CORRIGENDUM

Correction to "Integrated Conditional Moment Tests for Parametric Conditional Distributions"

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1 Introduction

Lemma 4 in our paper Bierens and Wang (2012) claims that, with $Z(\beta)$ a zero mean complex-valued continuous Gaussian process on a compact subset $\mathbf{B}$ of a Euclidean space and μ a probability measure on $\mathbf{B}$, $\int |Z(\beta)|^2\mu(d\beta) = \sum_{m=1}^{\infty}\lambda_m e_m' e_m$, where the λ_m's are the eigenvalues of the covariance function $E\left[Z(\beta_1)\overline{Z(\beta_2)}\right]$ and the e_m's are independently $N_2[0, I_2]$ distributed. However, it follows from Mercer's theorem that $E\left[\int |Z(\beta)|^2\mu(d\beta)\right] = \sum_{m=1}^{\infty}\lambda_m$, whereas Lemma 4 would imply that $E\left[\int |Z(\beta)|^2\mu(d\beta)\right] = 2\sum_{m=1}^{\infty}\lambda_m$. Apart from this obvious and embarrassing error, the flaw in the proof of Lemma 4 that has lead us to this erroneous result is the incorrect equation (A.6).

2 Lemma 4 revised

The following corrected version of Lemma 4 is closely related to Theorem 3 in Bierens and Ploberger (1997).

LEMMA 4 (Revised). *Let $Z(\beta)$ be a zero-mean complex-valued continuous Gaussian process on a compact subset $\mathbf{B}$ of a Euclidean space, and let μ be a probability measure on $\mathbf{B}$. There exists a non-negative sequence ω_m satisfying $\sum_{m=1}^{\infty}\omega_m < \infty$ such that $\int |Z(\beta)|^2\mu(d\beta) = \sum_{m=1}^{\infty}\omega_m\varepsilon_m^2$, where the ε_m's are independent standard normally distributed random variables.*

Proof. Let $\{\lambda_m\}_{m=1}^{\infty}$ be the sequence of eigenvalues of the covariance kernel $\Gamma(\beta_1, \beta_2) = E\left[Z(\beta_1)\overline{Z(\beta_2)}\right]$ with corresponding sequence $\{\psi_m(\beta)\}_{m=1}^{\infty}$ of orthonormal eigenfunctions (relative to μ). By the completeness of $\{\psi_m(\beta)\}_{m=1}^{\infty}$ we can write $Z(\beta) = \sum_{m=1}^{\infty} g_m \psi_m(\beta)$ a.s. μ,[1] where $g_m = \int Z(\beta)\overline{\psi_m(\beta)}\mu(d\beta)$. Consequently

$$\int |Z(\beta)|^2 \mu(d\beta) = \sum_{m=1}^{\infty} |g_m|^2. \tag{1}$$

Since $Z(\beta)$ is zero-mean Gaussian, the g_m's are jointly zero-mean complex-valued normally distributed. Moreover, by Mercer's theorem,

$$\begin{aligned}
E\left[\overline{g_k} g_m\right] &= \int\int \psi_k(\beta_2) E\left[\overline{Z(\beta_2)} Z(\beta_1)\right] \overline{\psi_m(\beta_1)}\mu(d\beta_1)\mu(d\beta_2) \\
&= \int\int \psi_k(\beta_2)\Gamma(\beta_1,\beta_2)\overline{\psi_m(\beta_1)}\mu(d\beta_1)\mu(d\beta_2) \\
&= \sum_{j=1}^{\infty} \lambda_j \int\int \psi_k(\beta_2)\psi_j(\beta_1)\overline{\psi_j(\beta_2)}.\overline{\psi_m(\beta_1)}\mu(d\beta_1)\mu(d\beta_2) \\
&= \sum_{j=1}^{\infty} \lambda_j \int \psi_k(\beta_2)\overline{\psi_j(\beta_2)}\mu(d\beta_2) \int \psi_j(\beta_1)\overline{\psi_m(\beta_1)}\mu(d\beta_1) \\
&= \sum_{j=1}^{\infty} \lambda_j \mathbf{1}(k=j).\mathbf{1}(m=j) \\
&= \lambda_m \mathbf{1}(k=m),
\end{aligned} \tag{2}$$

where $\mathbf{1}(.)$ is the indicator function. As is well-known, due to joint normality, (2) implies that the sequence $\{g_m\}_{m=1}^{\infty}$ is independent, and so is the sequence $G_m = (\mathrm{Re}[g_m], \mathrm{Im}[g_m])'$.[2]

Each G_m is bivariate zero mean normally distributed, i.e., $G_m \sim N_2[0, \Sigma_m]$. Using the well-known decomposition $\Sigma_m = Q_m \Omega_m Q_m'$, where $\Omega_m = \mathrm{diag}(\omega_{1,m}, \omega_{2,m})$ is the diagonal matrix of eigenvalues of Σ_m, and Q_m is the orthogonal matrix of the two corresponding eigenvectors, we can write

$$Q_m' G_m = \begin{pmatrix} \sqrt{\omega_{1,m}} e_{1,m} \\ \sqrt{\omega_{2,m}} e_{2,m} \end{pmatrix},$$

where the sequence $(e_{1,m}, e_{2,m})'$ is i.i.d. $N_2[0, I_2]$. Now

$$|g_m|^2 = g_m \overline{g_m} = G_m' G_m = G_m' Q_m Q_m' G_m = \omega_{1,m} e_{1,m}^2 + \omega_{2,m} e_{2,m}^2$$

[1] I.e., $\mu\left(\{\beta \in \mathbf{B} : Z(\beta) = \sum_{m=1}^{\infty} g_m \psi_m(\beta)\}\right) = 1$.

[2] See Bierens (2017, pp. 410-412) for a formal proof of the latter.

where $\omega_{1,m} + \omega_{2,m} = \lambda_m$, and by Mercer's theorem,

$$\sum_{m=1}^{\infty} \omega_{1,m} + \sum_{m=1}^{\infty} \omega_{2,m} = \sum_{m=1}^{\infty} E[g_m \overline{g_m}] = \sum_{m=1}^{\infty} \lambda_m < \infty.$$

Thus, (1) now reads

$$\int |Z(\beta)|^2 \mu(d\beta) = \sum_{m=1}^{\infty} \omega_{1,m} e_{1,m}^2 + \sum_{m=1}^{\infty} \omega_{2,m} e_{2,m}^2.$$

Finally, denoting for $m \in \mathbb{N}$, $\omega_{2m-1} = \omega_{1,m}$, $\omega_{2m} = \omega_{2,m}$, $\varepsilon_{2m-1} = e_{1,m}$, $\varepsilon_{2m} = e_{2,m}$, for example, the result of the revised Lemma 4 follows. ■

Note that the erroneous Lemma 4 was actually a side issue and has no consequences for the other results in the paper, except that the proof of the local power in section 2.6 needs to be adjusted.

However, the question remains whether more can be said about the variance matrices Σ_m. In particular, the question is whether the Σ_m's have a particular case-independent structure, apart from being variance matrices and satisfying $\sum_{k=1}^{\infty} \text{trace}[\Sigma_k] < \infty$. The following example shows that the answer is No! In other words, the revised Lemma 4 is complete.

3 An example

For $\beta \in [0, 1]$, let

$$\begin{aligned} Z(\beta) &= \sum_{m=1}^{\infty} (U_{1,m} + \mathbf{i}.U_{2,m}) \\ &\quad \times (\cos(2m\pi\beta) + \mathbf{i}.\sin(2m\pi\beta)) \\ &= \sum_{m=1}^{\infty} (U_{1,m}\cos(2m\pi\beta) - U_{2,m}\sin(2m\pi\beta)) \\ &\quad + \mathbf{i}.\sum_{m=1}^{\infty} (U_{1,m}\sin(2m\pi\beta) + U_{2,m}\cos(2m\pi\beta)), \end{aligned}$$

where the sequence $U_m = (U_{1,m}, U_{2,m})'$ is independently $N_2(0, \Sigma_m)$ distributed, with

$$\lambda_k = \text{trace}[\Sigma_k] > 0, \quad \sum_{m=1}^{\infty} \lambda_m < \infty.$$

The latter condition implies that $\int_0^1 |Z(\beta)|^2 d\beta < \infty$ a.s.

Denote

$$q_{1,m}(\beta) = \begin{pmatrix} \cos(2m\pi\beta) \\ -\sin(2m\pi\beta) \end{pmatrix}, \; q_{2,m}(\beta) = \begin{pmatrix} \sin(2m\pi\beta) \\ \cos(2m\pi\beta) \end{pmatrix},$$

so that

$$Z(\beta) = \sum_{m=1}^{\infty} \left(q_{1,m}(\beta)' U_{m.} + \mathbf{i}.q_{2,m}(\beta)' U_{m.} \right)$$

Clearly, $Z(\beta)$ is a zero-mean complex valued Gaussian process on $[0,1]$, with covariance function

$$\begin{aligned}
\Gamma(\beta_1, \beta_2) &= E\left[Z(\beta_1)\overline{Z(\beta_2)} \right] \\
&= \sum_{m=1}^{\infty} \left(q_{1,m}(\beta_1)' + \mathbf{i}.q_{2,m}(\beta_1)' \right) E[U_m U_m'] \\
&\quad \times \left(q_{1,m}(\beta_2) - \mathbf{i}.q_{2,m}(\beta_2) \right) \\
&= \sum_{m=1}^{\infty} \left(q_{1,m}(\beta_1)' + \mathbf{i}.q_{2,m}(\beta_1)' \right) \Sigma_m \left(q_{1,m}(\beta_2) - \mathbf{i}.q_{2,m}(\beta_2) \right) \\
&= \sum_{m=1}^{\infty} \left(q_{1,m}(\beta_1)' \Sigma_m q_{1,m}(\beta_2) + q_{2,m}(\beta_1)' \Sigma_m q_{2,m}(\beta_2) \right) \\
&\quad + \mathbf{i}. \sum_{m=1}^{\infty} \left(q_{2,m}(\beta_1)' \Sigma_m q_{1,m}(\beta_2) - q_{1,m}(\beta_1)' \Sigma_m q_{2,m}(\beta_2) \right).
\end{aligned}$$

To evaluate this expression further, suppose in first instance that

$$\Sigma_k = \lambda_k \begin{pmatrix} \omega & \rho\sqrt{\omega(1-\omega)} \\ \rho\sqrt{\omega(1-\omega)} & 1-\omega \end{pmatrix}.$$

where $\omega \in (0,1)$ and $\rho \in (-1,1)$. Then

$$\begin{aligned}
& q_{1,m}(\beta_1)' \Sigma_m q_{1,m}(\beta_2) \\
&= \lambda_k \Big(\omega \cos(2m\pi\beta_1)\cos(2m\pi\beta_2) - \rho\sqrt{\omega(1-\omega)} \cos(2m\pi\beta_1)\sin(2m\pi\beta_2) \\
&\quad - \rho\sqrt{\omega(1-\omega)} \sin(2m\pi\beta_1)\cos(2m\pi\beta_2) + (1-\omega)\sin(2m\pi\beta_1)\sin(2m\pi\beta_2) \Big)
\end{aligned}$$

and

$$
\begin{aligned}
&q_{2,m}(\beta_1)'\Sigma_m q_{2,m}(\beta_2) \\
&= \lambda_k \Big(\omega \sin(2m\pi\beta_1)\sin(2m\pi\beta_2) + \rho\sqrt{\omega(1-\omega)}\sin(2m\pi\beta_1)\cos(2m\pi\beta_2) \\
&+\rho\sqrt{\omega(1-\omega)}\cos(2m\pi\beta_1)\sin(2m\pi\beta_2) + (1-\omega)\cos(2m\pi\beta_1)\cos(2m\pi\beta_2)\Big),
\end{aligned}
$$

so that

$$
\begin{aligned}
&q_{1,m}(\beta_1)'\Sigma_m q_{1,m}(\beta_2) + q_{2,m}(\beta_1)'\Sigma_m q_{2,m}(\beta_2) \\
&\qquad = \lambda_k \left(\cos(2m\pi\beta_1)\cos(2m\pi\beta_2) + \sin(2m\pi\beta_1)\sin(2m\pi\beta_2)\right).
\end{aligned}
$$

Next, observe that

$$
\begin{aligned}
&q_{2,m}(\beta_1)'\Sigma_m q_{1,m}(\beta_2) \\
&= \lambda_k \Big(\omega \sin(2m\pi\beta_1)\cos(2m\pi\beta_2) - \rho\sqrt{\omega(1-\omega)}\sin(2m\pi\beta_1)\sin(2m\pi\beta_2) \\
&+\rho\sqrt{\omega(1-\omega)}\cos(2m\pi\beta_1)\cos(2m\pi\beta_2) - (1-\omega)\cos(2m\pi\beta_1)\sin(2m\pi\beta_2)\Big),
\end{aligned}
$$

which by swapping β_1 and β_2 yields

$$
\begin{aligned}
&q_{1,m}(\beta_1)'\Sigma_m q_{2,m}(\beta_2) \\
&= \lambda_k \Big(\omega \cos(2m\pi\beta_1)\sin(2m\pi\beta_2) - \rho\sqrt{\omega(1-\omega)}\sin(2m\pi\beta_1)\sin(2m\pi\beta_2) \\
&+\rho\sqrt{\omega(1-\omega)}\cos(2m\pi\beta_1)\cos(2m\pi\beta_2) - (1-\omega)\sin(2m\pi\beta_1)\cos(2m\pi\beta_2)\Big),
\end{aligned}
$$

so that

$$
\begin{aligned}
&q_{2,m}(\beta_1)'\Sigma_m q_{1,m}(\beta_2) - q_{1,m}(\beta_1)'\Sigma_m q_{2,m}(\beta_2) \\
&\qquad = \lambda_k \left(\sin(2m\pi\beta_1)\cos(2m\pi\beta_2) - \cos(2m\pi\beta_1)\sin(2m\pi\beta_2)\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Gamma(\beta_1,\beta_2) &= \sum_{m=1}^{\infty}\lambda_m \cos(2m\pi\beta_1)\cos(2m\pi\beta_2) \\
&\quad + \sum_{m=1}^{\infty}\lambda_m \sin(2m\pi\beta_1)\sin(2m\pi\beta_2) \\
&\quad + \mathbf{i}.\sum_{m=1}^{\infty}\lambda_m \sin(2m\pi\beta_1)\cos(2m\pi\beta_2) \\
&\quad - \mathbf{i}.\sum_{m=1}^{\infty}\lambda_m \cos(2m\pi\beta_1)\sin(2m\pi\beta_2).
\end{aligned}
$$

Obviously, this results carries over to the general case

$$\Sigma_k = \lambda_k \left(\begin{array}{cc} \omega_k & \rho_k \sqrt{\omega_k(1-\omega_k)} \\ \rho\sqrt{\omega(1-\omega)} & 1-\omega_k \end{array} \right),$$

where $\omega_k \in (0,1)$ and $\rho_k \in (-1,1)$ for all $k \in \mathbb{N}$.

The functions $\sqrt{2}\cos(2m\pi\beta)$, $\sqrt{2}\sin(2k\pi\beta)$, $m, k \in \mathbb{N}$, together with the constant function 1, are known as the Fourier series on $[0,1]$, which form a complete orthonormal sequence in the Hilbert space $L^2(0,1)$. Therefore the complex functions

$$\psi_k(\beta) = \cos(2k\pi\beta) + \mathbf{i}.\sin(2k\pi\beta),\ \ k \in \mathbb{N},$$

together with $\psi_0(\beta) \equiv 1$, form a complete orthonormal sequence in the Hilbert space $L^2_{\mathbb{C}}(0,1)$ of square integrable complex valued functions on $(0,1)$.

Moreover, it is easy to verify that for $k \geq 1$ the $\psi_k(\beta)$'s are the eigenfunctions of $\Gamma(\beta_1, \beta_2)$ with corresponding eigenvalues λ_k, whereas $\psi_0(\beta)$ is the eigenfunction corresponding to the (single) zero eigenvalue.

Recall that in this case $Z(\beta) = \sum_{m=1}^{\infty} g_m \psi_m(\beta)$ a.s. with respect to the uniform probability measure on $(0,1)$, where now

$$g_m = \int_0^1 Z(\beta)\overline{\psi_m(\beta)}d\beta = U_{1,m} + \mathbf{i}.U_{2,m},$$

hence, $E\left[|g_m|^2\right] = \text{trace}[\Sigma_m] = \lambda_m$, and

$$G_m = \left(\begin{array}{c} \text{Re}[g_m] \\ \text{Im}[g_m] \end{array} \right) = \left(\begin{array}{c} U_{1,m} \\ U_{2,m} \end{array} \right) \sim N_2(0, \Sigma_m),$$

as in the revised Lemma 4 above.

Since in this example the only conditions on the Σ_m's are that they are variance matrices and satisfy $\sum_{k=1}^{\infty}\text{trace}[\Sigma_k] < \infty$, the revised Lemma 4 above is indeed complete.

REFERENCES

Bierens, H. J. (2017) *Econometric Model Specification*: *Consistent Model Specification Tests and Semi-Nonparametric Modeling and Inference.* World Scientific Publishers.

Bierens, H. J. & L. Wang (2012) Integrated conditional moment tests for parametric conditional distributions. *Econometric Theory* 28, 328-362.

Bierens, H. J. & W. Ploberger (1997) Asymptotic theory of integrated conditional moment tests. *Econometrica* 56, 1129-1151.