TESTS OF NORMALITY OF REGRESSION ERRORS

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In this lecture note I will derive the Jarque-Bera (1980) and-Kiefer- Salmon (1983) tests of the normality of the regression errors. Some of the steps in the derivations follow from standard results, in particular Chapters 5 and 6 in Bierens (2004) and are therefore left as exercises, where the readers have to figure out for themselves which results in Bierens (2004) are applicable.

Consider the linear regression with an intercept

$$y_j = \theta^T x_j + u_j, \quad x_j^T = (1, x_{2,j}, \ldots, x_{k,j}), \quad j = 1,..,n, \tag{1}$$

where

ASSUMPTION 1: *The vectors* $(u_j, x_{2,j}, \ldots, x_{k,j})^T$ *are independent random drawings from a k-variate distribution, and* $E(u_j | x_{2,j}, \ldots, x_{k,j}) = 0$ *with probability* 1.

We want to test the null hypothesis that (conditionally on the *x* variables),

$$H_0\text{: } u_j | x_j \sim N(0, \sigma^2). \tag{2}$$

Under the null hypotheses we have:

$$E(u_j^3) = 0, \quad E(u_j^4) = 3\sigma^4, \tag{3}$$

and it will be more feasible to test the latter implication of the null hypothesis than the null hypothesis (2) itself.

If the errors u_j would be observable and the variance σ^2 known, we could test the hypothesis (3) on the basis of the random vector

$$\frac{1}{\sqrt{n}}\begin{pmatrix} \sum_{j=1}^{n} (u_j/\sigma)^3 \\ \sum_{j=1}^{n} \left((u_j/\sigma)^4 - 3\right) \end{pmatrix}, \quad (4)$$

which under the null hypothesis converges in distribution to the bivariate normal distribution with zero mean vector and variance matrix

$$\begin{pmatrix} E(u_1/\sigma)^6 & E(u_1/\sigma)^7 - 3E(u_1/\sigma)^3 \\ E(u_1/\sigma)^7 - 3E(u_1/\sigma)^3 & E(u_1/\sigma)^8 - 6E(u_1/\sigma)^4 + 9 \end{pmatrix} = \begin{pmatrix} 15 & 0 \\ 0 & 96 \end{pmatrix}, \quad (5)$$

where the equality involved follows from the fact that under the hypothesis (2),

$$\begin{aligned} E(u_j^{2m-1}) &= 0 \quad for\ m = 1,2,3,... \\ E(u_j^2) &= \sigma^2 \\ E(u_j^4) &= 3\sigma^4 \\ E(u_j^6) &= 15\sigma^6 \\ E(u_j^8) &= 105\sigma^8 \end{aligned} \quad (6)$$

The results in (6) can be derived using the moment generating function of the standard normal distribution. (*Exercise*: Try it!) Thus under the null hypothesis (2) we have

$$\frac{1}{\sqrt{n}}\begin{pmatrix} \sum_{j=1}^{n} (u_j/\sigma)^3 \\ \sum_{j=1}^{n} \left((u_j/\sigma)^4 - 3\right) \end{pmatrix} \to N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 15 & 0 \\ 0 & 96 \end{pmatrix}\right] \ in\ distr. \quad (7)$$

and consequently,

$$\frac{\left(\sum_{j=1}^{n} (u_j/\sigma)^3\right)^2}{15n} + \frac{\left(\sum_{j=1}^{n} \left((u_j/\sigma)^4 - 3\right)\right)^2}{96n} \to \chi_2^2 \ in\ distr. \quad (8)$$

(*Exercise*: Why?) However, we do not observe the u_j's and the variance σ^2, so that this test is not feasible.

The idea behind the Jarque-Bera (1980) and Kiefer-Salmon (1983) tests is to replace in (7) the u_j's by the least squares residuals

$$\hat{u}_j = y_j - \hat{\theta}^T x_j = u_j - (\hat{\theta}-\theta)^T x_j, \qquad \textbf{(9)}$$

where $\hat{\theta}$ is the OLS estimator of θ, and σ^2 by the estimator

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n} \hat{u}_j^2, \qquad \textbf{(10)}$$

and to prove that the asymptotic normality result (7) goes through, although with a different variance matrix.

Note that, due to the presence of an intercept,

$$\sum_{j=1}^{n} \hat{u}_j = 0. \qquad \textbf{(11)}$$

(*Exercise*: Why?). Therefore, we can also write

$$\hat{u}_j = u_j - \bar{u} - (\hat{\theta}-\theta)^T (x_j - \bar{x}), \qquad \textbf{(12)}$$

where $\bar{u} = (1/n)\Sigma_{j=1}^{n} u_j$ and $\bar{x} = (1/n)\Sigma_{j=1}^{n} x_j$. This will be convenient in proving Lemmas 2 and 3 below.

Next assume:

ASSUMPTION 2: *$Q = E(x_j x_j^T)$ is nonsingular, and $E(\|x_j\|^4) \le \infty$,*

where $\|x\| = \sqrt{x^T x}$ is the Euclidean norm. Then:

LEMMA 1: *Under Assumptions 1-2 and the null hypothesis* (2) *we have*

$$\text{plim}_{n\to\infty}[(1/\sqrt{n})\Sigma_{j=1}^{n}\hat{u}_j^2 - (1/\sqrt{n})\Sigma_{j=1}^{n} u_j^2] = 0.$$

Proof: It follows from (9) that

$$\sum_{j=1}^{n}\hat{u}_j^2 = \sum_{j=1}^{n}\left(u_j - (\hat{\theta}-\theta)^T x_j\right)^2 = \sum_{j=1}^{n}u_j^2 - 2(\hat{\theta}-\theta)^T\sum_{j=1}^{n}u_j x_j + (\hat{\theta}-\theta)^T\sum_{j=1}^{n}x_j x_j^T(\hat{\theta}-\theta)$$

$$= \sum_{j=1}^{n}u_j^2 - \sqrt{n}(\hat{\theta}-\theta)^T\left(\frac{1}{n}\sum_{j=1}^{n}x_j x_j^T\right)\sqrt{n}(\hat{\theta}-\theta). \tag{13}$$

(*Exercise*: Prove the last step). Moreover, under the conditions of the lemma we have

$$\sqrt{n}(\hat{\theta}-\theta) \to N_k(0,\sigma^2 Q^{-1}) \ \ in \ \ distr. \tag{14}$$

and by the law of large numbers,

$$\text{plim}_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}x_j x_j^T = Q. \tag{15}$$

(*Exercise*: Verify the conditions of the law of large numbers). These two results imply that

$$\sum_{j=1}^{n}u_j^2 - \sum_{j=1}^{n}\hat{u}_j^2 = \sqrt{n}(\hat{\theta}-\theta)^T\left(\frac{1}{n}\sum_{j=1}^{n}x_j x_j^T\right)\sqrt{n}(\hat{\theta}-\theta) \to \chi_k^2 \ \ in \ \ distr. \tag{16}$$

(*Exercise*: Which result in Bierens (2004) has been applied?). The lemma follows now from (16) (*Exercise*: Why?).

LEMMA 2: *Under Assumptions 1-2 and the null hypothesis* (2) *we have*

$$\text{plim}_{n\to\infty}\left[(1/\sqrt{n})\Sigma_{j=1}^{n}\hat{u}_j^3 - (1/\sqrt{n})\Sigma_{j=1}^{n}(u_j^3-3\sigma^2 u_j)\right] = 0.$$

Proof: Using (12) and the well-known equality $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$, we have

$$\sum_{j=1}^{n}\hat{u}_j^3 = \sum_{j=1}^{n}\left(u_j-\bar{u} - (\hat{\theta}-\theta)^T(x_j-\bar{x})\right)^3 = \sum_{j=1}^{n}(u_j-\bar{u})^3 - 3(\hat{\theta}-\theta)^T\sum_{j=1}^{n}(u_j-\bar{u})^2(x_j-\bar{x})$$

$$+ 3(\hat{\theta}-\theta)^T\sum_{j=1}^{n}(u_j-\bar{u})(x_j-\bar{x})(x_j-\bar{x})^T(\hat{\theta}-\theta) - \sum_{j=1}^{n}\left((\hat{\theta}-\theta)^T(x_j-\bar{x})\right)^3 \tag{17}$$

Since under the conditions of the lemma,

$$\text{plim}_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}(u_j-\bar{u})^2(x_j-\bar{x}) = 0 \tag{18}$$

and

$$\text{plim}_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}(u_j-\bar{u})(x_j-\bar{x})(x_j-\bar{x})^T = O \tag{19}$$

(*exercise*: why?), it follow that

$$\text{plim}_{n\to\infty}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\hat{u}_j^3 - \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(u_j-\bar{u}\right)^3\right) = 0. \qquad \textbf{(20)}$$

Next, we have

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(u_j-\bar{u})^3 &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}u_j^3 - 3\sqrt{n}\bar{u}\frac{1}{n}\sum_{j=1}^{n}u_j^2 + 2\sqrt{n}\bar{u}^3 \\ &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(u_j^3 - 3\sigma^2 u_j\right) - 3\sqrt{n}\bar{u}\frac{1}{n}\sum_{j=1}^{n}\left(u_j^2-\sigma^2\right) + 2\sqrt{n}\bar{u}^3,\end{aligned} \qquad \textbf{(21)}$$

hence

$$\text{plim}_{n\to\infty}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(u_j-\bar{u}\right)^3 - \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(u_j^3-3\sigma^2 u_j\right)\right) = 0. \qquad \textbf{(22)}$$

(*Exercise*: Why?). Combining (20) and (22), the lemma follows.

LEMMA 3: *Under Assumptions 1-2 and the null hypothesis* (2) *we have*

$$\text{plim}_{n\to\infty}\left((1/\sqrt{n})\Sigma_{j=1}^{n}\hat{u}_j^4 - (1/\sqrt{n})\Sigma_{j=1}^{n}u_j^4\right) = 0.$$

Proof: Again using (12), and the well-known equality $(a\text{-}b)^4 = a^4\text{-}4a^3b\text{+}6a^2b^2\text{-}4ab^3\text{+}b^4$, we have

$$\begin{aligned}\sum_{j=1}^{n}\hat{u}_j^4 &= \sum_{j=1}^{n}\left(u_j-\bar{u} - (\hat{\theta}-\theta)^T(x_j-\bar{x})\right)^4 \\ &= \sum_{j=1}^{n}(u_j-\bar{u})^4 - 4(\hat{\theta}-\theta)^T\sum_{j=1}^{n}(u_j-\bar{u})^3(x_j-\bar{x}) \\ &\quad + 6(\hat{\theta}-\theta)^T\sum_{j=1}^{n}(u_j-\bar{u})^2(x_j-\bar{x})(x_j-\bar{x})^T(\hat{\theta}-\theta) \\ &\quad -4\sum_{j=1}^{n}(u_j-\bar{u})\left((\hat{\theta}-\theta)^T(x_j-\bar{x})\right)^3 + \sum_{j=1}^{n}\left((\hat{\theta}-\theta)^T(x_j-\bar{x})\right)^4\end{aligned} \qquad \textbf{(23)}$$

Therefore, similar to the proof of Lemmas 2 we have:

$$\text{plim}_{n\to\infty}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\hat{u}_j^4 - \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(u_j-\bar{u}\right)^4\right) = 0. \quad \textbf{(24)}$$

(*Exercise*: Prove this). Moreover,

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(u_j-\bar{u})^4 = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}u_j^4 - 4\sqrt{n}\bar{u}\frac{1}{n}\sum_{j=1}^{n}u_j^3 + 6\frac{1}{\sqrt{n}}\left(\sqrt{n}\bar{u}\right)^2\frac{1}{n}\sum_{j=1}^{n}u_j^2 - 3\frac{1}{n\sqrt{n}}\left(\sqrt{n}\bar{u}\right)^4. \quad \textbf{(25)}$$

Since $\sqrt{n}\bar{u} \sim N(0,\sigma^2)$, $\text{plim}_{n\to\infty}(1/n)\Sigma_{j=1}^{n}u_j^3 = 0$ and $\text{plim}_{n\to\infty}(1/n)\Sigma_{j=1}^{n}u_j^2 = \sigma^2$, it follows now that

$$\text{plim}_{n\to\infty}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(u_j-\bar{u}\right)^4 - \frac{1}{\sqrt{n}}\sum_{j=1}^{n}u_j^4\right) = 0. \quad \textbf{(26)}$$

(*Exercise*: Why?). Combining (24) and (26), Lemma 3 follows.

LEMMA 4: *Under Assumptions 1-2 and the null hypothesis* (2) *we have*

$$\text{plim}_{n\to\infty}\left[(1/\sqrt{n})\Sigma_{j=1}^{n}(\hat{u}_j^4-3\hat{\sigma}^4) - (1/\sqrt{n})\Sigma_{j=1}^{n}(u_j^4 - 6\sigma^2u_j^2 + 3\sigma^4)\right] = 0.$$

Proof: It follows from (10), Lemma 1, and the law of large numbers and the central limit theorem that

$$\text{plim}_{n\to\infty}\hat{\sigma}^2 = \sigma^2, \quad \sqrt{n}\left(\hat{\sigma}^2-\sigma^2\right) \to N(0,2\sigma^4) \ \text{in distr}, \quad \textbf{(27)}$$

(*Exercise*: Why?) and thus

$$\text{plim}_{n\to\infty}\left(\sqrt{n}(\hat{\sigma}^4-\sigma^4) - 2\sigma^2\frac{1}{\sqrt{n}}\sum_{j=1}^{n}(u_j^2-\sigma^2)\right) = 0 \quad \textbf{(28)}$$

(*Exercise*: Why?). Combining the latter result with Lemma 3, Lemma 4 follows.

Finally, it follows from (6) that

$$E\begin{pmatrix} u_j^3-3\sigma^2u_j \\ u_j^4-6\sigma^2u_j^2+3\sigma^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad Var\begin{pmatrix} u_j^3-3\sigma^2u_j \\ u_j^4-6\sigma^2u_j+3\sigma^4 \end{pmatrix} = \begin{pmatrix} 6\sigma^6 & 0 \\ 0 & 24\sigma^8 \end{pmatrix} \quad \textbf{(29)}$$

hence, using the central limit theorem, it follows that

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\begin{pmatrix} u_j^3 - 3\sigma^2 u_j \\ u_j^4 - 6\sigma^2 u_j + 3\sigma^4 \end{pmatrix} \to N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6\sigma^6 & 0 \\ 0 & 24\sigma^8 \end{pmatrix}\right] \text{ in distr.} \quad \textbf{(30)}$$

Using this result and those of Lemmas 1-4, it follows that

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\begin{pmatrix} (\hat{u}_j/\hat{\sigma})^3 \\ (\hat{u}_j/\hat{\sigma})^4 - 3 \end{pmatrix} \to N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 24 \end{pmatrix}\right) \text{ in distr.} \quad \textbf{(31)}$$

hence

THEOREM 1: *Under Assumptions 1-2 and the null hypothesis* (2) *we have*

$$\hat{T}_N = n\left(\frac{[(1/n)\Sigma_{j=1}^{n}(\hat{u}_j/\hat{\sigma})^3]^2}{6} + \frac{[(1/n)\Sigma_{j=1}^{n}(\hat{u}_j/\hat{\sigma})^4 - 3)]^2}{24}\right) \to \chi_2^2 \text{ in distr.} \quad \textbf{(32)}$$

Proof: *Exercise*.

The statistic $\hat{T}_N$ is the test statistic of the Kiefer-Salmon (1983) test. The Jarque-Bera (1980) test is derived in a slightly different way, but is essentially the same.

REFERENCES

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