The Inverse of a Partitioned Matrix

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Consider a pair A, B of $n \times n$ matrices, partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \; B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where A_{11} and B_{11} are $k \times k$ matrices. Suppose that A is nonsingular and $B = A^{-1}$. In this note it will be shown how to derive the B_{ij}'s in terms of the A_{ij}'s, given that

$$\det(A_{11}) \neq 0 \text{ and } \det(A_{22}) \neq 0. \tag{1}$$

If $B = A^{-1}$ then

$$\begin{aligned} AB &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} \\ &= \begin{pmatrix} I_k & O_{k,n-k} \\ O_{n-k,k} & I_{n-k} \end{pmatrix}, \end{aligned} \tag{2}$$

where as usual I denotes the unit matrix and O a zero matrix, with sizes indicated by the subscripts involved.

To solve (2), we need to solve four matrix equations:

$$A_{11}B_{11} + A_{12}B_{21} = I_k \tag{3}$$
$$A_{11}B_{12} + A_{12}B_{22} = O_{k,n-k} \tag{4}$$
$$A_{21}B_{11} + A_{22}B_{21} = O_{n-k,k} \tag{5}$$
$$A_{21}B_{12} + A_{22}B_{22} = I_{n-k} \tag{6}$$

It follows from (4) and (5) that

$$B_{12} = -A_{11}^{-1} A_{12} B_{22}, \tag{7}$$
$$B_{21} = -A_{22}^{-1} A_{21} B_{11}, \tag{8}$$

so that (3) and (6) become

$$\begin{aligned} \left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right) B_{11} &= I_k \\ \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right) B_{22} &= I_{n-k} \end{aligned}$$

Hence

$$\begin{aligned} B_{11} &= \left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} \\ B_{22} &= \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} \end{aligned}$$

Substituting these solutions in (7) and (8) it follows that

$$\begin{aligned} B_{12} &= -A_{11}^{-1} A_{12} \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} \\ B_{21} &= -A_{22}^{-1} A_{21} \left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} \end{aligned}$$

Thus,

$$A^{-1} = \begin{pmatrix} \left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} & -A_{11}^{-1} A_{12} \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} \\ -A_{22}^{-1} A_{21} \left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} & \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} \end{pmatrix}$$

Moreover, since $A.A^{-1} = I_n$ implies $A^{-1} A = I_n$, we also have

$$A^{-1} = \begin{pmatrix} \left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} & -\left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} A_{12} A_{22}^{-1} \\ -\left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} A_{21} A_{11}^{-1} & \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} \end{pmatrix}$$