A NOTE ON THE INVARIANCE PRINCIPLE FOR ADDITIVE FUNCTIONS

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SUMMARY. In this note we give an alternative proof of a theorem due to W. Philipp (1971). Our proof is probabilistic in nature and does not involve many number theoretic calculations.

Let \( \{f_N, N \geq 1\} \) be a sequence of arithmetic functions. For \( n \geq 1 \) write \( B^2(N, n) = \sum_{p \leq n} f_N(p)/p \). Assume \( B(N) = B(N, N) \to \infty \). We define random functions \( h_N(t, m) \in D[0, 1] \) by \( h_N(t, m) = B^{-1}(N) \sum_{p \leq N} \left( \frac{f_N(p)}{p} \right) \left( \frac{\beta_p(m)}{1} \right) \) where the sum is extended over all primes \( p \leq N \) satisfying \( B^2(N, p) \leq tB^2(N) \) and \( \beta_p(m) = 1 \) or \( 0 \) according to whether or not \( p \mid m \). Throughout this paper, \( p, q, p_1, p_2, \ldots \) always denote prime numbers.

The following theorem is due to W. Philipp (1971).

Theorem: Let \( \{f_N\} \) be a sequence of arithmetic functions with \( B(N) \to \infty \). Suppose that for any \( \varepsilon > 0 \)

\[
\frac{1}{B^2(N)} \left\{ \sum_{\substack{p \leq N \\mid \, |f_N(p)| > \varepsilon B(N) \}} \frac{f_N(p)}{p} \right\} \to 0 \quad (N \to \infty).
\]

Then

\[
h_N(t, m) \xrightarrow{D} W
\]

where \( W \) is standard Brownian motion. Moreover, if \( B(N)/B(N, \sqrt{N}) \to 1 \) then (2) is also necessary.

For possible generalizations of the above theorem see Jogesh Babu (1972) and Philipp (1971).

We shall now give an alternative proof of this theorem which is probabilistic in nature and does not involve many number theoretic calculations.

Proof of the theorem: (1) is necessary for (2) to hold is shown in Philipp (1971).

To prove the sufficiency we note that the condition (1) implies the existence of an integer-valued function \( r = r(N) = N^{1/(2)} \) such that \( B(N)/B(N, r) \to 1 \) with \( \log N/\log r \to \infty \) (See Kubilius, 1964, p. 61). Let, for each \( N \), \( \{\xi_{NP} : p \leq N\} \) be a sequence of independent random variables such that

\[
P \left( \xi_{NP} = \frac{-f_N(p)}{pB(N)} \right) = \frac{1}{p}
\]

\[
P \left( \xi_{NP} = \frac{f_N(p)}{B(N)} \left( 1 - \frac{1}{p} \right) \right) = \frac{1}{p}
\]

Let

\[
\xi_{N}(t) = \sum_{\substack{p \leq N \\mid \, |f_N(p)| > \varepsilon B(N) \}} \xi_{NP}, \quad t \in [0, 1].
\]

Let \( X_{N}(t) = \sum_{\substack{p \leq N \\mid \, |f_N(p)| > \varepsilon B(N) \}} \xi_{NP}, \quad t \in [0, 1] \).

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Clearly $X_N(t) \in D[0, 1]$ (See Parthasarathy (1967)).

Define $Y_N \in C[0, 1]$, the space of all bounded continuous functions on the unit interval equipped with the uniform metric, by

$$Y_N(t) = \sum_{p \leq N} \xi_N(p), \quad \text{if } t(B(N))^2 = (B(N, q))^2, \quad q \leq N$$

and for other $t \in (0, 1]$, define $Y_N(t)$ by linear interpolation. In view of the condition (1), the sequence $\{Y_N\}$ satisfies the hypothesis of theorem 4.1 of Parthasarathy (1967, p. 221). Hence

$$Y_N \rightarrow W'$$

where $W'$ is the Wiener measure on $C[0, 1]$. Let $d$ denote the Skhorshod metric (Parthasarathy, 1967) on $D[0, 1]$. For any $\epsilon > 0$, we have

$$P(d(X_N, Y_N) > \epsilon) \leq P\left(\sup_{t \in (0, 1]} |Y_N(t) - X_N(t)| > \epsilon\right)$$

$$\leq P\left(\sup_{p \leq r(N)} |\xi_N(p)| > \frac{\epsilon}{2}\right) + P\left(\sum_{p \leq r(N)} |\xi_N(p)| > \frac{\epsilon}{2}\right)$$

$$\leq \frac{4}{\epsilon B^2(N)} \left(\sum_{p \leq r(N)} f_0^2(p) \left(\frac{1}{p} - \frac{1}{p^2}\right) + \frac{f_K^2(p)}{p^2} \left(1 - \frac{1}{p}\right)\right) + o(1)$$

by condition (1).

Hence as the Wiener measure of $(D[0, 1] - C[0, 1])$ is zero

$$X_N \overset{d}{\rightarrow} W \text{ as } N \rightarrow \infty.$$

Let $E_N = \{1, 2, \ldots, N\}$. For $p \leq r(N)$ and $t = 0, 1$; define

$$E(p, t, N) = \{1 \leq m \leq N : \delta_p(m) = t\}.$$

Also define

$$h_N(t, m, n) = B^{-1}(N) \sum_{p \leq n} f_N(p) \left(\delta_p(m) - \frac{1}{p}\right)$$

where the sum is extended over all primes $p \leq n$ satisfying $B^2(N, p) \leq t B^2(N)$.

Let $F_N$ be the smallest algebra of sets containing all $E(p, t, N)$. The algebra of sets and the function $\frac{1}{N} \text{ card } \{m \leq N : meA\}$ form a finite probability space, and the functions $h_N(\cdot, m, r(N))$ are measurable with respect to this space. For each square-free integer $1 \leq k \leq N$, let

$$E_N = \bigcap_{p \leq r(N)} E(p, \delta(k), N).$$
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Plainly, for different squarefree integers $k$, the classes $E_k$ have no element in common. Clearly if $A \subseteq F_N$ then $A = \bigcup_k E_k$, where the union is taken over certain $k$. Let

$$P(E(p, t, N)) = 1 - \frac{1}{p} \text{ if } t = 0 \text{ and } p \leq r(N),$$

$$= \frac{1}{p}, \text{ if } t = 1 \text{ and } p \leq r(N).$$

Let

$$P(A) = \sum_k \prod_{p \leq r(N)} P(E(p, \delta_p(k), N)).$$

If $(\log r(N)/\log N) \to 0$ as $N \to \infty$, then it is shown in Kubilius (1964). p.27 that

$$\frac{1}{N^k} \text{ card} \{ m \leq N : meA \} = P(A) + o(1) \text{ as } N \to \infty$$

where the estimate $o(1)$ is uniform with respect to all $A \subseteq F_N$. It is easy to see that the joint distribution of the random variables $\xi_{NP}(p \leq r(N))$ is equal to the product over $p \leq r(N)$ of the one dimensional distribution of the random variables $\xi_{NP}$. Hence it follows that, uniformly for every Borel set $A \subseteq [0, 1]$,

$$P(X_N(\cdot) \in A) = \frac{1}{N^k} \text{ card} \{ m \leq N : h_N(\cdot, m, r(N)) \in A \} + o(1). \quad \cdots \quad (3)$$

Let

$$f(p)(m) = \sum_{p \leq m} f_N(p).$$

Now if $f(p)(m)$ and $A(N, n)$ are defined as in the corollary and we show, for every $\varepsilon > 0$, that

$$\frac{1}{N^k} \text{ card} \{ \max_{r(N) \leq n \leq N} \left| \frac{f(p)(m) - f(p)(N)(m) - A(N, n) + A(N, r(N))}{B(N)} \right| > \varepsilon \} \to 0 \quad \cdots \quad (4)$$

as $N \to \infty$, then from (3) it follows easily that,

$$h_N(\cdot, m, N) \to W.$$ 

Note that, the left hand side of (4) is not more than

$$\frac{1}{N^k} \sum_{r(N) \leq n \leq N} \left| \frac{1}{B(N)} \sum_{p \leq n} f_N(p) \left( \delta_p(m) - \frac{1}{p} \right) \right|$$

$$\leq \frac{1}{\varepsilon B(N)} \sum_{r(N) \leq n \leq N} \frac{|f_N(p)|}{p}.$$ 

By using (1) find a sequence $\varepsilon(N)$ of positive real numbers tending to zero such that

$$\frac{1}{B^2(N)} \sum_{|f_N(p)| > \varepsilon(N) B(N)} \frac{1}{p} f(p) < \varepsilon(N).$$

So we can take $r(N) = N^{\varepsilon(N)}$.

Clearly

$$\frac{B(r(N))}{B(N)} \to 1 \text{ and } \frac{\log r(N)}{\log N} \cdot \varepsilon(N) \to 0.$$
Also,

\[
\left( \frac{1}{B(N)} \sum_{\eta(N) < p \leq N} \frac{|f_N(p)|^2}{p} \right) \leq \frac{1}{B^2(N)} \left( \sum_{\eta(N) < p \leq N} \frac{|f_N(p)|^2}{p} \right) \left( \frac{1}{n(N) < p \leq N} \frac{1}{p} \right)
\]

\[\leq (-\log \epsilon(N) + o(1))(\epsilon(N) - \epsilon(N)^2 \log \epsilon(N)) + o(1)) \to 0 \text{ as } N \to \infty.\]

So (4) holds. This completes the proof of the theorem.

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References


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