The Inverse of a Partitioned Matrix

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Consider a pair $A$, $B$ of $n \times n$ matrices, partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where $A_{11}$ and $B_{11}$ are $k \times k$ matrices. Suppose that $A$ is nonsingular and $B = A^{-1}$. In this note it will be shown how to derive the $B_{ij}$'s in terms of the $A_{ij}$'s, given that

$$\det(A_{11}) \neq 0 \text{ and } \det(A_{22}) \neq 0. \quad (1)$$

If $B = A^{-1}$ then

$$AB = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} I_k & O_{k,n-k} \\ O_{n-k,k} & I_{n-k} \end{pmatrix},$$

where as usual $I$ denotes the unit matrix and $O$ a zero matrix, with sizes indicated by the subscripts involved.

To solve (2), we need to solve four matrix equations:

$$A_{11}B_{11} + A_{12}B_{21} = I_k \quad (3)$$
$$A_{11}B_{12} + A_{12}B_{22} = O_{k,n-k} \quad (4)$$
$$A_{21}B_{11} + A_{22}B_{21} = O_{n-k,k} \quad (5)$$
$$A_{21}B_{12} + A_{22}B_{22} = I_{n-k} \quad (6)$$
It follows from (4) and (5) that

\[
B_{12} = -A_{11}^{-1}A_{12}B_{22}, \quad (7)
\]
\[
B_{21} = -A_{22}^{-1}A_{21}B_{11}, \quad (8)
\]

so that (3) and (6) become

\[
\begin{align*}
(A_{11} - A_{12}A_{22}^{-1}A_{21})B_{11} & = I_k \\
(A_{22} - A_{21}A_{11}^{-1}A_{12})B_{22} & = I_{n-k}
\end{align*}
\]

Hence

\[
B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}
\]
\[
B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}
\]

Substituting these solutions in (7) and (8) it follows that

\[
\begin{align*}
B_{12} & = -A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \\
B_{21} & = -A_{22}^{-1}A_{21}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}
\end{align*}
\]

Thus,

\[
A^{-1} = \begin{pmatrix}
(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \\
-A_{22}^{-1}A_{21}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}
\end{pmatrix}
\]

Moreover, since \(A.A^{-1} = I_n\) implies \(A^{-1}A = I_n\), we also have

\[
A^{-1} = \begin{pmatrix}
(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}A_{12}A_{22}^{-1} \\
-\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}
\end{pmatrix}
\]