ for $X = \mathbb{P}^2$.

Set $\nu^\perp = \{ u \in K(\mathbb{P}^2) \mid \chi(u \cdot \nu) = 0 \} \simeq \mathbb{Z}^2$.

Donaldson homomorphism $\nu^\perp \xrightarrow{\lambda} \text{Pic}(M(\nu))$ gives a natural way to construct line bundles on $M(\nu)$.

**Definition**

Vector bundle $E$ is exceptional if

- $\text{Hom}(E, E) = \mathbb{C}$,
- $\text{Ext}^i(E, E) = 0$ for $i > 0$.

**Definition**

Suppose $\nu$ lies on the DLP curve. There is a unique exceptional bundle $E$ with $\chi([E], \nu) = 0$ and $0 \leq \mu(E) - \mu(\nu) < 3$.

We say $E$ is associated to $\nu$.

**Theorem (Drézet ’88)**

Let $\nu = (r, \nu, \Delta)$ be a character with $\dim M(\nu) > 0$.

1. If $\Delta > \text{DLP}(\nu)$, then $\lambda$ is an isomorphism and $\text{Pic}(M(\nu)) \simeq \mathbb{Z}^2$.

2. If $\Delta = \text{DLP}(\nu)$, then $\lambda$ is surjective, $\text{Pic}(M(\nu)) \simeq \mathbb{Z}$, and $\ker(\lambda) = \mathbb{Z}[E^*]$, where $E$ is associated to $\nu$. 
The $X = \mathbb{P}^1 \times \mathbb{P}^1$ case

**Theorem (Rudakov ’94)**

\[ \dim M(\nu) > 0 \]

\[ \Delta \geq \text{DLP}(\nu) \]

Note that $\nu = \frac{c_1}{r} = aE + bF$, $a, b \in \mathbb{Q}$.

**Definition**

Suppose $\nu$ lies on the DLP surface. Exceptional $E$ is associated to $\nu$ if

$\chi([E], \nu) = 0$ and $0 \leq \mu(E) - \mu(\nu) < 4$.

Set $\nu^\perp = \{ u \in K(\mathbb{P}^1 \times \mathbb{P}^1) \mid \chi(u \cdot \nu) = 0 \} \cong \mathbb{Z}^3$.

The Picard number is no longer controlled only by associated exceptional bundles.

**Theorem (P. ’20)**

Let $\nu = (r, \nu, \Delta)$ be a character with $\dim M(\nu) > 0$. Let $\nu^\perp \xrightarrow{\lambda} \text{Pic}(M(\nu))$ be the Donaldson homomorphism.

1. If $\Delta \geq \text{DLP}(\nu) + \frac{1}{r}$, then $\lambda$ is an isomorphism and $\text{Pic}(M(\nu)) \cong \mathbb{Z}^3$.

2. If $\Delta = \text{DLP}(\nu)$ and $\nu$ has a single associated bundle $E$, then $\lambda$ is surjective, but there are $\nu$ with

   a) $\text{Pic}(M(\nu)) \cong \mathbb{Z}^2$, $\ker(\lambda) = \mathbb{Z}[E]$,

   b) $\text{Pic}(M(\nu)) \cong \mathbb{Z}$, $\ker(\lambda) \supseteq \mathbb{Z}[E]$. (!)

3. If $\Delta = \text{DLP}(\nu)$ and $\nu$ has two associated bundles $E_1, E_2$, then $\lambda$ is surjective with

   $\text{Pic}(M(\nu)) \cong \mathbb{Z}$, $\ker(\lambda) \cong \mathbb{Z}[E_1] + \mathbb{Z}[E_2]$. (!)
New behavior: example

Take $v = (r, \nu, \Delta) = (4, -\frac{1}{4}E - \frac{1}{4}F, \frac{9}{16})$; line bundle $\mathcal{O}$ is associated to $v$.

Vector bundles $\mathcal{V} \in M(v)$ admit a Beilinson-type resolution

$$
0 \longrightarrow \mathcal{O}(-1, -1)^2 \xrightarrow{\varphi} \mathcal{O}(-1, 0)^3 \oplus \mathcal{O}(0, -1)^3 \longrightarrow \mathcal{V} \longrightarrow 0
$$

Set $\mathbb{H} = \text{Hom}(A, B)$. Then $\mathbb{H} \setminus \{\varphi \mid \text{coker}(\varphi) \text{ is unstable}\} \to M(v)$ is a geometric quotient by a certain group action.

New divisorial locus of unstable bundles

In this case, the locus $\{\varphi \mid \text{coker}(\varphi) \text{ is unstable}\} \subset \mathbb{H}$ has a divisorial component, which makes $\text{Pic}(M(v)) \cong \mathbb{Z}$ instead of $\mathbb{Z}^2$. For general $\varphi$ in this component, the corresponding bundle $\mathcal{V} = \text{coker}(\varphi)$ admits the Harder-Narasimhan filtration $0 \subset F_1 \subset F_2 = \mathcal{V}$ whose quotients have characters

$$
v_1 = (2, -\frac{1}{2}E, \frac{1}{2}), \quad v_2 = (2, -\frac{1}{2}F, \frac{1}{2}).
$$

Thank you!