The matrix norm $||A|| = \sqrt{\text{trace}\,(AA')}$

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Let A be an $k \times m$ matrix. Define the matrix norm $||.||$ by

$$||A|| = \sqrt{\text{trace}\,(AA')} \tag{1}$$

Recall that a norm needs to satisfy three conditions

$$\begin{aligned} ||A|| &= 0 \text{ if and only if } A = O, & (2)\\ ||c.A|| &= |c|.||A|| \text{ for any scalar } c, & (3)\\ ||A+B|| &\leq ||A|| + ||B|| \text{ (triangular inequality)}, & (4) \end{aligned}$$

where of course in the latter case the matrix B has the same size as A. Conditions (2) and (3) follow trivially from (1), but condition (4) is not obvious and will be shown below. We also have

$$||A|| = ||A'|| \tag{5}$$

because trace(AA') = trace$(A'A)$. Moreover, it will be shown that for conformable matrices A and B,

$$||AB|| \leq ||A||.||B|| \tag{6}$$

and thus for $b \in \mathbb{R}^m$,

$$||Ab|| \leq ||A||.||b||.$$

Proof of (4): To check the triangular inequality, let B be a $k \times m$ matrix. Then

$$||A+B||^2 = \text{trace}\,((A+B)(A'+B'))$$

$$
\begin{aligned}
&= \text{trace}(AA') + \text{trace}(AB') + \text{trace}(BA') + \text{trace}(BB') \\
&= ||A||^2 + 2.\text{trace}(B'A) + ||B||^2 \\
&\leq ||A||^2 + 2.||A||.||B|| + ||B||^2 \\
&= (||A|| + ||B||)^2
\end{aligned}
$$

which implies (4). The inequality follows from Schwarz inequality, applied twice, as follows. Let $a_{i.j}$ and $b_{i,j}$ be the typical elements of A and B, respectively. Then

$$
\begin{aligned}
|\text{trace}(B'A)| &= \left|\sum_{j=1}^{m}\sum_{i=1}^{k} b_{j,i}a_{i,j}\right| \leq \sum_{j=1}^{m} k \left|\frac{1}{k}\sum_{i=1}^{k} b_{j,i}a_{i,j}\right| \\
&\leq \sum_{j=1}^{m} k \sqrt{\frac{1}{k}\sum_{i=1}^{k} b_{j,i}^2}\sqrt{\frac{1}{k}\sum_{i=1}^{k} a_{i,j}^2} = \sum_{j=1}^{m} \sqrt{\sum_{i=1}^{k} b_{j,i}^2}\sqrt{\sum_{i=1}^{k} a_{i,j}^2} \\
&\leq \sqrt{\sum_{j=1}^{m}\sum_{i=1}^{k} b_{j,i}^2}\sqrt{\sum_{j=1}^{m}\sum_{i=1}^{k} a_{i,j}^2} = \sqrt{\text{trace}(BB')}\sqrt{\text{trace}(AA')} \\
&= ||B||.||A||
\end{aligned}
$$

Proof of (6): Let B be an $m \times \ell$ matrix. Then

$$
||AB||^2 = \text{trace}\left(ABB'A'\right) = \text{trace}\left((BB')(A'A)\right)
$$

We can write $BB' = \sum_{i=1}^{m} \lambda_i q_i q_i'$, where the λ_i's are the (nonnegative) eigenvalues of BB' and the q_i's are the corresponding orthonormal eigenvectors. Note that $\sum_{i=1}^{m} q_i q_i' = I_m$ and $\text{trace}(BB') = \sum_{i=1}^{m} \lambda_i \text{trace}(q_i q_i') = \sum_{i=1}^{m} \lambda_i (q_i' q_i) = \sum_{i=1}^{m} \lambda_i$. Then

$$
\begin{aligned}
||AB||^2 &= \text{trace}\left((BB')(A'A)\right) \\
&= \sum_{i=1}^{m} \lambda_i \text{trace}(q_i q_i' A'A) = \sum_{i=1}^{m} \lambda_i q_i' A'A q_i \\
&\leq \sum_{i=1}^{m} \lambda_i \sum_{i=1}^{m} q_i' A'A q_i \\
&= \text{trace}(BB') \sum_{i=1}^{m} \text{trace}(A'A q_i q_i')
\end{aligned}
$$

$$
\begin{aligned}
&= \text{trace}(BB').\text{trace}\left(A'A\sum_{i=1}^{m} q_i q_i'\right) \\
&= \text{trace}(BB').\text{trace}\left(A'A\right) \\
&= \text{trace}(BB').\text{trace}\left(AA'\right) = ||B||^2.||A||^2
\end{aligned}
$$