Martingale Limit Theory and Its Application by P. Hall and C. C. Heyde, Academic Press, New York,

"\ldots\ldots without limit theorems it is impossible to understand the real content of the primary concept of all our sciences—the concept of probability".
—B. V. Gnedenko and A. N. Kolmogorov.

Recent years have seen rapid development of martingale limit theory. The authors themselves are responsible for some major contributions to the theory.

As the authors point out in the preface, "Historically, the first martingale limit theorems were motivated by a desire to extend the theory for sums of independent random variables. Very little attention was paid to possible applications, and it is only in much more recent times that applied probability and mathematical statistics have been a real force behind the development of martingale theory". Limit results for independent random variables are inadequate for handling present developments in many fields. On the other hand, relevant martingales can, most often, be constructed to handle these.

The major portion of the book is devoted to the three 'key limit laws of probability': the strong law of large numbers, the central limit theorem and the law of iterated logarithm. As far as I know this is the first book on martingale limit theory.

The first four chapters contain basic inequalities and various theorems on strong law, CLT, law of the iterated logarithm, invariance principles and CLT for reverse martingales. In the same spirit, random CLT for martingales should have been included.

In chapter 5, the CLT and other limit laws are proved for stationary processes by approximating them by martingales.

Various applications to maximum likelihood estimators, conditional least squares, subadditive processes, Hawkins random sieve etc., are given in the last two chapters.

The book should serve as a good reference work. However, with regret I should point out that the book suffers from some serious mistakes. For example, Theorem A on page 100 and Theorem 7.16 on page 242, as stated, are false. The theorems are stated below for easy reference, along with the counterexamples.

Theorem A. Let $W(t)$, $t \geq 0$, be standard Brownian motion and $T_n$, $n \geq 1$, be positive r.v. Let $W_1$ denote the restriction of $W$ to $[0, 1]$ and write $\xi_n(t) = W(t T_n)/\sqrt{T_n}$, $t \in [0, 1]$. If there exist constants $c_n$ such that $T_n/c_n \sim \gamma^2 > 0$ a.s., then $\xi_n \overset{d}{\to} W_1$ in the sense $(C, \rho)$. 
Counter example. Take $B = \{ f \in C[0, 1] : f(1) < 0, f(\frac{1}{2}) > 0 \}$, $T_n = T = \eta^2 = I(W(1) < 0 \text{ and } W(\frac{1}{2}) > 0) + 1$, $c_n = 1$ for all $n$. Now

$$P(W(T)\sqrt{T} \in B) = P(W(T) < 0, W(T)^2 > 0)$$
$$= P(W(1) < 0, W(\frac{1}{2}) > 0, T = 1) + P(W(2) < 0, W(1) > 0, T = 2)$$
$$= 0 < P(W \in B).$$

So the proofs of Theorems 4.1 and 4.3, which depend on this theorem, are incorrect.

**Theorem 7.16.** Let $\{X_n\}, \{T_n = T_n(X_1, ..., X_n)\}$, and $\{Y_n = Y_n(X_1, ..., X_n)\}$ be sequences of random variables with $X_1$ arbitrary and $X_{n+1} = T_n + Y_n$, where $E(Y_n | X_1, ..., X_n) = 0$ a.s. and $\sum E(Y_n^2) < \infty$.

Suppose also that $|T_n| \leq \max (\alpha_n, 1 + \beta_n, |X_n| + \gamma_n)$, where $\alpha_n, \beta_n$ and $\gamma_n$ are non-negative random variables such that $\alpha_n \rightarrow 0$ a.s., $\beta_n < \infty$ a.s., and $\sum \gamma_n = \infty$ a.s. Then $X_n \rightarrow 0$ a.s. as $n \rightarrow \infty$.

**Note:** There is some confusion regarding the notation here. In the earlier chapters $Y(X)$ is taken to mean that $Y$ is a function of $X$. So $Y(X)$ is measurable with respect to the $\sigma$-field generated by $X$. On the other hand, when applying Theorem 7.16, $Y(X)$ is taken to mean something else. Here for each realization $X = x$, $Y(x)$ is a random variable. If this definition is used then Theorem 7.16 is false.

Counter example. Let $\{Z_n\}$ be i.i.d. with $P(Z_n = 1) = P(Z_n = -1) = \frac{1}{2}$. Put $Y_n = \frac{1}{n} Z_n, X_1 = 1, T_n = \{1 + \frac{n-2}{n} | X_n | - \frac{1}{n} \} \text{ Sign } Y_n, \alpha_n = 0, X_{n+1} = T_n + Y_n = (1 + n^{-2}) | X_n | Z_n$ and $\gamma_n = \frac{1}{n}$.

Clearly, $|T_n| \leq (1 + n^{-2}) | X_n | - \frac{1}{n}$ but

$$|X_{n+1}| = \prod_{i=1}^{n} \left(1 + \frac{1}{i} \right) \not\rightarrow 0.$$

Some misprints are listed below:

Page 100...in Section 3.4 should read as...in Section 3.3... Page 129... $P(\phi(t) \leq 1 | \beta(t) - W(t) | > (2^{t/2} + (2^t)^{1/2})$ for some $t > \lambda$) should read as $P(\phi(0) \leq 1 | \beta(t) - W(t) | < (2^{t/2} + (2^t)^{1/2})$ for all $t > \lambda$)... Page 136... $\mu_k = T^{-k}(x)$... should read as... $\mu_k = T^{-k}(M_x)$... Page 239... $\sup \text{ Var } Y(x) \rightarrow \infty$ should read as... $\sup \text{ Var } Y(x) \rightarrow \infty$... Page 240... $2K_1 A > 1$... should read as $2K_1 A > 1$.

In spite of these drawbacks this would serve as a good reference book.

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