SPURIOUS REGRESSION

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Consider two independent unit root processes, $\Delta y_t = u_t$ and $\Delta x_t = v_t$, where the u_t's and the v_t's are independent, say:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} \sim \; i.i.d. \; N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right). \tag{1}$$

Then y_t, with all leads and lags, is independent of x_t, with all leads and lags. If we regress y_t on x_t for $t = 1,..,n$, one would therefore expect the slope coefficient to be insignificant, but as we show now, that is not true. First, consider the OLS regression of y_t on x_t without an intercept. The OLS estimate of the slope is:

$$\begin{aligned}\hat{\gamma}_0 &= \frac{\sum_{t=1}^n x_t y_t}{\sum_{t=1}^n x_t^2} = \frac{(1/n)\sum_{t=1}^n \left(x_0/\sqrt{n} + (1/\sqrt{n})\sum_{i=1}^t v_i\right)\left(y_0/\sqrt{n} + (1/\sqrt{n})\sum_{j=1}^t u_j\right)}{(1/n)\sum_{t=1}^n \left(x_0/\sqrt{n} + (1/\sqrt{n})\sum_{i=1}^t v_i\right)^2} \\ &= \frac{(1/n)\sum_{t=1}^n \left(x_0/\sqrt{n} + W_{x,n}(t/n)\right)\left(y_0/\sqrt{n} + W_{y,n}(t/n)\right)}{(1/n)\sum_{t=1} \left(x_0/\sqrt{n} + W_{x,n}(t/n)\right)^2},\end{aligned} \tag{2}$$

where

$$W_{x,n}(r) = (1/\sqrt{n})\sum_{j=1}^{[nr]} v_j \;\; if \;\; r \in [n^{-1},1], \quad W_{x,n}(r) = 0 \;\; if \;\; r \in [0,n^{-1}), \tag{3}$$

$$W_{y,n}(r) = (1/\sqrt{n})\sum_{j=1}^{[nr]} u_j \;\; if \;\; r \in [n^{-1},1], \quad W_{y,n}(r) = 0 \;\; if \;\; r \in [0,n^{-1}), \tag{4}$$

with $[rn]$ the largest natural number $\leq rn$. Since these functions are step functions, and both $W_{x,n}(1)$ and $W_{y,n}(1)$ are standard normally distributed, hence $W_{x,n}(1) = O_p(1)$ and $W_{y,n}(1) = O_p(1)$, we have

$$(1/n)\sum_{t=1}^{n} W_{x,n}(t/n)^2 \;=\; (1/n)\sum_{t=0}^{n-1}\int_{t}^{t+1} W_{x,n}(z/n)^2 dz \;+\; W_{x,n}(1)^2/n$$

$$=\; (1/n)\int_{0}^{n} W_{x,n}(z/n)^2 dz \;+\; W_{x,n}(1)^2/n \;=\; \int W_{x,n}(r)^2 dr \;+\; O_p(1/n)\,, \tag{5}$$

and similarly

$$(1/n)\sum_{t=1}^{n} W_{y,n}(t/n)^2 \;=\; \int W_{y,n}(r)^2 dr \;+\; O_p(1/n)\,, \tag{6}$$

$$(1/n)\sum_{t=1}^{n} W_{x,n}(t/n)W_{y,n}(t/n) \;=\; \int W_{x,n}(r)W_{y,n}(r)dr \;+\; O_p(1/n)\,. \tag{7}$$

The integrals in (5) through (7), and in the sequel, are taken over the unit interval [0,1], unless otherwise indicated.

It can be shown that the integrals in (5) through (7) converge jointly in distribution, due to the functional central limit theorem and the continuous mapping theorem:

LEMMA 1. $\left(\int W_{x,n}(r)^2 dr, \int W_{y,n}(r)^2 dr, \int W_{x,n}(r)W_{y,n}(r)dr\right)^T$ *converges in distribution to* $\left(\int W_x(r)^2 dr,\ \int W_y(r)^2 dr,\ \int W_x(r)W_y dr\right)^T$, *where* $W_x(r)$ *and* $W_y(r)$ *are independent standard Wiener processes*.

For the proof of Lemma 1, see Billingsley (1968)[1].

Using Lemma 1, we now have

$$\hat{\gamma}_0 \;=\; \frac{\int W_{y,n}(r)W_{x,n}(r)dr}{\int W_{x,n}(r)^2 dr} \;+\; o_p(1) \;\rightarrow\; \gamma_0 \;=\; \frac{\int W_y(r)W_x(r)dr}{\int W_x(r)^2 dr} \tag{8}$$

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[1] Billingsley, Patric: *Convergence of Probability Measures*, John Wiley, New York, 1968

in distribution. Note that the limiting random variable γ_0 is continuously distributed, and in particular, $P(\gamma_0 = 0) = 0$.

Along the same lines, and using (8), it follows that the residual sum of squares, RSS_0, of the regression involved, divided by n^2, satisfies

$$\begin{aligned} \frac{RSS_0}{n^2} &= \frac{1}{n^2}\sum_{t=1}^{n} y_t^2 - \hat{\gamma}_0^2 \frac{1}{n^2}\sum_{t=1}^{n} x_t^2 = \int W_{y,n}(r)^2 dr - \hat{\gamma}_0^2 \int W_{x,n}(r)^2 dr + o_p(1) \\ &\to \int W_y(r)^2 dr - \gamma_0^2 \int W_x(r)^2 dr \ \ in\ \ distr. \end{aligned} \tag{9}$$

hence the t-value $\hat{t}_0$, say, of the slope, divided by $\sqrt{n}$, satisfies:

$$\begin{aligned} \frac{\hat{t}_0}{\sqrt{n}} &= \frac{\hat{\gamma}_0 \sqrt{(1/n)\Sigma_{t=1}^{n} x_t^2}}{\sqrt{RRS_0/(n-1)}} = \sqrt{n/(n-1)} \frac{\hat{\gamma}_0 \sqrt{(1/n^2)\Sigma_{t=1}^{n} x_t^2}}{\sqrt{RRS_0/n^2}} \\ &= \frac{\hat{\gamma}_0 \sqrt{\int W_{x,n}(r)^2 dr}}{\sqrt{\int W_{y,n}(r)^2 dr - \hat{\gamma}_0^2 \int W_{x,n}(r)^2 dr}} + o_p(1) \to \frac{\gamma_0 \sqrt{\int W_x(r)^2 dr}}{\sqrt{\int W_y(r)^2 dr - \gamma_0^2 \int W_x(r)^2 dr}} \end{aligned} \tag{10}$$

in distribution. This result, together with $P(\gamma_0 = 0) = 0$, implies that $\text{plim}_{n\to\infty}|\hat{t}_0| = \infty$.

Similar results as in (8) and (10) hold for slope parameter $\hat{\gamma}_1$ and corresponding t-value $\hat{t}_1$ of the regression of y_t on x_t with intercept:

$$\hat{\gamma}_1 \to \gamma_1 = \frac{\int W_y(r)W_x(r)dr - \left(\int W_y(r)dr\right)\left(\int W_x(r)dr\right)}{\int W_x(r)^2 dr - \left(\int W_x(r)dr\right)^2} \tag{11}$$

and

$$\frac{\hat{t}_1}{\sqrt{n}} \to \gamma_1 \frac{\sqrt{\int W_x(r)^2 dr - \left(\int W_x(r)dr\right)^2}}{\sqrt{\int W_y(r)^2 dr - \left(\int W_y(r)dr\right)^2 - \gamma_1^2 \int W_x(r)^2 dr + \gamma_1^2\left(\int W_x(r)dr\right)^2}} \tag{12}$$

in distribution. Moreover, the R^2 of the regression involved satisfies:

$$R^2 \to \gamma_1^2 \frac{\int W_x(r)^2 dr - \left(\int W_x(r) dr\right)^2}{\int W_y(r)^2 dr - \left(\int W_y(r) dr\right)^2} \qquad \textbf{(13)}$$

in distribution

The conclusion from these results is that one should be very cautious when conducting standard econometric analysis using time series. If the time series involved are unit root processes, naive application of regression analysis may yield nonsense results.