Separate Appendix to:

NONPARAMETRIC COINTEGRATION ANALYSIS

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Following Phillips (1987), we use throughout this appendix the symbol "⇒" to indicate weak convergence (cf. Billingsley 1968), convergence in distribution, or convergence in probability. From the context it will be clear which mode of convergence applies.

Proof of Lemma 1: Denoting the partial sums associated with v_t and w_t by

$$\begin{aligned} S_n^v(x) &= 0 \ \ if \ x \in [0,n^{-1}); \ S_n^v(x) = \sum_{t=1}^{[xn]} v_t \ \ if \ x \in [n^{-1},1], \\ S_n^w(x) &= 0 \ \ if \ x \in [0,n^{-1}); \ S_n^w(x) = \sum_{t=1}^{[xn]} w_t \ \ if \ x \in [n^{-1},1]. \end{aligned} \tag{A.1}$$

respectively, it follows easily that

$$\frac{1}{\sqrt{n}}\begin{pmatrix} S_n^v \\ S_n^w \end{pmatrix} \Rightarrow \begin{pmatrix} I \\ D(1) \end{pmatrix} W, \tag{A.2}$$

where W is a q-variate standard Wiener process. Next, denote the partial sums associated with z_t and Δz_t by

$$\begin{aligned} S_n^z(x) &= 0 \ \ if \ x \in [0,n^{-1}); \ S_n^z(x) = \sum_{t=1}^{[xn]} z_t \ \ if \ x \in [n^{-1},1], \\ S_n^{\Delta z}(x) &= 0 \ \ if \ x \in [0,n^{-1}); \ S_n^{\Delta z}(x) = \sum_{t=1}^{[xn]} \Delta z_t \ \ if \ x \in [n^{-1},1], \end{aligned} \tag{A.3}$$

respectively. Then it follows from (3) and (A.2) that

$$\begin{pmatrix} \frac{S_n^z(x)}{n\sqrt{n}} \\ \frac{S_n^{\Delta z}(x)}{\sqrt{n}} \end{pmatrix} \Rightarrow \begin{pmatrix} C(1)\int_0^x W(y)dy \\ C(1)W(x) \end{pmatrix}. \tag{A.4}$$

It follows from Lemma 9.6.3 in Bierens (1994, p.200) that

$$\sum_{t=1}^{n} F_k(t/n)z_t = F_k(1)S_n^z(1) - \int f_k(x)S_n^z(x)dx, \tag{A.5}$$

where f_k is the derivative of F_k, and similarly for Δz_t. Using (A.4), (A.5), and the straightforward equalities

$$\begin{aligned} &\mathrm{Var}\left(F_k(1)W(1) - \int f_k(x)W(x)dx\right) \\ &= \left(F_k(1)^2 - 2F_k(1)\int x f_k(x)dx + \int\int f_k(x)f_k(y)\min(x,y)\,dx\,dy\right) \cdot I_q = \left(\int F_k(x)^2 dx\right) \cdot I_q, \end{aligned} \tag{A.6}$$

$$F_k(1)\int W(x)dx - \int f_k(x)\int_0^x W(y)dydx = \int F_k(x)W(x)dx, \tag{A.7}$$

$$\mathrm{Var}\left(\int F_k(x)W(x)dx\right) = \int\int F_k(x)F_k(y)\min(x,y)\,dx\,dy \cdot I_q, \tag{A.8}$$

$$\begin{aligned} &\mathrm{Cov}\left[\left(\int F_k(x)W(x)dx\right), \left(F_k(1)W(1) - \int f_k(x)W(x)dx\right)\right] \\ &= \left(F_k(1)\int xF_k(x)dx - \int\int F_k(x)f_k(y)\min(x,y)\,dx\,dy\right) \cdot I_q \\ &= \left(\int F_k(x)\left(\int_0^x F_k(y)dy\right)dx\right) \cdot I_q = \tfrac{1}{2}\left(\int F_k(x)dx\right)^2 \cdot I_q, \end{aligned} \tag{A.9}$$

it follows that (11) holds. The independence of the random vectors X_k and Y_k over k follows from:

$$\mathrm{Cov}\left(\left(\int F_i(x)W(x)dx\right),\left(\int F_j(y)W(y)dy\right)\right)$$
$$= \left(\int\int F_i(x)F_j(y)\min(x,y)\,dx\,dy\right)\cdot I_q = O \text{ for } i \neq j, \tag{A.10}$$

$$\mathrm{Cov}\left(\left(\int F_i(x)W(x)dx\right),\left(F_j(1)W(1)-\int f_j(y)W(y)dy\right)\right)$$
$$= \left(F_j(1)\int xF_i(x)dx - \int\int F_i(x)f_j(y)\min(x,y)\,dx\,dy\right)\cdot I_q$$
$$= \left(\int F_j(x)\left(\int_0^x F_i(y)dy\right)dx\right)\cdot I_q = O \text{ for } i \neq j, \tag{A.11}$$

$$\mathrm{Cov}\left(\left(F_i(1)W(1) - \int f_i(x)W(x)dx\right),\left(F_j(1)W(1) - \int f_j(y)W(y)dy\right)\right)$$
$$= \Big(F_i(1)F_j(1) - F_i(1)\int xf_j(x)dx - F_j(1)\int xf_i(x)dx$$
$$+ \int\int f_i(x)f_j(y)\min(x,y)\,dx\,dy\Big) \times I_q$$
$$= \Big(-F_i(1)F_j(1) + F_i(1)\int F_j(x)dx + F_j(1)\int F_i(x)dx \tag{A.12}$$
$$+ \int\int f_i(x)f_j(y)\min(x,y)\,dx\,dy\Big) \times I_q$$
$$= \left(\int\int f_i(x)f_j(y)\min(x,y)\,dx\,dy - F_i(1)F_j(1)\right) \times I_q$$
$$= \int F_i(x)F_j(x)dx \times I_q = O \text{ for } i \neq j.$$

Q.E.D.

Proof of Lemma 2: Let F be a typical function F_k, with derivative f, and let ξ be a cointegrating vector. Using Lemma 9.6.3 in Bierens (1994, p.200) it follows now that

$$\sqrt{n}\,\xi^T M_n^z(F) = \xi^T\left(F(1)\frac{S_n^w(1)}{\sqrt{n}} - \int f(x)\frac{S_n^w(x)}{\sqrt{n}}dx \right) + \xi^T(z_0 - w_0)\sqrt{n}\left(F(1) - \int f(x)\frac{[nx]}{n}dx \right) \tag{A.13}$$

Note that

$$F(1) - \int x f(x)dx = \int F(x)dx = 0, \tag{A.14}$$

hence

$$|F(1) - \int \frac{[nx]}{n} f(x)dx| \le \tfrac{1}{n}\int x|f(x)|dx \tag{A.15}$$

and consequently equation (A.13) then becomes

$$\sqrt{n}\,\xi^T M_n^z(F) = \xi^T\left(F(1)\frac{S_n^w(1)}{\sqrt{n}} - \int f(x)\frac{S_n^w(x)}{\sqrt{n}}dx \right) + O\left(\frac{\xi^T(z_0 - w_0)}{\sqrt{n}} \right). \tag{A.16}$$

Moreover,

$$\mathrm{Var}\left(\int f(x) w_{[nx]} dx\right) = \int\int f(x)f(y)\mathrm{Cov}(w_{[nx]}, w_{[ny]})\,dx\,dy \to \int\int f(x)f(y)I(x{=}y)\,dx\,dy\,\mathrm{Var}(w_0) = 0. \tag{A.17}$$

and

$$\xi^T S_n^{\Delta z}(x) = \xi^T w_{[nx]} - \xi^T w_0, \tag{A.18}$$

and consequently

$$n\,\xi^T M_n^{\Delta z}(F) = \xi^T\left(F(1)(w_n - w_0) - \int f(x)(w_{[nx]} - w_0)dx\right) = \xi^T\left(F(1)w_n - \int f(x)w_{[nx]}dx\right) = F(1)\,\xi^T w_n + o_p(1). \tag{A.19}$$

The last equality in (A.19) follows from the fact that by the dominated convergence theorem, Furthermore, denoting

$$s_n(F) = F(1)\frac{S_n^w(1)}{\sqrt{n}} - \int f(x)\frac{S_n^w(x)}{\sqrt{n}}dx, \tag{A.20}$$

and using the easy equality

$$\sum_{t=1}^{n} E(w_t w_n^T) = \sum_{t=1}^{n} E(w_0 w_{n-t}^T) = \sum_{j=0}^{n-1} E(w_0 w_j^T), \tag{A.21}$$

Assumption 1 implies that $s_n(F)$ and w_n are jointly normally distributed with covariance

$$\text{Covar}\left(s_n(F), w_n\right) = \int f(x)\frac{\sum_{j=0}^{n-[nx]-1} E(w_0 w_j^T)}{\sqrt{n}}dx = O(1/\sqrt{n}). \tag{A.22}$$

Finally, (A.2) implies that

$$s_n(F) \Rightarrow D(1)\left(F(1)W(1) - \int f(x)W(x)dx\right), \tag{A.23}$$

whereas

$$w_n \sim N_q(0, D_* D_*^T) \tag{A.24}$$

cf. (4). Lemma 2 now easily follows from these results. Q.E.D.

Proof of Lemma 3: Let $z = \exp(2ik\pi/n) = \cos(2k\pi/n) + i.\sin(2k\pi/n)$, and observe that $z^n = 1$. Then

$$\sum_{t=1}^{n} z^t = z\frac{z^n - 1}{z-1} = 0 \tag{A.25}$$

and

$$\sum_{t=1}^{n} tz^{t} = z\frac{d}{dz}\sum_{t=1}^{n} z^{t} = \frac{nz}{z-1} = \tfrac{1}{2}n\left(1 - i\frac{\cos(k\pi/n)}{\sin(k\pi/n)}\right) \tag{A.26}$$

Thus, taking the real part, we have

$$\sum_{t=1}^{n}\cos(2k\pi t/n) = 0, \quad \sum_{t=1}^{n} t\cos(2k\pi t/n) = \tfrac{1}{2}n, \tag{A.27}$$

which proves the conditions (6) and (7). The other condition follow from the proof of Lemma 6 below. Q.E.D.

Proof of Lemma 4: We only prove (17); the other parts of Lemma 4 follow straightforwardly from Lemmas 1-2. It is a standard exercise in linear algebra to verify that

$$\left(R^{T}\hat{A}_{m}R\right)^{-1} = \begin{pmatrix} R_{q-r}^{T}\hat{A}_{m}R_{q-r} & R_{q-r}^{T}\hat{A}_{m}R_{r} \\ R_{r}^{T}\hat{A}_{m}R_{q-r} & R_{r}^{T}\hat{A}_{m}R_{r} \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{A}_{m}^{11} & \tilde{A}_{m}^{12} \\ \tilde{A}_{m}^{21} & \tilde{A}_{m}^{22} \end{pmatrix} \tag{A.28}$$

where

$$\begin{aligned} \tilde{A}_{m}^{11} &= \left(R_{q-r}^{T}\hat{A}_{m}R_{q-r} - (nR_{q-r}^{T}\hat{A}_{m}R_{r})(n^{2}R_{r}^{T}\hat{A}_{m}R_{r})^{-1}(nR_{r}^{T}\hat{A}_{m}R_{q-r})\right)^{-1} \\ \tilde{A}_{m}^{22} &= n^{2}\left(n^{2}R_{r}^{T}\hat{A}_{m}R_{r} - (nR_{r}^{T}\hat{A}_{m}R_{q-r})(R_{q-r}^{T}\hat{A}_{m}R_{q-r})^{-1}(nR_{q-r}^{T}\hat{A}_{m}R_{r})\right)^{-1} \\ \tilde{A}_{m}^{12} &= -n(R_{q-r}^{T}\hat{A}_{m}R_{q-r})^{-1}(nR_{q-r}^{T}\hat{A}_{m}R_{r})(\tilde{A}_{m}^{22}/n^{2}) = (\tilde{A}_{m}^{21})^{T} \end{aligned} \tag{A.29}$$

Therefore,

$$n^{-2}\left(R^{T}\hat{A}_{m}R\right)^{-1} = \begin{pmatrix} O_{p}(n^{-2}) & O_{p}(n^{-1}) \\ O_{p}(n^{-1}) & n^{-2}\tilde{A}_{m}^{22} \end{pmatrix} \Rightarrow \begin{pmatrix} O & O \\ O & V_{r,m}^{-1} \end{pmatrix}, \tag{A.30}$$

where the latter result follows from (15). Q.E.D.

Proof of Theorem 1: It follows from (15), (16), (17) and (18) that

$$R^T\hat{A}_m R \overset{D}{\to} \begin{pmatrix} \sum_{k=1}^{m} X_k^* X_k^{*\,T} & O \\ O & O \end{pmatrix}$$

$$R^T\hat{B}_m R \overset{D}{\to} \begin{pmatrix} \sum_{k=1}^{m} Y_k^* Y_k^{*\,T} & O \\ O & O \end{pmatrix}$$

and

$$R^T\left(\hat{B}_m + n^{-2}\hat{A}_m^{-1}\right)R \overset{D}{\to} \begin{pmatrix} \sum_{k=1}^{m} Y_k^* Y_k^{*\,T} & O \\ O & V_{r,m}^{-1} \end{pmatrix}$$

The solutions of the generalized eigenvalue problem $\det\left[\hat{A}_m - \lambda\hat{B}_m\right] = \det\left[R^T\hat{A}_m R - \lambda R^T\hat{B}_m R\right] =$ 0 are asymptotically ill-defined because the matrix $R^T\hat{B}_m R$ converges in distribution to a singular matrix. This is one of the reasons for working with the generalized eigenvalue problem (19). The other reason is the result of Andersen, Brons and Jensen (1983), which states that if for symmetric $k\times k$ matrices $\hat{P}$ and $\hat{Q}$, $\hat{P} \overset{D}{\to} P$, $\hat{Q} \overset{D}{\to} Q$, *and* $\det(Q) \neq 0$, then the ordered solutions of the generalized eigenvalue problem $\det\left[\hat{P} - \lambda\hat{Q}\right] = 0$ converge in distribution to the ordered solutions of the generalized eigenvalue problem $\det\left[P - \lambda Q\right] = 0$. The conclusion of Theorem 1 now follows from this result. Q.E.D.

Proof of Lemma 5: By Chebishev inequality:

$$P(\lambda_{1,m}^* \leq n\sqrt{K_{\alpha,q-r,m}}) \geq 1 - \frac{E(\lambda_{1,m}^*)}{n\sqrt{K_{\alpha,q-r,m}}} \geq 1 - \frac{E[\mathrm{trace}(V_{r+1,m}^*)]}{n\sqrt{K_{\alpha,q-r,m}}}. \qquad \text{(A.31)}$$

Moreover, it follows easily from (23), by first conditioning on the X_k^*'s, that

$$E(V^*_{r,m}) = \sum_{k=1}^{m}\gamma_k^2 I_r - \sum_{j=1}^{m}\gamma_j^2 E\left(X_j^{*T}\left(\sum_{k=1}^{m}X_k^*X_k^{*T}\right)^{-1}X_j^*\right)I_r$$

$$= \sum_{k=1}^{m}\gamma_k^2 I_r - \sum_{j=1}^{m}\gamma_j^2 E\,\text{trace}\left((1/m)\left(\sum_{k=1}^{m}X_k^*X_k^{*T}\right)^{-1}\sum_{i=1}^{m}X_i^*X_i^{*T}\right)I_r \tag{A.32}$$

$$= \sum_{k=1}^{m}\gamma_k^2 I_r - \frac{q-r}{m}\sum_{k=1}^{m}\gamma_k^2 I_r,$$

where the second equality follows from fact that the X_k^*'s are i.i.d., hence it follows from (22) that

$$E[\text{trace}(V_{r+1,m})] = \left(1 - \frac{q - r - 1}{m}\right)\left(\sum_{k=1}^{m}\gamma_k^2\right)\text{trace}\left(R_{r+1}^T D(1)D(1)^T R_{r+1}\right). \tag{A.33}$$

Q.E.D.

Proof of Lemma 6: It follows from Fourier analysis that we can write without loss of generality:

$$F_k(x) = \sum_{-\infty<j<\infty} c_{j,k}\exp(2i\pi jx), \text{ where } c_{j,k} = \int\exp(2i\pi jx)F_k(x)dx. \tag{A.34}$$

Note that, since F_k is real valued, we can also represent F_k by

$$F_k(x) = \alpha_{0,k} + \sum_{j=1}^{\infty}\left(\alpha_{j,k}\cos(2\pi jx) + \beta_{j,k}\sin(2\pi jx)\right), \tag{A.35}$$

where

$$c_{0,k} = \alpha_{0.k}; \quad \text{for } j \geq 1: \quad c_{j,k} = \frac{\alpha_{j,k} - i\beta_{j,k}}{2}, \quad c_{-j,k} = \frac{\alpha_{j,k} + i\beta_{j,k}}{2}. \tag{A.36}$$

Since

$$\int\exp(2i\pi jx)dx = I(j=0), \tag{A.37}$$

it follows that

$$\int F_k(x)dx = 0 \text{ implies } c_{0,k} = \alpha_{0,k} = 0, \tag{A.38}$$

hence

$$F_k(x) = \sum_{j \neq 0} c_{j,k} \exp(2i\pi jx). \tag{A.39}$$

Next, observe that

$$\int_0^x \exp(2i\pi jy)dy = \frac{\exp(2i\pi jx) - 1}{2i\pi j} I(j \neq 0) + xI(j=0) \tag{A.40}$$

and

$$\int_0^x y\exp(2i\pi jy)dy = \left(\frac{x\exp(2i\pi jx)}{2i\pi j} - \frac{\exp(2i\pi jx) - 1}{(2i\pi j)^2} \right) I(j \neq 0) + \tfrac{1}{2}x^2 I(j=0), \tag{A.41}$$

hence, for $j_1 \neq 0, j_2 \neq 0$,

$$\begin{aligned} &\int\int \exp(2i\pi j_1 x)\exp(2i\pi j_2 y)\min(x,y)dxdy \\ &= \int \exp(2i\pi j_1 x) \int_0^x y\exp(2i\pi j_2 y)dydx - \int x\exp(2i\pi j_1 x)\int_0^x \exp(2i\pi j_2 y)dydx \\ &= \int \frac{x\exp(2i\pi(j_1+j_2)x)}{2i\pi j_2}dx - \int \frac{\exp(2i\pi(j_1+j_2)x)}{(2i\pi j_2)^2}dx + \int \frac{\exp(2i\pi j_1 x)}{(2i\pi j_2)^2}dx \\ &\quad - \int \frac{x\exp(2i\pi(j_1+j_2)x)}{2i\pi j_2}dx + \int \frac{x\exp(2i\pi j_1 x)}{2i\pi j_2}dx \\ &= -\frac{1}{4\pi^2 j_1 j_2} + \frac{I(j_1+j_2=0)}{4\pi^2 j_2^2} \end{aligned} \tag{A.42}$$

and

$$\begin{aligned} \int \exp(2i\pi j_1 x)\int_0^x \exp(2i\pi j_2 y)dydx &= \int \frac{\exp(2i\pi(j_1+j_2)x)}{2i\pi j_2}dx - \int \frac{\exp(2i\pi j_1 x)}{2i\pi j_2}dx \\ &= \frac{I(j_1+j_2=0)}{2i\pi j_2}. \end{aligned} \tag{A.43}$$

It follows now from (A.38) and (A.42) that

$$\int\int F_k(x)F_m(y)\min(x,y)dxdy = \frac{1}{4\pi^2}\left(\sum_{j\neq 0}\frac{c_{j,k}c_{-j,m}}{j^2} - \left(\sum_{j\neq 0}\frac{c_{j,k}}{j}\right)\left(\sum_{j\neq 0}\frac{c_{j,m}}{j}\right)\right)$$
$$= \frac{1}{4\pi^2}\left(\sum_{j=1}^{\infty}\frac{c_{j,k}c_{-j,m}}{j^2} + \sum_{j=1}^{\infty}\frac{c_{j,m}c_{-j,k}}{j^2} - \left(\sum_{j=1}^{\infty}\frac{c_{j,k}}{j} - \sum_{j=1}^{\infty}\frac{c_{-j,k}}{j}\right)\left(\sum_{j=1}^{\infty}\frac{c_{j,m}}{j} - \sum_{j=1}^{\infty}\frac{c_{-j,m}}{j}\right)\right) \tag{A.44}$$
$$= \frac{1}{4\pi^2}\left(\frac{1}{2}\sum_{j=1}^{\infty}\frac{\alpha_{j,k}\alpha_{j,m}}{j^2} + \frac{1}{2}\sum_{j=1}^{\infty}\frac{\beta_{j,k}\beta_{j,m}}{j^2} + \left(\sum_{j=1}^{\infty}\frac{\beta_{j,k}}{j}\right)\left(\sum_{j=1}^{\infty}\frac{\beta_{j,m}}{j}\right)\right)$$

and it follows from (A.38) and (A.43) that

$$\int F_k(x)\int_0^x F_m(y)dydx = \frac{1}{2i\pi}\sum_{j\neq 0}\frac{c_{j,k}c_{-j,m}}{j} = \frac{1}{4\pi}\left(\sum_{j=1}^{\infty}\frac{\alpha_{j,k}\beta_{j,m}}{j} - \sum_{j=1}^{\infty}\frac{\alpha_{j,m}\beta_{j,k}}{j}\right) \tag{A.45}$$

Moreover,

$$\int F_k(x)F_m(x)dx = \sum_{j\neq 0}c_{j,k}c_{-j,m} = \frac{1}{2}\sum_{j=1}^{\infty}\alpha_{j,k}\alpha_{j,m} + \frac{1}{2}\sum_{j=1}^{\infty}\beta_{j,k}\beta_{m,k} \tag{A.46}$$

Finally,

$$\int xF_k(x)dx = \sum_{j\neq 0}\frac{c_{j,k}}{2i\pi j} = -\frac{1}{2\pi}\sum_{j=1}^{\infty}\frac{\beta_{j,k}}{j} \tag{A.47}$$

Q.E.D.

Proof of Lemma 7: Note that the set of solutions of eigenvalue problem (34) is a subset of the set of solutions of eigenvalue problem

$$\det\left[\begin{pmatrix} R_{q-r}^{T}C(1)\sum_{k=1}^{m} X_k X_k^{T} C(1)^T R_{q-r} & O \\ O & O \end{pmatrix} - \lambda\begin{pmatrix} \left(R_{q-r}^{T}C(1)\sum_{k=1}^{m} X_k X_k^{T} C(1)^T R_{q-r}\right)^{-1} & O \\ O & V_{r,m} \end{pmatrix}\right] = 0, \tag{A.48}$$

because the matrix in (34) is singular only if the matrix in (A.48) is singular. Moreover, the non-zero eigenvalues of (A.48) are just the solutions of the eigenvalue problem

$$\det\left[R_{q-r}^{T}C(1)\sum_{k=1}^{m} X_k X_k^{T} C(1)^T R_{q-r} - \lambda\left(R_{q-r}^{T}C(1)\sum_{k=1}^{m} X_k X_k^{T} C(1)^T R_{q-r}\right)^{-1}\right] = 0. \tag{A.49}$$

Therefore, the non-zero solutions of eigenvalue problem (34) are bounded from below by the minimum solution of eigenvalue problem (A.49), and so is $T_{1,m}(H)$. Using the notation (18), it is easy to verify that this minimum solution is the squared minimum solution of the eigenvalue problem

$$\det\left[\sum_{k=1}^{m} X_k^{*} X_k^{*\,T} - \lambda\left(R_{q-r}^{T}C(1)C(1)^T R_{q-r}\right)^{-1}\right] = 0, \tag{A.50}$$

where the X_i^{*}'s are i.i.d. $N_{q-r}(0,I_{q-r})$, and the latter minimum solution is equal to, and bounded from below by

$$\inf_{\eta}\frac{\eta^T\left(\sum_{k=1}^{m} X_k^{*} X_k^{*\,T}\right)\eta}{\eta^T\left(R_{q-r}^{T}C(1)C(1)^T R_{q-r}\right)^{-1}\eta} \geq \inf_{\eta}\frac{\eta^T\left(\sum_{k=1}^{m} X_k^{*} X_k^{*\,T}\right)\eta}{\eta^T\eta}\ \inf_{\eta}\frac{\eta^T R_{q-r}^{T}C(1)C(1)^T R_{q-r}\eta}{\eta^T\eta}$$
$$= \lambda_{\min}\left(\sum_{k=1}^{m} X_k^{*} X_k^{*\,T}\right)\lambda_{\min}\left(R_{q-r}^{T}C(1)C(1)^T R_{,q-r}\right). \tag{A.51}$$

This proves the inequality involved. Since

$$\lambda_{\min}\left(\sum_{k=1}^{m} X_k^* X_k^{*\,T}\right) \leq \lambda_{\min}\left(\sum_{k=1}^{m+1} X_k^* X_k^{*\,T}\right), \tag{A.52}$$

and $M_{\alpha,s,q-r,m}$ is decreasing in *m*, it follows now that the right-hand side lower bound involved increases with *m*. Q.E.D.

Proof of Lemma 8: For $k > 0$ we can write

$$2\sum_{t=1}^{n} t\cos(2k\pi(t-0.5)/n) = g_n(2k\pi/n) + g_n(-2k\pi/n), \tag{A.53}$$

where

$$\begin{aligned} g_n(x) &= e^{-0.5ix}\sum_{t=1}^{n} te^{ixt} = \frac{1}{i}e^{-0.5ix}\frac{d}{dx}\sum_{t=1}^{n}\left(e^{ix}\right)^t = \frac{1}{i}e^{-0.5ix}\frac{d}{dx}\left(e^{ix}\frac{1-e^{inx}}{1-e^{ix}}\right) \\ &= e^{0.5ix}\left(\frac{1-e^{inx}}{\left(e^{-0.5ix}-e^{0.5ix}\right)^2}+\frac{e^{-0.5ix}\left(1-(n+1)e^{inx}\right)}{e^{-0.5ix}-e^{0.5ix}}\right) \\ &= \frac{\cos(0.5x)-i\sin(0.5x)}{-4\sin^2(0.5x)}\left(1-e^{inx}\right) + \frac{2in\sin(0.5x)e^{inx}}{-4\sin^2(0.5x)} \\ &= \frac{\cos(0.5x)(1-\cos(nx)) - (2n-1)\sin(0.5x)\sin(nx)}{-4\sin^2(0.5x)} \\ &\quad + i\,\frac{(2n-1)\sin(0.5x)\cos(nx) - \cos(0.5x)\sin(nx) + \sin(0.5x)}{-4\sin^2(0.5x)}. \end{aligned} \tag{A.54}$$

Thus,

$$g_n(x) + g_n(-x) = \frac{\cos(0.5x)(1-\cos(nx)) - (2n-1)\sin(0.5x)\sin(nx)}{-2\sin^2(0.5x)}. \tag{A.55}$$

Since $\cos(2k\pi) = 1$ and $\sin(2k\pi) = 0$, the second equality in (38) follows. The proof of the first equality goes similarly, and (39) is trivial. Q.E.D.

Proof of Lemma 9: Observe that

$$\sum_{t=1}^{K} \exp\big(i(xt + y)\big) = \exp\big(i(y + 0.5x)\big)\frac{1 - \exp(iKx)}{\exp(-0.5ix) - \exp(0.5ix)}$$

$$= \frac{\big(\cos(y + 0.5x) + i\sin(y + 0.5x)\big)\big(1 - \cos(Kx) - i\sin(Kx)\big)}{-2i\sin(0.5x)}$$

$$= i\frac{\cos(y + 0.5x)\big(1 - \cos(Kx)\big) + \sin(y + 0.5x)\sin(Kx)}{2\sin(0.5x)} \qquad \text{(A.56)}$$

$$+ \frac{\cos(y + 0.5x)\sin(Kx) - \sin(y + 0.5x)\big(1 - \cos(Kx)\big)}{2\sin(0.5x)},$$

hence

$$\sum_{t=1}^{K} \cos(xt + y) = \frac{\cos(y + 0.5x)\sin(Kx) - \sin(y + 0.5x)\big(1 - \cos(Kx)\big)}{4\sin(0.5x)}. \qquad \text{(A.57)}$$

Substituting $K = [(n-\tau)/s]$, $x = 2k\pi s/n$, $y = 2k\pi(\tau-0.5)/n$ it follows that

$$K \sim \frac{n}{s}, \qquad \text{(A.58)}$$

$$1 - \cos(Kx) \sim \frac{2k^2\pi^2\tau^2}{n^2}, \qquad \text{(A.59)}$$

$$\sin(Kx) \sim -\frac{2k\pi\tau}{n}, \qquad \text{(A.60)}$$

$$\sin(y + 0.5x) \sim \frac{2k\pi(\tau + 0.5s - 0.5)}{n}, \quad \cos(y + 0.5x) \sim 1, \qquad \text{(A.61)}$$

$$\sin(0.5x) \sim \frac{k\pi s}{n}, \quad \cos(0.5x) \sim 1, \qquad \text{(A.62)}$$

hence

$$\sum_{j=1}^{[(n-\tau)/s]} \cos\bigl(2k\pi(js + \tau - 0.5)/n\bigr)$$
$$\sim \frac{-2k\pi\tau/n - \bigl(2k\pi(\tau+0.5s-0.5)/n\bigr)\bigl(2k^2\pi^2\tau^2/n^2\bigr)}{4k\pi s/n} \sim -\frac{\tau}{2s} \tag{A.63}$$

and consequently

$$\lim_{n\to\infty}\sum_{t=1}^{n} d_t \cos\bigl[2k\pi(t - 0.5)/n\bigr] = -\frac{\tau}{2s}. \tag{A.64}$$

Next, observe that

$$\sum_{t=1}^{n}\left(\sum_{j=1}^{t} d_j\right)\cos\bigl[2k\pi(t - 0.5)/n\bigr] = \sum_{t=1}^{n}\left[\frac{t - \tau}{s}\right]\cos\bigl[2k\pi(t - 0.5)/n\bigr]$$
$$= \sum_{j=1}^{[(n-\tau)/s]} j\cos\bigl[(2k\pi s/n)j + 2k\pi(\tau - 0.5)/n\bigr] \tag{A.65}$$
$$= \tfrac{1}{2}h_{[(n-\tau)/s]}(2k\pi s/n, 2k\pi(\tau - 0.5)/n) + \tfrac{1}{2}h_{[(n-\tau)/s]}(-2k\pi s/n, -2k\pi(\tau - 0.5)/n),$$

where

$$h_K(x,y) = e^{iy}\sum_{t=1}^{K} t\,e^{ixt} = e^{i(y + 0.5x)}g_K(x), \tag{A.66}$$

with g_K defined by (A.54). Thus

$$h_K(x,y) + h_K(-x,-y)$$
$$= \cos(y+0.5x)\frac{\cos(0.5x)(1 - \cos(Kx) - (2K-1)\sin(0.5x)\sin(Kx)}{-2\sin^2(0.5x)} \tag{A.67}$$
$$- \sin(y+0.5x)\frac{(2K-1)\sin(0.5x)\cos(Kx) - \cos(0.5x)\sin(Kx) + \sin(0.5x)}{-2\sin^2(0.5x)}.$$

Again substituting $K = [(n-\tau)/s]$, $x = 2k\pi s/n$, $y = 2k\pi(\tau-0.5)/n$ it follows that

$$\begin{aligned}\frac{h_K(x,y) + h_K(-x,-y)}{n} &\sim \frac{2k^2\pi^2\tau^2/n^2 - (2K-1)(k\pi s/n)(2k\pi\tau/n)}{-(2k^2\pi^2 s^2/n^2)n} \\ -\frac{2k\pi(\tau+0.5s-0.5)}{n} &\times \frac{(2K-1)(k\pi s/n) - (2k\pi\tau/n) + (k\pi s/n)}{-2(k^2\pi^2 s^2/n^2)n} \\ &\sim \frac{4\tau + s - 1}{s^2}\end{aligned} \tag{A.68}$$

and thus

$$\lim_{n\to\infty}(1/n)\sum_{t=1}^{n}\left(\sum_{j=1}^{t} d_j\right)\cos[2k\pi(t - 0.5)/n] = \frac{4\tau + s - 1}{s^2}. \tag{A.69}$$

This completes the proof of the second part of Lemma 9. The proof of the first part goes similarly. Q.E.D.

TABLES

Table A.1: Fractiles of the lambda-min test statistic:

q-r	m	20 %	10 %	5 %	m	20 %	10 %	5 %
1	1	.10927	.02490	.00598	2	.24145	.11106	.05416
	3	.34138	.18732	.11052	4	.40009	.24428	.15818
	5	.44898	.29513	.19710	6	.47848	.32682	.23884
	7	.52024	.36133	.25962	8	.54094	.38633	.28506
	9	.55668	.41246	.31644	10	.57481	.42687	.33316
	11	.60317	.45547	.35829	12	.60966	.46685	.37801
	13	.62288	.48238	.39139	14	.63239	.49416	.40401
	15	.64366	.51156	.42575	16	.65304	.51782	.42674
	17	.65581	.52710	.43630	18	.67318	.54237	.44917
	19	.66888	.54293	.46049	20	.68914	.56646	.47641
2					2	.01680	.00451	.00115
	3	.07695	.03429	.01691	4	.13448	.07598	.04622
	5	.18198	.11266	.07456	6	.22009	.14202	.10115
	7	.25860	.18104	.12877	8	.28385	.20510	.15549
	9	.30867	.22996	.17613	10	.33487	.25390	.19884
	11	.35873	.27751	.22201	12	.37111	.29057	.23197
	13	.39327	.31205	.24751	14	.40791	.32389	.26502
	15	.41789	.33326	.27669	16	.42895	.34733	.29278
	17	.44201	.36213	.30543	18	.45990	.37446	.31541
	19	.46495	.38068	.32593	20	.47274	.39531	.34064
3	3	.00647	.00148	.00035	4	.03702	.01696	.00842
	5	.07389	.04309	.02562	6	.10887	.06916	.04553
	7	.13921	.09427	.06512	8	.17107	.12133	.09162
	9	.19590	.14465	.10972	10	.21724	.16459	.12784
	11	.23632	.18167	.14391	12	.25775	.19926	.16162
	13	.27382	.21732	.17629	14	.29270	.23328	.19104
	15	.30309	.24746	.20532	16	.31875	.26103	.21643
	17	.33175	.27222	.23120	18	.34300	.28312	.24172
	19	.35170	.29115	.24852	20	.36621	.30316	.25803
4					4	.00318	.00077	.00018
	5	.02337	.01107	.00543	6	.04804	.02784	.01696
	7	.07363	.04634	.03141	8	.10015	.06783	.04832
	9	.12201	.08748	.06562	10	.14265	.10626	.08136
	11	.16247	.12395	.09599	12	.18079	.13703	.11037
	13	.19592	.15346	.12478	14	.21619	.17281	.13926
	15	.22979	.18428	.15159	16	.24584	.19860	.16613
	17	.25364	.20715	.17483	18	.27008	.22414	.18747
	19	.28262	.23834	.20272	20	.29298	.24514	.21046
5	5	.00202	.00050	.00012	6	.01506	.00722	.00357
	7	.03318	.01952	.01192	8	.05301	.03377	.02289
	9	.07287	.05087	.03662	10	.09383	.06725	.04988
	11	.11163	.08272	.06363	12	.12721	.09663	.07651
	13	.14343	.11074	.08839	14	.15954	.12627	.10145
	15	.17604	.14098	.11458	16	.18660	.15171	.12562
	17	.20273	.16348	.13633	18	.21158	.17482	.14828
	19	.22441	.18684	.15770	20	.23545	.19856	.17235

Table A.2: Fractiles of the trace test statistic:

q-r	s	m	20%	10%	5%
1	1	1	≈ ∞	≈ ∞	≈ ∞
1	1	2	10.27089	40.45604	185.15271
1	1	3	2.79425	5.15514	10.01774
1	1	4	1.89590	2.81468	4.42990
1	1	5	1.57446	2.12710	2.98437
1	1	6	1.43377	1.79242	2.30820
1	1	7	1.33653	1.60759	1.94689
1	1	8	1.28459	1.50687	1.77884
1	1	9	1.24592	1.43055	1.66712
1	1	10	1.21811	1.38056	1.56992
1	1	11	1.19692	1.33283	1.50327
1	1	12	1.16973	1.29442	1.44038
1	1	13	1.15015	1.25579	1.38437
1	1	14	1.13847	1.23552	1.35100
1	1	15	1.12841	1.22230	1.33338
1	1	16	1.12008	1.20687	1.30285
1	1	17	1.11127	1.19487	1.28560
1	1	18	1.10456	1.17804	1.25982
1	1	19	1.09660	1.16585	1.24079
1	1	20	1.09114	1.15560	1.22561
1	2	2	≈ ∞	≈ ∞	≈ ∞
1	2	3	26.53175	109.22668	423.04843
1	2	4	5.98635	11.15879	22.49172
1	2	5	3.93237	5.62639	8.26385
1	2	6	3.27373	4.20552	5.52463
1	2	7	2.90537	3.53926	4.29752
1	2	8	2.70794	3.12661	3.64454
1	2	9	2.59083	2.95510	3.38588
1	2	10	2.49265	2.79090	3.14160
1	2	11	2.42987	2.66328	2.92706
1	2	12	2.38458	2.59271	2.83659
1	2	13	2.33520	2.50875	2.70296
1	2	14	2.31191	2.47380	2.64671
1	2	15	2.28288	2.42507	2.58518
1	2	16	2.26040	2.39302	2.52879
1	2	17	2.23829	2.35403	2.49195
1	2	18	2.22214	2.33179	2.45291
1	2	19	2.20783	2.31083	2.42204
1	2	20	2.19858	2.29481	2.40497
1	3	3	≈ ∞	≈ ∞	≈ ∞
1	3	4	44.17178	172.78804	653.78790
1	3	5	9.27488	16.16469	31.74145
1	3	6	6.09519	8.50167	12.45088
1	3	7	4.90693	6.18520	8.08074
1	3	8	4.34888	5.17899	6.32292
1	3	9	4.04032	4.66316	5.41461
1	3	10	3.86540	4.31744	4.85558
1	3	11	3.73838	4.11202	4.56015
1	3	12	3.64370	3.94585	4.29261
1	3	13	3.56688	3.82893	4.14216
1	3	14	3.50144	3.73125	3.96340
1	3	15	3.45025	3.64729	3.85352
1	3	16	3.41261	3.58986	3.80299
1	3	17	3.37611	3.53963	3.62190
1	3	18	3.35126	3.50311	3.66461
1	3	19	3.32970	3.46050	3.60918
1	3	20	3.30271	3.42830	3.55566
1	4	4	≈ ∞	≈ ∞	≈ ∞

1	4	5	59.58699	228.30013	850.87280
1	4	6	12.24434	22.28531	40.43592
1	4	7	7.93184	10.98499	15.53134
1	4	8	6.43190	8.11059	10.35581
1	4	9	5.79224	6.82991	8.23696
1	4	10	5.38720	6.07237	6.92696
1	4	11	5.14141	5.69175	6.33817
1	4	12	4.94268	5.39732	5.91662
1	4	13	4.83353	5.21590	5.64867
1	4	14	4.73080	5.05050	5.42302
1	4	15	4.66360	4.93383	5.24837
1	4	16	4.59474	4.83169	5.08249
1	4	17	4.53581	4.75111	4.97513
1	4	18	4.48886	4.68240	4.88423
1	4	19	4.45383	4.61883	4.81243
1	4	20	4.42914	4.58026	4.74957
2	1	2	$\approx \infty$	$\approx \infty$	$\approx \infty$
2	1	3	24.50277	91.44987	362.12466
2	1	4	5.13644	10.08217	20.34912
2	1	5	2.87585	4.55994	7.61196
2	1	6	2.22376	3.09964	4.40473
2	1	7	1.91091	2.50828	3.22750
2	1	8	1.72206	2.15846	2.75744
2	1	9	1.59818	1.96306	2.41139
2	1	10	1.49204	1.75951	2.09143
2	1	11	1.43400	1.67211	1.94222
2	1	12	1.37626	1.57523	1.80628
2	1	13	1.33670	1.51398	1.70210
2	1	14	1.31538	1.47124	1.66277
2	1	15	1.27958	1.42164	1.58065
2	1	16	1.25560	1.39149	1.54149
2	1	17	1.24056	1.36140	1.49320
2	1	18	1.22119	1.33515	1.45630
2	1	19	1.21052	1.31243	1.42101
2	1	20	1.20187	1.30334	1.40523
2	2	2	$\approx \infty$	$\approx \infty$	$\approx \infty$
2	2	3	$\approx \infty$	$\approx \infty$	$\approx \infty$
2	2	4	61.91753	258.81168	1150.01965
2	2	5	10.59734	20.28767	38.36969
2	2	6	6.05734	9.04116	13.70440
2	2	7	4.52866	6.07478	8.24005
2	2	8	3.82651	4.77489	5.90008
2	2	9	3.40032	4.06216	4.96908
2	2	10	3.15162	3.64828	4.26856
2	2	11	2.97918	3.40804	3.84954
2	2	12	2.84370	3.21368	3.65350
2	2	13	2.73075	3.04280	3.38203
2	2	14	2.65728	2.91700	3.21621
2	2	15	2.59119	2.83387	3.07198
2	2	16	2.53118	2.72154	2.94469
2	2	17	2.48894	2.68427	2.88348
2	2	18	2.45685	2.62200	2.80744
2	2	19	2.43068	2.58588	2.74454
2	2	20	2.39975	2.53892	2.68444
2	3	3	$\approx \infty$	$\approx \infty$	$\approx \infty$
2	3	4	$\approx \infty$	$\approx \infty$	$\approx \infty$
2	3	5	103.36205	413.47913	1705.09412
2	3	6	16.34617	31.27909	60.42951
2	3	7	9.20393	13.75572	19.36834
2	3	8	6.73397	8.87307	11.72758

2	3	9	5.71744	7.18353	9.01382
2	3	10	5.05852	5.97450	7.11445
2	3	11	4.69140	5.35596	6.18987
2	3	12	4.42334	4.96870	5.61262
2	3	13	4.21246	4.64371	5.19294
2	3	14	4.07089	4.46895	4.86464
2	3	15	3.95648	4.28268	4.62844
2	3	16	3.85624	4.15692	4.47847
2	3	17	3.77542	4.02964	4.30385
2	3	18	3.70934	3.93142	4.17179
2	3	19	3.65618	3.86633	4.09166
2	3	20	3.60936	3.81094	4.01248
2	4	4	≈ ∞	≈ ∞	≈ ∞
2	4	5	≈ ∞	≈ ∞	≈ ∞
2	4	6	148.74661	624.72162	2301.92505
2	4	7	22.37093	42.32471	79.57790
2	4	8	12.08182	18.03877	27.56934
2	4	9	9.00571	11.59995	15.18499
2	4	10	7.47450	9.09957	11.14851
2	4	11	6.73648	7.84779	9.17053
2	4	12	6.20259	7.05255	8.09714
2	4	13	5.84929	6.53957	7.32500
2	4	14	5.61266	6.16357	6.76585
2	4	15	5.40286	5.86386	6.38816
2	4	16	5.25239	5.62673	6.07742
2	4	17	5.10956	5.44657	5.84971
2	4	18	5.02389	5.32631	5.63055
2	4	19	4.94531	5.21366	5.47029
2	4	20	4.86233	5.09956	5.35324
3	1	3	≈ ∞	≈ ∞	≈ ∞
3	1	4	38.47602	147.90552	628.65021
3	1	5	7.26398	14.48385	29.58936
3	1	6	3.87222	6.12236	10.13583
3	1	7	2.84031	4.07944	5.79377
3	1	8	2.39632	3.20831	4.24671
3	1	9	2.06276	2.66343	3.38151
3	1	10	1.86216	2.31567	2.87495
3	1	11	1.71717	2.06395	2.51029
3	1	12	1.62446	1.92934	2.25253
3	1	13	1.56046	1.82195	2.12588
3	1	14	1.49405	1.72750	1.97979
3	1	15	1.44824	1.64387	1.85722
3	1	16	1.41740	1.61066	1.81048
3	1	17	1.37483	1.54018	1.71611
3	1	18	1.34887	1.49986	1.64932
3	1	19	1.32023	1.45358	1.58759
3	1	20	1.30487	1.43662	1.58724
3	2	3	≈ ∞	≈ ∞	≈ ∞
3	2	4	≈ ∞	≈ ∞	≈ ∞
3	2	5	93.00429	361.06729	1479.62976
3	2	6	16.13880	31.22144	61.88641
3	2	7	7.91967	12.53577	19.37587
3	2	8	5.80985	8.07796	11.15446
3	2	9	4.71049	5.98588	7.66056
3	2	10	4.08611	4.98691	6.11596
3	2	11	3.66900	4.36060	5.17738
3	2	12	3.42223	3.95770	4.58891
3	2	13	3.22444	3.67791	4.15968
3	2	14	3.06499	3.44421	3.88629
3	2	15	2.96360	3.28493	3.63179

3	2	16	2.86478	3.15767	3.44659
3	2	17	2.77741	3.02203	3.27700
3	2	18	2.71687	2.96573	3.20931
3	2	19	2.66058	2.86360	3.08757
3	2	20	2.62449	2.81048	2.99621
3	3	3	≈ ∞	≈ ∞	≈ ∞
3	3	4	≈ ∞	≈ ∞	≈ ∞
3	3	5	≈ ∞	≈ ∞	≈ ∞
3	3	6	166.69800	687.85577	2673.08301
3	3	7	24.12948	47.16924	88.97661
3	3	8	12.41689	18.96752	30.17323
3	3	9	8.64259	11.73291	15.78372
3	3	10	7.04850	8.94094	11.10475
3	3	11	6.04714	7.21111	8.70557
3	3	12	5.44534	6.36663	7.50848
3	3	13	5.05691	5.77774	6.58924
3	3	14	4.75932	5.34315	5.99465
3	3	15	4.53830	5.03726	5.55357
3	3	16	4.38823	4.78769	5.22870
3	3	17	4.23783	4.61019	4.98024
3	3	18	4.13387	4.46193	4.80153
3	3	19	4.03362	4.32947	4.61589
3	3	20	3.96285	4.21554	4.48116
3	4	4	≈ ∞	≈ ∞	≈ ∞
3	4	5	≈ ∞	≈ ∞	≈ ∞
3	4	6	≈ ∞	≈ ∞	≈ ∞
3	4	7	244.25818	957.57916	4024.66016
3	4	8	32.70491	62.47633	125.55256
3	4	9	16.49575	24.41030	36.64227
3	4	10	11.54159	15.27315	20.31912
3	4	11	9.31754	11.61103	14.59455
3	4	12	7.94072	9.40790	11.16500
3	4	13	7.24958	8.41942	9.71438
3	4	14	6.71290	7.52136	8.46355
3	4	15	6.34575	7.00670	7.78777
3	4	16	6.04001	6.62328	7.27576
3	4	17	5.79484	6.29086	6.85427
3	4	18	5.61419	6.03428	6.48686
3	4	19	5.47009	5.83943	6.24385
3	4	20	5.36221	5.71664	6.04735
4	1	4	≈ ∞	≈ ∞	≈ ∞
4	1	5	56.66291	236.45728	856.82471
4	1	6	9.51424	18.69047	37.44304
4	1	7	4.85499	8.09154	12.45082
4	1	8	3.49984	5.19103	7.40119
4	1	9	2.76747	3.79929	5.06636
4	1	10	2.40223	3.11572	4.03353
4	1	11	2.13557	2.71290	3.37059
4	1	12	1.96666	2.42428	2.92038
4	1	13	1.83161	2.19769	2.57913
4	1	14	1.73539	2.05232	2.44148
4	1	15	1.65182	1.91702	2.22406
4	1	16	1.60059	1.83099	2.08077
4	1	17	1.54215	1.75600	1.97929
4	1	18	1.48518	1.67564	1.86772
4	1	19	1.45582	1.63666	1.80892
4	1	20	1.42796	1.58790	1.75401
4	2	4	≈ ∞	≈ ∞	≈ ∞
4	2	5	≈ ∞	≈ ∞	≈ ∞
4	2	6	129.57700	555.50806	2153.23486

4	2	7	20.84252	42.16933	82.84001
4	2	8	10.25564	16.31891	24.81494
4	2	9	6.98886	9.88485	13.56721
4	2	10	5.51381	7.06309	9.38965
4	2	11	4.70261	5.80228	7.13175
4	2	12	4.18702	5.02169	5.96311
4	2	13	3.84164	4.50865	5.25043
4	2	14	3.61155	4.14485	4.68903
4	2	15	3.40158	3.83342	4.36209
4	2	16	3.25422	3.62673	4.02613
4	2	17	3.13366	3.48036	3.88130
4	2	18	3.02209	3.32310	3.62977
4	2	19	2.92286	3.19585	3.48005
4	2	20	2.86429	3.10779	3.35501
4	3	4	≈ ∞	≈ ∞	≈ ∞
4	3	5	≈ ∞	≈ ∞	≈ ∞
4	3	6	≈ ∞	≈ ∞	≈ ∞
4	3	7	231.10181	910.61505	3823.96753
4	3	8	32.02048	64.73278	128.13597
4	3	9	15.59013	23.91462	37.02358
4	3	10	10.45466	14.19038	19.48011
4	3	11	8.27987	10.53883	13.10155
4	3	12	7.06594	8.55731	10.34639
4	3	13	6.22437	7.36279	8.60385
4	3	14	5.73507	6.60219	7.57303
4	3	15	5.31539	5.98667	6.72832
4	3	16	5.02716	5.57712	6.20682
4	3	17	4.78402	5.29643	5.78535
4	3	18	4.65216	5.09873	5.56232
4	3	19	4.45656	4.83824	5.20532
4	3	20	4.34131	4.66775	4.99190
4	4	4	≈ ∞	≈ ∞	≈ ∞
4	4	5	≈ ∞	≈ ∞	≈ ∞
4	4	6	≈ ∞	≈ ∞	≈ ∞
4	4	7	≈ ∞	≈ ∞	≈ ∞
4	4	8	294.06204	1052.47070	4441.59863
4	4	9	43.23587	84.39127	158.66469
4	4	10	20.47857	31.14259	49.13243
4	4	11	13.99785	18.77476	24.94080
4	4	12	10.89622	13.62592	16.74797
4	4	13	9.32626	11.15563	13.26725
4	4	14	8.25198	9.58426	11.09686
4	4	15	7.53886	8.56490	9.65555
4	4	16	7.04893	7.86310	8.77622
4	4	17	6.61319	7.34334	8.07292
4	4	18	6.39867	7.00531	7.59860
4	4	19	6.12313	6.62820	7.16563
4	4	20	5.90675	6.35843	6.82489

Table A.3: Unit root and trend stationarity tests for the extended Nelson-Plosser data

Test:		Phillips-Perron		Bierens-Guo Cauchy tests (abs. values)						Bierens' higher-order sample autocorrelation tests (*=detr.)				
Variable	n	PP1	PP2	BG1	BG2	BG3	BG4	BG5	BG6	B(1,1)	B(2,2)	B*(1,1)	B*(2,2)	concl.
LN[CPI]	129	2.29	-0.79	51.24	71.10	54.85	25.64	16.69	20.67	-16641	-6.15	-1.68	-8.49	UR?
LN[GNPDEFL]	100	1.43	-5.12	62.54	89.62	125.46	51.56	36.05	7.46	-10000	-3.85	-3.24	-9.71	UR?
LN[EMPLOY]	99	-0.27	-10.87	56.07	98.98	73.78	71.74	0.47	0.47	-0.86	-4.73	-43.20	-45.25	TST
LN[UNEMPLOY]	99	-21.35	-21.49	1.58	1.58	1.66	1.68	0.68	3.50	-2057	-2121	-2067	-2136	TST
LN[GNP]	80	0.56	-5.22	42.90	79.67	116.43	69.42	11.60	181.97	-0.04	-4.40	-2.87	-4.48	UR
LN[GNPPCAP]	80	0.16	-9.66	39.78	80.00	24.51	27.90	2.72	1.26	-0.91	-4.21	-90.71	-93.55	TST
LN[RealGNP]	80	0.17	-9.12	40.97	80.00	70.56	79.91	3.87	1.33	-0.27	-3.21	-133.33	-137.51	TST
LN[WAGE]	89	0.47	-6.96	46.26	88.80	148.13	76.72	10.08	5.60	-0.01	-3.22	-5.89	-8.93	UR
LN[RealWAGE]	89	-0.73	-4.95	35.04	88.29	60.68	49.80	9.55	1.64	-0.63	-1.82	-10.79	-26.87	?
LN[INDPROD]	129	-0.48	-16.99	81.62	128.75	128.77	151.27	7.07	1.70	-0.67	-3.48	-16641	-16641	TST
LN[MONEY]	100	0.22	-9.21	60.68	99.99	421.71	158.28	6.00	2.14	-0.08	-3.58	-10.31	-14.18	?
INTEREST	89	-1.52	-4.39	112.42	86.44	5.12	5.03	24.49	4.64	-7921	-16.32	-1.80	-27.37	?
LN[STOCKPR]	118	1.42	-6.50	84.65	115.95	27.10	29.56	17.55	7.67	-13924	-5.51	-2.92	-5.66	UR?
LN[VELOCITY]	120	-4.19	-2.82	9.23	10.25	8.19	14.18	21.56	46.71	-3.20	-3.26	-2.08	-5.47	UR
LN[RealM]	100	-1.09	-7.66	54.73	97.32	188.25	195.08	8.44	5.17	-0.97	-3.81	-14.87	-19.14	TST?
RealINTEREST	89	-29.23	-29.30	4.03	4.03	4.05	4.05	0.39	0.81	-7921	-7921	-7921	-7921	ST
INFLATION	128	-46.30	-40.10	1.53	1.53	1.60	1.63	0.42	1.44	-13.63	-15.90	-17.91	-19.53	ST
5% R.R.		<-14.0	<-21.5	>12.71	>12.71	>12.71	>12.71	>12.71	>12.71	<-14.0	<-15.7	<-20.6	<-22.4	
10% R.R.		<-11.2	<-18.1	>6.31	>6.31	>6.31	>6.31	>6.31	>6.31	<-11.2	<-13.1	<-17.1	<-18.9	
H_0:		UR	UR	ST	ST	ST	ST	TST	TST	UR	UR	UR	UR	
H_1:		ST	TST	UR	UR	UR	UR	UR	UR	ST	ST	TST	TST	

Remarks: The first two test are the Phillips-Perron tests Z_a of the null hypothesis H_0: y(t) = y(t-1) + u(t), E[u(t)] = 0, u(t) is alpha-mixing, against the alternatives y(t) = c + u(t) and y(t) = c + b.t + u(t), respectively. The next six tests are the Bierens-Guo's (1993) tests of the null hypothesis H_0: y(t) = c (+ d.t) + u(t), E[u(t)]=0, u(t) is alpha-mixing, against the unit root (with drift) hypothesis. The Phillips-Perron tests and Bierens-Guo's test no. 4 employ a Newey and West (1987) type variance estimator with truncation parameter $m = [5n^{0.2}]$. The last four tests are Bierens (1993) unit root tests on the basis of higher order sample autocorrelations. The first two test the unit root hypothesis against stationarity, and the last two have as alternative linear trend stationarity. These four tests depend on parameters $\mu > 0$, $\alpha > 0$, and $0 < \delta < 1$, and the lag length is: $m = 1+[\alpha n^{\delta\mu/(3\mu+2)}]$. The default values $\mu = 2$, $\alpha = 5$ and $\delta = .5$ are employed.

Table A.4: *Nonparametric tests of H_r against H_{r+1}*

r	test statistic	critical regions		conclusion
0	0.00060	10%:	(0,.005)	reject
	0.00425	5%:	(0,.017)	reject
1	1.20899	10%:	(0,.111)	accept
	1.20899	5%:	(0,.054)	accept

Table A.5: *Test of the hypothesis that the space of cointegrating vectors is spanned by the column of a* 2 × 1 *matrix H (Nonparametric)*:

H^T:	test stat.	conclusions 10%	5%
(1,-0.40)	8.13	reject	reject
(1,-0.50)	3.92	reject	accept
(1,-0.60)	1.65	accept	accept
(1,-0.65)	1.15	accept	accept
(1,-0.70)	1.01	accept	accept
(1,-0.75)	1.18	accept	accept
(1,-0.80)	1.63	accept	accept
(1,-0.90)	3.18	reject	accept
(1,-1.00)	5.37	reject	reject

Table A.6: *Johansen's test results for the number* (*r*) *of cointegrating vectors* (*intercept present, but linear trend absent*)

p	r	test stat.	crit. val. 10%	crit. val. 5%	conclusions: 10%	conclusions: 5%	test type	table(*)
2	0	7.8	12.1	14.0	accept	accept	lambda-max	A.1
	1	0.9	2.8	4.0	accept	accept	''	''
	1	0.9	2.8	4.0	accept	accept	trace	''
	0	8.7	13.3	15.2	accept	accept	''	''
	0	7.8	12.8	14.6	accept	accept	lambda-max	A.2
	1	0.9	6.7	8.1	accept	accept	''	''
	1	0.9	6.7	8.1	accept	accept	trace	''
	0	8.7	15.6	17.8	accept	accept	''	''
	0	16.4	13.8	15.8	reject	reject	lambda-max	A.3
	1	6.1	7.6	9.1	accept	accept	''	''
	1	6.1	7.6	9.1	accept	accept	trace	''
	0	22.5	18.0	20.2	reject	reject	''	''
	1	8.63	2.71	3.84	reject	reject	interc. restr.	$\chi^2(1)$
					r = 0	r = 0		
4	0	15.2	12.1	14.0	reject	reject	lambda-max	A.1
	1	2.4	2.8	4.0	accept	accept	''	''
	1	2.4	2.8	4.0	accept	accept	trace	''
	0	17.6	13.3	15.2	reject	reject	''	''
	0	15.2	12.8	14.6	reject	reject	lambda-max	A.2
	1	2.4	6.7	8.1	accept	accept	''	''
	1	2.4	6.7	8.1	accept	accept	trace	''
	0	17.6	15.6	17.8	reject	accept	''	''
	0	18.5	13.8	15.8	reject	reject	lambda-max	A.3
	1	11.9	7.6	9.1	reject	reject	''	''
	1	11.9	7.6	9.1	reject	reject	trace	''
	0	30.5	18.0	20.2	reject	reject	''	''
	1	3.36	2.71	3.84	reject	accept	interc. restr.	$\chi^2(1)$
					r = 1	r = 2		
6	0	14.7	12.1	14.0	reject	reject	lambda-max	A.1
	1	2.2	2.8	4.0	accept	accept	''	''
	1	2.2	2.8	4.0	accept	accept	trace	''
	0	16.9	13.3	15.2	reject	reject	''	''
	0	14.7	12.8	14.6	reject	reject	lambda-max	A.2
	1	2.2	6.7	8.1	accept	accept	''	''
	1	2.2	6.7	8.1	accept	accept	trace	''
	0	16.9	15.6	17.8	reject	accept	''	''
	0	19.0	13.8	15.8	reject	reject	lambda-max	A.3
	1	6.7	7.6	9.1	accept	accept	''	''
	1	6.7	7.6	9.1	accept	accept	trace	''
	0	25.7	18.0	20.2	reject	reject	''	''
	1	4.31	2.71	3.84	reject	reject	interc. restr.	$\chi^2(1)$
					r = 1	r = 1		

(*) Cf. Johansen and Juselius (1990). Table A.3 applies if cointegration restrictions have been imposed on the intercept parameters, whereas tables A.1 and A.2 apply if no cointegration restrictions are imposed. Table A.2 applies if these cointegration restrictions actually hold, and table A.1 applies if not. The $\chi^2(1)$ tests test the null hypothesis that cointegration restrictions on the intercept parameters hold, given $r = 1$, i.e., that the cointegration relation contains an intercept rather than the error correction model itself.

Table A.7: *Johansen's LR test of the hypothesis that the space of cointegrating vectors is spanned by the column of a* 2 × 1 *matrix H* (*intercept present, trend absent)*:

H^T:	test stat.	conclusions 10%	5%
(1,-0.40)	11.74	reject	reject
(1,-0.50)	11.55	reject	reject
(1,-0.60)	11.02	reject	reject
(1,-0.65)	10.26	reject	reject
(1,-0.70)	7.75	reject	reject
(1,-0.75)	0.21	accept	accept
(1,-0.80)	11.22	reject	reject
(1,-0.90)	12.55	reject	reject
(1,-1.00)	12.49	reject	reject

Table A.8: *Cointegration regressions for* ln[wages].

Regressors:	OLS estimates:		
LN[GNP]	0.64577	0.74564	0.68591
1		-1.27404	-0.86678
time (1860=1)			0.00385
R^2:	0.97845	0.99657	0.99684

n = 80 (1909-1988)

Table A.9: *Johansen's test results for the number* (*r*) *of cointegrating vectors: intercept and time trend present, with cointegration restrictions on the trend parameters imposed*

p	r	test stat.	crit. val. 10%	crit. val. 5%	conclusions: 10%	conclusions: 5%	test type	table(*)
6	0	18.2	16.9	19.2	reject	accept	lambda-max	V
	1	6.7	10.6	23.5	accept	accept	''	''
	1	6.7	10.6	12.5	accept	accept	trace	''
	0	24.9	23.0	25.4	reject	accept	''	''
	1	0.00	2.71	3.84	accept	accept	trend restr.	$\chi^2(1)$
					r = 1	r = 0		
8	0	27.2	16.9	19.2	reject	reject	lambda-max	V
	1	7.9	10.6	23.5	accept	accept	''	''
	1	7.9	10.6	12.5	accept	accept	trace	''
	0	35.1	23.0	25.4	reject	reject	''	''
	1	2.06	2.71	3.84	accept	accept	trend restr.	$\chi^2(1)$
					r = 1	r = 1		

(*) Cf. Johansen (1994). Table V applies if cointegration restrictions have been imposed on the trend parameters. The $\chi^2(1)$ test tests for cointegration restriction on the trend parameters, i.e., the hypothesis that there is a linear trend in the cointegration relation rather than in the error correction model itself.

Table A.10: *Johansen's LR test of the hypothesis that the space of cointegrating vectors is spanned by the column of a* 2 × 1 *matrix H* (*linear trend present*)

H^T:	test stat.	conclusions 10%	conclusions 5%
(1,-0.40)	18.84	reject	reject
(1,-0.50)	16.32	reject	reject
(1,-0.60)	9.71	reject	reject
(1,-0.65)	3.86	reject	reject
(1,-0.70)	0.00	accept	accept
(1,-0.75)	3.78	reject	accept
(1,-0.80)	10.41	reject	reject
(1,-0.90)	17.65	reject	reject
(1,-1.00)	20.06	reject	reject

Additional reference:

Newey,W.K. and K.D.West (1987), "A Simple Positive Definite Heteroskedasticity and Autocorrelation Consistent Covariance Matrix", *Econometrica* **55**, 703-708.