

## CORRIGENDUM

*Correction to "Integrated Conditional Moment Tests for Parametric Conditional Distributions"*

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## 1 Introduction

Lemma 4 in our paper Bierens and Wang (2012) claims that, with  $Z(\beta)$  a zero mean complex-valued continuous Gaussian process on a compact subset  $\mathbf{B}$  of a Euclidean space and  $\mu$  a probability measure on  $\mathbf{B}$ ,  $\int |Z(\beta)|^2 \mu(d\beta) = \sum_{m=1}^{\infty} \lambda_m e'_m e_m$ , where the  $\lambda_m$ 's are the eigenvalues of the covariance function  $E \left[ Z(\beta_1) \overline{Z(\beta_2)} \right]$  and the  $e_m$ 's are independently  $N_2[0, I_2]$  distributed. However, it follows from Mercer's theorem that  $E \left[ \int |Z(\beta)|^2 \mu(d\beta) \right] = \sum_{m=1}^{\infty} \lambda_m$ , whereas Lemma 4 would imply that  $E \left[ \int |Z(\beta)|^2 \mu(d\beta) \right] = 2 \sum_{m=1}^{\infty} \lambda_m$ . Apart from this obvious and embarrassing error, the flaw in the proof of Lemma 4 that has led us to this erroneous result is the incorrect equation (A.6).

## 2 Lemma 4 revised

The following corrected version of Lemma 4 is closely related to Theorem 3 in Bierens and Ploberger (1997).

**LEMMA 4 (Revised).** *Let  $Z(\beta)$  be a zero-mean complex-valued continuous Gaussian process on a compact subset  $\mathbf{B}$  of a Euclidean space, and let  $\mu$  be a probability measure on  $\mathbf{B}$ . There exists a non-negative sequence  $\omega_m$  satisfying  $\sum_{m=1}^{\infty} \omega_m < \infty$  such that  $\int |Z(\beta)|^2 \mu(d\beta) = \sum_{m=1}^{\infty} \omega_m \varepsilon_m^2$ , where the  $\varepsilon_m$ 's are independent standard normally distributed random variables.*

**Proof.** Let  $\{\lambda_m\}_{m=1}^{\infty}$  be the sequence of eigenvalues of the covariance kernel  $\Gamma(\beta_1, \beta_2) = E \left[ Z(\beta_1) \overline{Z(\beta_2)} \right]$  with corresponding sequence  $\{\psi_m(\beta)\}_{m=1}^{\infty}$  of or-

thonormal eigenfunctions (relative to  $\mu$ ). By the completeness of  $\{\psi_m(\beta)\}_{m=1}^{\infty}$  we can write  $Z(\beta) = \sum_{m=1}^{\infty} g_m \psi_m(\beta)$  a.s.  $\mu$ ,<sup>1</sup> where  $g_m = \int Z(\beta) \overline{\psi_m(\beta)} \mu(d\beta)$ . Consequently

$$\int |Z(\beta)|^2 \mu(d\beta) = \sum_{m=1}^{\infty} |g_m|^2. \quad (1)$$

Since  $Z(\beta)$  is zero-mean Gaussian, the  $g_m$ 's are jointly zero-mean complex-valued normally distributed. Moreover, by Mercer's theorem,

$$\begin{aligned} E[\overline{g_k} g_m] &= \int \int \psi_k(\beta_2) E \left[ \overline{Z(\beta_2)} Z(\beta_1) \right] \overline{\psi_m(\beta_1)} \mu(d\beta_1) \mu(d\beta_2) \\ &= \int \int \psi_k(\beta_2) \Gamma(\beta_1, \beta_2) \overline{\psi_m(\beta_1)} \mu(d\beta_1) \mu(d\beta_2) \\ &= \sum_{j=1}^{\infty} \lambda_j \int \int \psi_k(\beta_2) \psi_j(\beta_1) \overline{\psi_j(\beta_2)} \overline{\psi_m(\beta_1)} \mu(d\beta_1) \mu(d\beta_2) \\ &= \sum_{j=1}^{\infty} \lambda_j \int \psi_k(\beta_2) \overline{\psi_j(\beta_2)} \mu(d\beta_2) \int \psi_j(\beta_1) \overline{\psi_m(\beta_1)} \mu(d\beta_1) \\ &= \sum_{j=1}^{\infty} \lambda_j \mathbf{1}(k=j) \cdot \mathbf{1}(m=j) \\ &= \lambda_m \mathbf{1}(k=m), \end{aligned} \quad (2)$$

where  $\mathbf{1}(\cdot)$  is the indicator function. As is well-known, due to joint normality (2) implies that the sequence  $\{g_m\}_{m=1}^{\infty}$  is independent, and so is the sequence  $G_m = (\text{Re}[g_m], \text{Im}[g_m])'$ .

Each  $G_m$  is bivariate zero mean normally distributed, i.e.,  $G_m \sim N_2[0, \Sigma_m]$ . Using the well-known decomposition  $\Sigma_m = Q_m \Omega_m Q_m'$ , where  $\Omega_m = \text{diag}(\omega_{1,m}, \omega_{2,m})$  is the diagonal matrix of eigenvalues of  $\Sigma_m$  and  $Q_m$  is the orthogonal matrix of the two corresponding eigenvectors, we can write

$$Q_m' G_m = \begin{pmatrix} \sqrt{\omega_{1,m}} e_{1,m} \\ \sqrt{\omega_{2,m}} e_{2,m} \end{pmatrix},$$

where the sequence  $(e_{1,m}, e_{2,m})'$  is i.i.d.  $N_2[0, I_2]$ . Now

$$|g_m|^2 = g_m \overline{g_m} = G_m' G_m = G_m' Q_m Q_m' G_m = \omega_{1,m} e_{1,m}^2 + \omega_{2,m} e_{2,m}^2$$

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<sup>1</sup>I.e.,  $\mu(\{\beta \in \mathbf{B} : Z(\beta) = \sum_{m=1}^{\infty} g_m \psi_m(\beta)\}) = 1$ .

where  $\omega_{1,m} + \omega_{2,m} = \lambda_m$ , and by Mercer's theorem,

$$\sum_{m=1}^{\infty} \omega_{1,m} + \sum_{m=1}^{\infty} \omega_{2,m} = \sum_{m=1}^{\infty} E[g_m \overline{g_m}] = \sum_{m=1}^{\infty} \lambda_m < \infty.$$

Thus, (1) now reads

$$\int |Z(\beta)|^2 \mu(d\beta) = \sum_{m=1}^{\infty} \omega_{1,m} e_{1,m}^2 + \sum_{m=1}^{\infty} \omega_{2,m} e_{2,m}^2.$$

Finally, denoting for  $m \in \mathbb{N}$ ,  $\omega_{2m-1} = \omega_{1,m}$ ,  $\omega_{2m} = \omega_{2,m}$ ,  $\varepsilon_{2m-1} = e_{1,m}$ ,  $\varepsilon_{2m} = e_{2,m}$ , for example, the result of the revised Lemma 4 follows.  $\blacksquare$

Note that the erroneous Lemma 4 was actually a side issue and has no consequences for the other results in the paper, except that the proof of the local power in section 2.6 needs to be adjusted.

However, the question remains whether more can be said about the variance matrices  $\Sigma_m$ . In particular, the question is whether the  $\Sigma_m$ 's have a particular case-independent structure, apart from being variance matrices and satisfying  $\sum_{k=1}^{\infty} \text{trace}[\Sigma_k] < \infty$ . The following example shows that the answer is No! In other words, the revised Lemma 4 is complete.

### 3 An example

For  $\beta \in [0, 1]$ , let

$$\begin{aligned} Z(\beta) &= \sum_{m=1}^{\infty} (U_{1,m} + \mathbf{i}U_{2,m}) \\ &\quad \times (\cos(2m\pi\beta) + \mathbf{i} \sin(2m\pi\beta)) \\ &= \sum_{m=1}^{\infty} (U_{1,m} \cos(2m\pi\beta) - U_{2,m} \sin(2m\pi\beta)) \\ &\quad + \mathbf{i} \sum_{m=1}^{\infty} (U_{1,m} \sin(2m\pi\beta) + U_{2,m} \cos(2m\pi\beta)) \end{aligned}$$

where the sequence  $(U_{1,m}, U_{2,m})'$  is independently  $N_2(0, \Sigma_m)$  distributed, with

$$\lambda_k = \text{trace}[\Sigma_k] > 0, \quad \sum_{m=1}^{\infty} \lambda_m < \infty.$$

The latter condition implies that  $\int_0^1 |Z(\beta)|^2 d\beta < \infty$  a.s.

Clearly,  $Z(\beta)$  is a zero-mean complex valued Gaussian process on  $[0, 1]$ , with covariance function

$$\begin{aligned} \Gamma(\beta_1, \beta_2) &= E \left[ Z(\beta_1) \overline{Z(\beta_2)} \right] \\ &= \sum_{m=1}^{\infty} \lambda_m \cos(2m\pi\beta_1) \cos(2m\pi\beta_2) \\ &\quad + \sum_{m=1}^{\infty} \lambda_m \sin(2m\pi\beta_1) \sin(2m\pi\beta_2) \\ &\quad + \mathbf{i} \cdot \sum_{m=1}^{\infty} \lambda_m \sin(2m\pi\beta_1) \cos(2m\pi\beta_2) \\ &\quad - \mathbf{i} \cdot \sum_{m=1}^{\infty} \lambda_m \cos(2m\pi\beta_1) \sin(2m\pi\beta_2) \end{aligned}$$

The functions  $\sqrt{2} \cos(2m\pi\beta)$ ,  $\sqrt{2} \sin(2k\pi\beta)$ ,  $m, k \in \mathbb{N}$ , together with the constant function 1, are known as the Fourier series on  $[0, 1]$ , which form a complete orthonormal sequence in the Hilbert space  $L^2(0, 1)$ . Therefore the complex functions

$$\psi_k(\beta) = \cos(2k\pi\beta) + \mathbf{i} \cdot \sin(2k\pi\beta), \quad k \in \mathbb{N},$$

together with  $\psi_0(\beta) \equiv 1$ , form a complete orthonormal sequence in the Hilbert space  $L^2_{\mathbb{C}}(0, 1)$  of square integrable complex valued functions on  $(0, 1)$ .

Moreover, it is easy to verify that for  $k \geq 1$  the  $\psi_k(\beta)$ 's are the eigenfunctions of  $\Gamma(\beta_1, \beta_2)$  with corresponding eigenvalues  $\lambda_k$ , whereas  $\psi_0(\beta)$  is the eigenfunction corresponding to the (single) zero eigenvalue.

Recall that in this case  $Z(\beta) = \sum_{m=1}^{\infty} g_m \psi_m(\beta)$  a.s. with respect to the uniform probability measure on  $(0, 1)$ , where now

$$g_m = \int_0^1 Z(\beta) \overline{\psi_m(\beta)} d\beta = U_{1,m} + \mathbf{i} \cdot U_{2,m},$$

hence,  $E[|g_m|^2] = \text{trace}[\Sigma_m] = \lambda_m$ , and

$$G_m = \begin{pmatrix} \text{Re}[g_m] \\ \text{Im}[g_m] \end{pmatrix} = \begin{pmatrix} U_{1,m} \\ U_{2,m} \end{pmatrix} \sim N_2(0, \Sigma_m),$$

as in the revised Lemma 4 above.

Since in this example the only conditions on the  $\Sigma_m$ 's are that they are variance matrices and satisfy  $\sum_{k=1}^{\infty} \text{trace}[\Sigma_k] < \infty$ , the revised Lemma 4 above is indeed complete.

#### *REFERENCES*

- Bierens, H. J. & L. Wang (2012) Integrated conditional moment tests for parametric conditional distributions. *Econometric Theory* 28, 328-362.
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