Testing the unit root with drift hypothesis against nonlinear trend stationarity, with an application to the U.S. price level and interest rate[1]

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Abstract

This paper is concerned with testing the unit root with drift hypothesis against a very general trend stationarity hypothesis, namely the alternative that the time series is stationary about an almost arbitrary deterministic function of time. Our approach employs the fact that any function of time can be approximated arbitrarily closely by a linear function of Chebishev polynomials. We propose various tests on the basis of an Augmented Dickey-Fuller auxiliary regression with linear and nonlinear deterministic trends, where the nonlinear deterministic trend is approximated by detrended Chebishev time polynomials. Also, we propose a model-free test. We apply our tests to the GNP deflator, the consumer price index, and the interest rate for the USA, taken from the extended Nelson-Plosser data set. The results indicate that these series are nonlinear trend stationary.

Key words: Unit root, nonlinear trend, Chebishev polynomials

JEL Codes: C12, C14, C22

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1. Introduction

Since the seminal paper by Nelson and Plosser (1982), who conducted tests for unit roots in fourteen macroeconomic time series, econometricians have become increasingly aware that quite a few macroeconomic time series of interest have a unit root (also referred to as being integrated of order 1, and denoted by $I(1)$), or at least behave like having a unit root. The presence of a unit root may render standard asymptotic distribution theory inapplicable. See Fuller (1976), Dickey and Fuller (1979, 1981), Evans and Savin (1981, 1984), Said and Dickey (1984), Dickey, Hasza and Fuller (1984), Phillips (1987), Phillips and Perron (1988), Hylleberg and Mizon (1989), Kahn and Ogaki (1990), Bierens (1993), and Haldrup and Hylleberg (1995), among others, for various unit root tests and Schwert (1989) for a Monte Carlo analysis of the power of some of these tests. Also, see the special 1991 issue of the *Journal of Applied Econometrics* (vol. 6) and the references therein on the Bayesian approach to unit root testing.

If a time series is trend stationary rather than $I(1)$, and if the unit root hypothesis is tested against the stationarity hypothesis rather than the trend stationarity hypothesis, the unit root hypothesis may prevail because trend stationary processes and unit root processes may look quite similar. Recently, tests have been developed to test the unit root hypothesis against the linear trend stationarity hypothesis. See Phillips and Perron (1988), Perron (1988), Said (1991), Bierens (1993), and Haldrup (1995), among others. Moreover, Perron (1989, 1990), Perron and Vogelsang (1992) and Zivot and Andrews (1992) have shown that also structural breaks in the mean of a time series process may render the outcomes of unit root tests misleading: before taking structural breaks into account the unit root hypothesis is often accepted, whereas it is rejected when possible structural breaks are accounted for.

In this paper we consider a very general trend stationarity hypothesis as an alternative to the unit root hypothesis, namely that the time series is stationary about an almost arbitrary deterministic function of time. The tests we propose are further elaborations of the unit root tests of Dickey and Fuller (1979, 1981), using Chebishev polynomials to approximate a nonlinear deterministic time trend. Thus, our tests are (somewhat) in the spirit of the unit root test of Ouliaris, Park and Phillips (1989), but differ in that we use Chebishev time polynomials rather than regular time polynomials, a parametric specification of the dynamics rather than using a Newey-West (1987) type long-run variance estimator, and our null hypothesis is the unit root with constant drift hypothesis rather than the unit root with nonlinear trended drift hypothesis. The Chebishev polynomials have substantial advantages over regular time polynomials because they are orthogonal (with a closed form) and bounded. In the Ouliaris, Park and Phillips (1989) approach the highest order time polynomial will dominate, which affects the power of their test if this order is specified too high or too low. Also, we propose a model-free test.

We apply these test to the series for the GNP deflator, the consumer price index (CPI), and the interest rate, taken from the Nelson-Plosser (1982) data set, extended by Schotman and Van Dijk (1991) to 1988. The reason for choosing the two price series is that Perron (1989) found that, after correcting for trend breaks, the unit root hypothesis was rejected for the log of the GNP deflator but not for the log of CPI. Since the plots of these series look very much alike over the last 100 years, why would their data generating processes be so different? The empirical application aims to solve this puzzle. Perron (1989) could also not reject the unit root hypothesis for the interest rate. Therefore we analyze this series as well.

Most of the proofs of our results are given in the appendix.

2. Detrended Chebishev Time Polynomials

Orthogonal Chebishev polynomials $P_{k,n}(t)$ are defined as follows. For $t = 1,..,n,\ k = 1,..,n-1$, let

$$P_{0,n}(t) = 1, \quad P_{k,n}(t) = (\sqrt{2})\cos[k\pi(t-0.5)/n]. \quad \textbf{(1)}$$

Then [cf. Hamming (1973)]

Lemma 1. For $k,m = 0,1,...,n-1$, $(1/n)\sum_{t=1}^{n} P_{k,n}(t)P_{m,n}(t) = I(k=m)$, *where* $I(.)$ *is the indicator function.*

Note that $P_{k,n}(t)$ is a polynomial of order k in $\cos[\pi(t-0.5)/n]$, for

$$\cos(k \cdot a) = \frac{1}{2}\sum_{j=0}^{[k/2]} \binom{j}{k} (-1)^j (1-\cos^2(a))^j (\cos(a))^{k-2j},$$

where here and in the sequel $[x]$ denotes the largest integer $\leq x$.

Owing to the orthogonality property, and because of the fact that Chebishev polynomials have a closed form, these polynomials are particularly suitable for approximating highly nonlinear trend functions. Any function $g(t)$ of time $t = 1,...,n$ can be written as

$$g(t) = \sum_{k=0}^{n-1} \xi_{k,n} P_{k,n}(t) \text{ for } t = 1,\dots,n, \text{ where } \xi_{k,n} = (1/n)\sum_{t=1}^{n} g(t) P_{k,n}(t).$$

If the trend function g is reasonably smooth we can approximate $g(t)$ quite well by a linear combination of a modest number m+1, say, of Chebishev polynomials:

$$g_{m,n}(t) = \sum_{k=0}^{m} \xi_{k,n} P_{k,n}(t) \approx g(t), \; t = 1,\dots,n.$$

Note that this is a special case of a Fourier approximation. However, also a linear trend can be approximated quite well by (1) for relatively small values of m. Therefore, in order to distinguish linear and nonlinear time trends, we shall now transform the Chebishev polynomials such that they become orthogonal to t. For even $k \geq 2$ they already are:

Lemma 2. For $k = 1,2,...,[n/2]-1$, $\Sigma_{t=1}^{n} t P_{2k,n}(t) = 0$.

However, for odd $k \geq 1$ the Chebishev polynomials are correlated with t:

Lemma 3. For k = 1,2,...,[n/2],

$$(1/n)\sum_{t=1}^{n}(t/n)P_{2k-1,n}(t) \;=\; -\frac{\cos[0.5(2k-1)\pi/n]}{n^2\sqrt{2}\sin^2[0.5(2k-1)\pi/n]} \;\rightarrow\; \frac{-2\sqrt{2}}{(2k-1)^2\pi^2}$$

as $n \rightarrow \infty$.

These results suggest the following transformations:

$$P^*_{0,n}(t) \;=\; 1,$$

$$P^*_{1,n}(t) \;=\; \frac{t \;-\; (n+1)/2}{\sqrt{(n^2-1)/12}},$$

$$P^*_{2k,n}(t) \;=\; \frac{P_{2k-1,n}(t)-\alpha_{k,n}-\sum_{j=1}^{k-1}\beta_{k,j,n}P_{2j-1,n}(t)-\gamma_{k,n}(t/n)}{c_{k,n}}, \tag{2}$$

$$P^*_{2k+1,n}(t) \;=\; P_{2k,n}(t).$$

for k = 1,2,...,[n/2], where $\alpha_{k,n}$, $\beta_{k,j,n}$, and $\gamma_{k,n}$ are the least squares coefficients of the regression of $P_{2k-1,n}(t)$ on 1, $P_{2j-1,n}(t)$, j =1,..,$k-1$, and t/n, respectively, and the $c_{k,n}$'s are norming constant such that $(1/n)\Sigma_{t=1}^{n}[P^*_{2k,n}(t)]^2 \;=\; 1$.

Of course, this is not the only way to orthogonalize the Chebishev time polynomials and the time trend. Alternatively, one could also orthogonalize the time trend t w.r.t. the $P_{k,n}(t)$'s, but the above approach has the advantage that we now can distinguish linear trend stationarity from nonlinear trend stationarity.

3. Asymptotic OLS distribution theory of an ADF-type auxiliary regression under the unit root with drift hypothesis

Let z_t be a univariate time series. Consider the null hypothesis

$$H_0: \; z_t = z_{t-1} + \mu + u_t$$

where μ is a constant drift parameter and u_t is a stationary $AR(p)$ process:

Assumption 1. $u_t - \sum_{j=1}^{p}\phi_j u_{t-1} = \phi(L)u_t = \varepsilon_t$, *where* ε_t *is i.i.d. with zero mean, variance* σ_ε^2 *and finite fourth moment, and* $\phi(L)$ *is a p-order lag polynomial with roots all outside the unit circle.*

We shall propose tests of this null hypothesis against the alternative of nonlinear trend stationarity:

$$H_1: \; z_t = g(t) + u_t$$

where $g(t)$ is a possibly nonlinear trend function. Following Dickey and Fuller (1979, 1981), Said and Dickey (1984) and Said (1991), these tests will be based on an Augmented Dickey-Fuller (ADF) type auxiliary regression model

$$\Delta z_t = \alpha z_{t-1} + \sum_{j=1}^{p}\phi_j \Delta z_{t-j} + \theta^T P_{t,n}^{(m)} + \varepsilon_t, \qquad \textbf{(3)}$$

where

$$P_{t,n}^{(m)} = (P_{0,n}^*(t), P_{1,n}^*(t), \ldots., P_{m,n}^*(t))^T.$$

Note that under H_0, $\alpha = 0$ and the last m components of θ are zero.

In order to describe the limiting distribution of the OLS estimates of the parameters in model (3), we need the following notation:

$$X_0 = W(1), \quad X_1 = \sqrt{3}\{W(1) - 2\int_0^1 W(x)dx\},$$

$$X_{2k+1} = \sqrt{2}\{W(1) + 2k\pi\int_0^1 \sin(2k\pi x)W(x)dx\},$$

$$X_{2k} = c_k^{-1}\{-\sqrt{2}W(1) + \sqrt{2}(2k-1)\pi\int_0^1 \sin((2k-1)\pi x)W(x)dx$$

$$-\alpha_k W(1) - \gamma_k W(1) + \gamma_k\int_0^1 W(x)dx$$

$$+ \sum_{j=1}^{k-1} \beta_{k,j}\sqrt{2}[W(1) - (2j-1)\pi\int_0^1 \sin((2j-1)\pi x)W(x)dx]\},$$

$$Y_0 = \int_0^1 W(x)dx, \quad Y_1 = \sqrt{3}\int_0^1 (2x - 1)W(x)dx,$$

$$Y_{2k+1} = \sqrt{2}\int_0^1 \cos(2k\pi x)W(x)dx,$$

$$Y_{2k} = c_k^{-1}\{\sqrt{2}\int_0^1 \cos((2k-1)\pi x)W(x)dx - \alpha_k\int_0^1 W(x)dx - \gamma_k\int_0^1 xW(x)dx$$

$$-\sqrt{2}\sum_{j=1}^{k-1} \beta_{k,j}\int_0^1 \cos((2j-1)\pi x)W(x)dx\},$$

for $k = 1,2,...,[n/2]$, where α_k, $\beta_{k,j}$, γ_k, and c_k are the limits of $\alpha_{k,n}$, $\beta_{k,j,n}$, $\gamma_{k,n}$ and $c_{k,n}$, respectively, and $W()$ is a standard Wiener process. Moreover, denote

$$X^{(m)} = (X_0, X_1, ..., X_m)^T, \quad Y^{(m)} = (Y_0, Y_1, ..., Y_m)^T.$$

Theorem 1. Let $\hat{\alpha}$*,* $\hat{\phi}$ *and* $\hat{\theta}$ *be the OLS estimators of* α*,* ϕ *and* θ*, respectively, in model* (3)*, let* $\hat{\varepsilon}_{m,t}$ *be the OLS residual, and let* $\hat{t}(m)$ *be the t-statistic of* α*. Moreover, let* $\tilde{\phi}$ *and* $\tilde{\theta}$ *be the OLS estimators of* ϕ *and* θ*, respectively, in model* (3) *with* $\alpha = 0$ (*thus the regression* (3) *without the term* z_{t-1}) *, and let* $\tilde{\varepsilon}_{k,t}$ *be the OLS residual. Under* H_0 *the following convergence results hold jointly:*

$$\frac{n\hat{\alpha}}{\hat{\phi}(1)} \Rightarrow \frac{\frac{1}{2} W(1)^2 - 1 - 2Y^{(m)\,T}X^{(m)}}{\int_0^1 W(x)^2 dx - Y^{(m)\,T}Y^{(m)}} = a_m, \text{ say},$$

$$\sqrt{n}(\hat{\phi} - \phi) \Rightarrow \sigma_\varepsilon [E(U_0 U_0^T)]^{-1/2} b_p, \text{ where } b_p \sim N_p(0, I_p),$$

$$\sqrt{n}(\hat{\theta} - \theta) \Rightarrow \sigma_\varepsilon [X^{(m)} - a_m Y^{(m)}],$$

$$\hat{t}(m) \Rightarrow \frac{\frac{1}{2} W(1)^2 - 1 - 2Y^{(m)\,T}X^{(m)})^2}{\sqrt{\int_0^1 W(x)^2 dx - X^{(m)\,T}X^{(m)}}} = t_m, \text{ say},$$

$$\sum_{t=1}^{n} \hat{\varepsilon}_{m,t}^2 - \sum_{t=1}^{n} \varepsilon_t^2 \Rightarrow -\sigma_\varepsilon^2 [t_m^2 + b_p^T b_p + X^{(m)\,T}X^{(m)}].$$

$$\sqrt{n}(\tilde{\phi} - \phi) \Rightarrow \sigma_\varepsilon [E(U_0 U_0^T)]^{-1/2} b_p,$$

$$\sqrt{n}(\tilde{\theta} - \theta_*) \Rightarrow \sigma_\varepsilon X^{(m)}$$

$$\sum_{t=1}^{n} \tilde{\varepsilon}_{k,t}^2 - \sum_{t=1}^{n} \varepsilon_t^2 \Rightarrow -\sigma_\varepsilon^2 \left[b_p^T b_p + X^{(m)\,T}X^{(m)} \right].$$

("⇒" means "converges weakly to". Cf. Billingsley (1968). We shall use this symbol to indicate weak convergence of random functions, as well as convergence in distribution of random variables and vectors).

4. Unit root tests based on an ADF-type auxiliary regression

4.1 The tests and their asymptotic null distribution

Theorem 1 suggests to use either the t-test $\hat{t}(m)$ or the test statistic

$$\hat{A}(m) = \frac{n\hat{\alpha}}{1 - \sum_{j=1}^{p}\hat{\phi}_j},$$

where the $\hat{\phi}_j$'s are the components of $\hat{\phi}$ (or $\hat{\phi}$), for testing H_0 against H_1.

Tables 1 and 2 contain the most important fractiles of the null distribution of a_m and t_m, for m = 1,..,20. These results are derived on the basis of 10,000 replications of a Gaussian random walk with sample size 500.

<Insert Tables 1 and 2 about here>

A disadvantage of these two tests is that they do not take all available information into account, in particular the information that under H_0 the last m components of the parameter vector θ in model (3) are zero. Therefore, we also propose the usual F-test of the joint hypothesis involved:

$$\hat{F}(m) = \frac{\left(\sum_{t=1}^{n}\tilde{\varepsilon}_{0,t}^2 - \sum_{t=1}^{n}\hat{\varepsilon}_{m,t}^2\right)/(m+1)}{s^2},$$

where

$$s^2 = \frac{1}{n-p-m-1}\sum_{t=1}^{n}\hat{\varepsilon}_t^2 .$$

Denoting

$$X^{(i,m)} = \left(X_i, \ldots, X_m\right)^T, \; i = 1,2,..,m,$$

it follows straightforwardly from Theorem 1 that

Theorem 2. Under H_0, $\hat{F}(m) \Rightarrow [t_m^2 + X^{(1,m)\,T}X^{(1,m)}]/(m+1) = F_m$, *say.*

Note that the limiting distribution F_m is *not* an F distribution.

Table 3 contains the most important fractiles of the distribution of F_m, again based on 10,000 replications of a Gaussian random walk with sample size 500.

<Insert Table 3 about here>

An alternative to the F-test is the following χ^2 test. Let

$$Y^{(i,m)} = \left(Y_i, \ldots, Y_m\right)^T, \quad \hat{\theta}^{(i,m)} = \left(\hat{\theta}_i, \ldots, \hat{\theta}_m\right)^T, \; P_{t,n}^{(i,m)} = (P_{i,n}^*(t), \ldots, P_{m,n}^*(t))^T, \quad i = 1, 2, \ldots, m.$$

and denote:

$$\hat{T}_i(m) = n \frac{\left(\hat{\alpha}\frac{1}{n}\sum_{t=1}^{n} z_t P_{t,n}^{(i,m)} + \hat{\theta}^{(i,m)}\right)^T \left(\hat{\alpha}\frac{1}{n}\sum_{t=1}^{n} z_t P_{t,n}^{(i,m)} + \hat{\theta}^{(i,m)}\right)}{s^2}.$$

Then:

Theorem 3. Under H_0, $\hat{T}_i(m) \Rightarrow X^{(i,m)\,T}X^{(i,m)} \sim \chi^2_{m-i+1}, \quad i = 1, 2, \ldots, m.$

Although this test is not based on conventional testing principles, we mention it because, first, it has a standard null distribution, and second, since we are dealing with unit root processes, these conventional testing principles may no longer be optimal, so we should not confine ourselves to the conventional framework.

4.2. Linear trend stationarity

The linear trend stationarity hypothesis is:

$$H_1^L: z_t = \lambda_0 + \lambda_1 t + u_t,$$

where we assume again that u_t obeys Assumption 1. Under this linear trend stationarity hypothesis auxiliary regression (3) reads as

$$\phi(L)z_t = [1 - (1 + \alpha^*)L - \sum_{j=1}^{p} \phi_j^* L^j(1-L)]z_t = \theta^T P_{t,n}^{(m)} + \varepsilon_t, \quad \textbf{(4)}$$

where $\alpha^* = \text{plim}_{n\to\infty}\hat{\alpha}$, $\phi_j^* = \text{plim}_{n\to\infty}\hat{\phi}_j$, $j = 1,\ldots,p$, and $\phi(1) = -\alpha^*$. Then:

Theorem 4. Under H_1^L, $\text{plim}_{n\to\infty}\hat{A}(m)/n = -1$, $\text{plim}_{n\to\infty}\hat{t}(m)/\sqrt{n} < 0$, $\text{plim}_{n\to\infty}\hat{F}(m)/n > 0$,

$\text{plim}_{n\to\infty}\hat{T}_2(m) = 0$, $\text{plim}_{n\to\infty}\hat{T}_1(m) = 0$ *if* $\lambda_1 = 0$, *and*

$$\text{plim}_{n\to\infty}\hat{T}_1(m)/n^3 = \frac{\lambda_1^2(1+\alpha_*)^2}{12\sigma_\varepsilon^2} \text{ if } \lambda_1 \neq 0.$$

Theorem 4 suggests that both the t-test and the χ^2 test $T_2(m)$ may be conducted left-sided. However, as we will argue below, this is only the case for the alternative of *linear* trend stationarity.

4.3 Nonlinear trend stationarity

We have seen that under the unit root with drift hypothesis as well as under linear trend stationarity the OLS estimators of the coefficients of the lag polynomial $\phi(L)$ are consistent. However, this is no longer the case under nonlinear trend stationarity, i.e., the hypothesis

$$H_1^{NL}: z_t = g(t) + u_t = \lambda_0 + \lambda_1 t + f(t) + u_t, \quad \textbf{(5)}$$

say, where we assume that $f(t)$ is a nonconstant deterministic function of time such that

$$\lim_{n\to\infty}(1/n)\sum_{t=1}^{n} f(t) = 0; \quad \lim_{n\to\infty}(1/n)\sum_{t=1}^{n} t f(t) = 0. \qquad \textbf{(6)}$$

To see this, consider the lag polynomial $\phi(L)$ in equation (4), which under nonlinear trend stationarity now becomes:

$$\psi(L|\alpha,\phi) = 1 - (1 + \alpha)L - \sum_{j=1}^{p}\phi_j L^j(1-L) \; ; \quad \phi(L) = \psi(L|\alpha^*,\phi^*). \qquad \textbf{(7)}$$

Denoting $\theta_{k,n}(\alpha,\phi) = (1/n)\Sigma_{t=1}^{n}P_{k,n}^{*}(t)[\psi(L|\alpha,\phi)g(t)]$ and $\theta = \left(\theta_0,\theta_1,\ldots,\theta_m\right)^T$, it follows that under the alternative hypothesis (5),

$$E\left[(1/n)\sum_{t=1}^{n}\left(\psi(L|\alpha,\phi)z_t - \theta^T P_{t,n}^{(m)}\right)^2\right] = (1/n)\sum_{t=1}^{n}\left(\psi(L|\alpha,\phi)g(t) - \theta^T P_{t,n}^{(m)}\right)^2$$

$$+ (1/n)\sum_{t=1}^{n}E\left([\psi(L|\alpha,\phi) - \psi(L|\alpha^*,\phi^*)]u_t\right)^2 + \sigma_\varepsilon^2.$$

$$= \sum_{k=0}^{m}(\theta_k - \theta_{k,n}(\alpha,\phi))^2$$

$$+ \sum_{k=m+1}^{n-1}\theta_{k,n}(\alpha,\phi)^2 + (1/n)\sum_{t=1}^{n}E\left([\psi(L|\alpha,\phi) - \psi(L|\alpha^*,\phi^*)]u_t\right)^2 + \sigma_\varepsilon^2.$$

This function is minimal at $\alpha = \alpha^*$ and $\phi = \phi^*$ only if $\Sigma_{k=m+1}^{n-1}\theta_{k,n}(\alpha^*,\phi^*)^2 = 0$, which, however, is in general not the case if $g(t)$ is nonlinear. Consequently, for fixed m the probability limit of $\hat{\alpha}$ may be positive, and then the probability limit of the t-statistic of $\hat{\alpha}$ will be plus infinity rather than minus infinity. However, there is no guarantee that this will happen. Therefore, in testing the unit root with drift hypothesis against nonlinear trend stationarity the t-test should be conducted two-sided. If the t-value of $\hat{\alpha}$ turns out the be greater than the right-side critical value then we have an indication that nonlinear trend stationarity is the alternative, but left-sided rejection does not provide information about the nature of the alternative.

Next, we consider the limiting behavior of the χ^2 test under nonlinear trend stationarity. Observe from (5) and (7) that

$$\hat{\theta}^{(2,m)} = (1/n)\sum_{t=1}^{n} \psi(L|\hat{\alpha},\hat{\phi})z_t P_{t,n}^{(2,m)}$$
$$= (1/n)\sum_{t=1}^{n} \psi(L|\hat{\alpha},\hat{\phi})f(t)P_{t,n}^{(2,m)} + (1/n)\sum_{t=1}^{n} \psi(L|\hat{\alpha},\hat{\phi})u_t P_{t,n}^{(2,m)}$$

$$\hat{\alpha}(1/n)\sum_{t=1}^{n} z_t P_{t,n}^{(2,m)} = \hat{\alpha}(1/n)\sum_{t=1}^{n} f(t)P_{t,n}^{(2,m)} + \hat{\alpha}(1/n)\sum_{t=1}^{n} u_t P_{t,n}^{(2,m)}$$

and

$$\hat{\alpha} + \psi(L|\hat{\alpha},\hat{\phi}) = (1 - L)\left(1 + \hat{\alpha} - \sum_{j=1}^{p} \hat{\phi}_j L^j\right).$$

Moreover, it follows from Theorem A.1 in the appendix and (7) that

$$(1/\sqrt{n})\sum_{t=1}^{n} \left(\hat{\alpha} + \psi(L|\hat{\alpha},\hat{\phi})\right)u_t P_{t,n}^{(2,m)} \Rightarrow 0.$$

Thus we have:

$$\hat{\alpha}(1/n)\sum_{t=1}^{n} z_t P_{t,n}^{(2,m)} + \hat{\theta}^{(2,m)}$$
$$= (1/n)\sum_{t=1}^{n} \left\{\left(1 + \hat{\alpha} - \sum_{j=1}^{p} \hat{\phi}_j L^j\right)(f(t) - f(t-1))\right\}P_{t,n}^{(2,m)} + o_p(1/\sqrt{n})$$

Therefore, the test statistic $T_2(m)$ converges in probability to infinity if

$$\text{plim}_{n\to\infty} n\sum_{k=2}^{m}\left((1/n)\sum_{t=1}^{n}\left\{\left(1 + \hat{\alpha} - \sum_{j=1}^{p}\hat{\phi}_j L^j\right)(f(t) - f(t-1))\right\}P^*_{k,n}(t)\right)^2 = \infty .$$

By a similar argument we can derive the condition for $T_1(m)$ to converge in probability to infinity. However, if m is too small it may not. Thus the χ^2 tests involved are not consistent against all forms of nonlinear trend stationarity, but this inconsistency is common to all tests for model misspecification based on a finite number of sample moments. For example, if we test the null hypothesis $\alpha = 1$ against the alternative $\alpha < 1$ in the model $y_t = \alpha y_{t-1} + u_t$, where u_t is i.i.d. $N(0,\sigma^2)$, using the Dickey-Fuller test $n(\hat{\alpha}-1)$ with $\hat{\alpha}$ the OLS estimator of α, while the actual alternative is $y_t = t + u_t$, then $\text{plim}_{n\to\infty} n(\hat{\alpha}-1) =$ 3/2, hence the test will have no power against this alternative. Another example is the case where the nonlinear time trend is a cumulative sum of a chaotic deterministic process (like a pseudo-random number generator). It will be virtually impossible to distinguish it from a random process with a unit root. Anyhow, our argument shows that also the χ^2 tests should be conducted two-sided, but now the rejection of the null provides information about the alternative: in the case of test $T_2(m)$ left-sided rejection indicates linear trend stationarity and right-sided rejection indicates nonlinear trend stationarity, whereas in the case of test $T_1(m)$ left-sided rejection indicates stationarity about a constant and right-sided rejection indicates linear or nonlinear trend stationarity.

Finally, if m is sufficiently large the F-test statistic will likely converge in probability to infinity, but also the F-test is not consistent against all forms of nonlinear trend stationarity.

5. A model-free unit root test

A disadvantage of the above tests is that we have to specify the lag length p in the auxiliary regression (3). Therefore we now propose a model-free alternative to the χ^2 test, based on the following specification of the data-generating process:

$$\Delta z_t = -\rho z_{t-1} + \lambda_0 + \rho\lambda_1 t + f(t) + u_t, \quad \rho \in \{0,1\},$$

where $f(t)$ satisfies the conditions in (6), and u_t is a zero-mean process that obeys the functional central limit theorem:

Assumption 2. Let $S_n(x) = 0$ *if* $x \in [0, n^{-1})$; $S_n(x) = \sum_{t=1}^{[xn]} u_t$ *if* $x \in [n^{-1}, 1]$. *Then* $S_n/\sqrt{n} \Rightarrow \sigma W$*, where W is a standard Wiener process and* σ^2 *is the long-run variance of the process* u_t.

Now the unit root with drift hypothesis corresponds to the hypothesis

$$H_0: \ \rho = 0, \ f(t) \equiv 0,$$

and the alternatives of linear and non-linear trend stationarity correspond to

$$H_1^L: \ \rho = 1, \ f(t) \equiv 0, \qquad H_1^{NL}: \ \rho = 1$$

respectively. Our aim is to design a test that distinguishes these three hypotheses in a similar way as the χ^2 tests, but without assuming an AR structure for the u_t process.

In order to motivate our new test, let us first assume that the process z_t is linear trend stationary. Then for $k = 1,..,m$,

$$\sum_{t=1}^{n} \Delta z_t P_{k,n}^*(t) = \sum_{t=1}^{n} \Delta u_t P_{k,n}^*(t) = \sum_{t=1}^{n} u_t \left(P_{k,n}^*(t) - P_{k,n}^*(t+1) \right) + u_n P_{k,n}^*(n+1) - u_0 P_{k,n}^*(1). \quad \textbf{(8)}$$

In order to get rid of the last two terms in (8), regress the left-hand side of (8) on $P_{k,n}^*(n+1)$ and $P_{k,n}^*(1)$, provided $m > 2$, with least squares coefficients $\hat{\xi}_1$ and $\hat{\xi}_2$. Then it can be shown that under linear trend stationarity and Assumption 2,

$$\sum_{t=1}^{n} \Delta z_t P_{t,n}^{(2,m)} - \hat{\xi}_1 P_{n+1,n}^{(2,m)} - \hat{\xi}_2 P_{1,n}^{(2,m)} = O_p(1/\sqrt{n}). \quad \textbf{(9)}$$

Next, assume that the unit root with drift hypothesis holds. Then it can be shown that

$$(1/\sqrt{n})\left(\sum_{t=1}^{n} \Delta z_t P_{t,n}^{(1,m)} - \hat{\xi}_1 P_{n+1,n}^{(1,m)} - \hat{\xi}_2 P_{1,n}^{(1,m)}\right) \Rightarrow \sigma \Sigma_m X^{(1,m)}, \qquad \textbf{(10)}$$

where Σ_m is an idempotent matrix, with rank $m-2$.

Furthermore, denoting

$$\tilde{\theta}^{(m)} = (1/n)\sum_{t=1}^{n} z_t P_{t,n}^{(m)},$$

and assuming

Assumption 3. $\text{plim}_{n\to\infty}(1/n)\Sigma_{t=1}^{n} u_t^2 = \sigma_u^2 \in (0, \infty),$

it can be shown under the linear trend stationarity hypothesis and Assumptions 2-3,

$$\text{plim}_{n\to\infty}(1/n)\sum_{t=1}^{n}\left(z_t - \tilde{\theta}^{(m)T} P_{t,n}^{(m)}\right)^2 = \sigma_u^2, \qquad \textbf{(11)}$$

whereas under the unit root with drift hypothesis and Assumption 2,

$$(1/n^2)\sum_{t=1}^{n}\left(z_t - \tilde{\theta}^{(m)T} P_{t,n}^{(m)}\right)^2 \Rightarrow \sigma^2\left(\int_0^1 W(x)^2 dx - Y^{(m)T} Y^{(m)}\right). \qquad \textbf{(12)}$$

These results now suggest the following test statistic:

$$\tilde{T}(m) = \frac{\left(\sum_{t=1}^{n} \Delta z_t P_{t,n}^{(1,m)} - \hat{\xi}_1 P_{n+1,n}^{(1,m)} - \hat{\xi}_2 P_{1,n}^{(1,m)}\right)^T \left(\sum_{t=1}^{n} \Delta z_t P_{t,n}^{(1,m)} - \hat{\xi}_1 P_{n+1,n}^{(1,m)} - \hat{\xi}_2 P_{1,n}^{(1,m)}\right)}{(1/n)\sum_{t=1}^{n}\left(z_t - \tilde{\theta}^{(m)T} P_{t,n}^{(m)}\right)^2}. \qquad \textbf{(13)}$$

Theorem 5. Let m > 2. *Under Assumption 2 and the unit root with drift hypothesis,*

$$\tilde{T}(m) \Rightarrow \frac{X^{(1,m)\,T}\Sigma_m X^{(1,m)}}{\int_0^1 W(x)^2 dx - Y^{(m)\,T}Y^{(m)}} = T_m\,,\ say,$$

whereas under Assumptions 2-3 and the linear trend stationarity hypothesis, $\tilde{T}(m) = O_p(1/n)$.

Table 4 contains the most important fractiles of the distribution of T_m for $m = 3,..,20$. These results are again derived on the basis of 10,000 replications of a Gaussian random walk with sample size 500.

<Insert Table 4 about here>

Finally, assume that the nonlinear trend stationarity hypothesis holds, and moreover, assume that

Assumption 4. Conditions (6) *and* $\limsup_{n\to\infty}(1/n)\Sigma_{t=1}^n f(t)^2 < \infty$ *hold.*

Then it is not too hard to verify that, owing to the latter condition,

$$(1/n)\sum_{t=1}^{n}\left(z_t - \tilde{\theta}^{(m)\,T}P_{t,n}^{(m)}\right)^2 = O_p(1).$$

Furthermore, denote for $k > 0$,

$$\theta_{k,n}(\Delta f) = (1/n)\sum_{t=1}^{n}\left(f(t) - f(t-1)\right)P_{k,n}^{*}(t),$$

and let

$$\theta_n^{(1,m)}(\Delta f) = \left(\theta_{1,n}(\Delta f), \ldots., \theta_{m,n}(\Delta f)\right)^T.$$

Then it follows easily that

$$\sum_{t=1}^{n} \Delta z_t P_{t,n}^{(2,m)} - \hat{\xi}_1 P_{n+1,n}^{(2,m)} - \hat{\xi}_2 P_{1,n}^{(2,m)} = n\theta_n^{(2,m)}(\Delta f) + O_p(1/\sqrt{n}).$$

Thus the test statistic (13) converges in probability to infinity if

$$\lim_{n\to\infty} \| n\theta_n^{(2,m)} \| = \infty .$$

Again, this condition may not hold if m is too small. Whether this condition is reasonable or not depends on how well the function

$$\Delta f_m(t) = \sum_{k=2}^{m} \theta_{k,n}(\Delta f) P_{k,n}^{*}(t)$$

approximates $\Delta f(t) = f(t) - f(t-1)$. Since

$$\Delta f(t) = \sum_{k=0}^{n-1} \theta_{k,n}(\Delta f) P_{k,n}^{*}(t) = \sum_{k=2}^{n-1} \theta_{k,n}(\Delta f) P_{k,n}^{*}(t) + o(1),$$

where the last conclusion follows from Assumption 4, and

$$(1/n)\sum_{t=1}^{n} \Delta f(t)\Delta f_m(t) = \sum_{k=2}^{m} \theta_{k,n}(\Delta f)^2 = (1/n)\sum_{t=1}^{n} \left(\Delta f_m(t)\right)^2,$$

we have

$$(1/n)\sum_{t=1}^{n} \left(\Delta f(t) - \Delta f_m(t)\right)^2 = (1/n)\sum_{t=1}^{n} \left(\Delta f(t)\right)^2 - (1/n)\sum_{t=1}^{n} \left(\Delta f_m(t)\right)^2$$

$$= (1/n)\sum_{t=1}^{n} \left(\Delta f(t)\right)^2 - \|\theta_n^{(2,m)}\|^2,$$

Therefore, if

$$\liminf_{n\to\infty}(1/n)\sum_{t=1}^{n}\left(\Delta f(t)\right)^2 \;>\; 0$$

and

$$\lim_{n\to\infty} n^2\left[1 \;-\; \frac{(1/n)\sum_{t=1}^{n}\left(\Delta f(t) \;-\; \Delta f_m(t)\right)^2}{(1/n)\sum_{t=1}^{n}\left(\Delta f(t)\right)^2}\right] \;=\; \infty\,,$$

then under nonlinear trend stationarity,

$$\text{plim}_{n\to\infty}\tilde{T}(m) \;=\; \infty\,.$$

Thus again, left rejection indicates linear trend stationarity, and right rejection indicates nonlinear trend stationarity.

6. Are the logs of GNP deflator and CPI unit root with drift processes?

6.1 *Introduction*

The log of the price level is often considered to be a unit root with drift process, or even an I(2) process. See Johansen and Juselius (1990) for the latter. On the other hand, Perron (1989) found for the Nelson-Plosser (1982) data that if one corrects for a trend break then the unit root hypothesis is rejected for the log of the GNP deflator but not for the log of the consumer price index (CPI). Since a trend break is a special case of a nonlinear trend, it is therefore challenging to investigate whether or not these variables are able to withstand our new tests. The source of the time series involved is the Nelson-Plosser (1982) data set of 14 macroeconomic annual time series of the US, extended by Schotman and Van Dijk (1991) to 1988. The GNP deflator series has length n = 100, from t = 1889 to t = 1988, and the CPI series has length n = 129, from t = 1860 to 1988. We shall denote the logs of these series by LNDEF(100) and LNCPI(129), respectively. Also, we shall investigate the CPI series over the same period as the GNP deflator, and we denote the log of it by LNCPI(100). The latter two series are plotted together in Figure 1.

<Insert Figure 1 about here>[3]

First, we have conducted various unit root and linear trend stationarity tests to the levels and first differences of LNDEF(100), LNCPI(129) and LNCPI(100):

<Insert Table 5 about here>

The first set of three tests consists of the Phillips (1987) and Phillips-Perron (1988) tests of the unit root hypothesis against the alternatives $z_t = u_t$, $z_t = \beta + u_t$ and $z_t = \beta + \gamma t + u_t$, respectively, where u_t is a zero mean stationary process. These tests employ a Newey-West (1987) type variance estimator with truncation parameter $k = [cn^r]$, where $c > 0$, $0 < r < 1/3$. We have used the values $c = 5$, $r = .2$

The next two tests are the augmented Dickey-Fuller t-tests, including a linear time trend in the auxiliary regression. The first ADF test employs a lag width that is the same as the truncation parameter of the Newey-West variance estimator used in the Phillips and Phillips-Perron tests. Cf. Said (1991). The second ADF test employs a pretested auxiliary regression, where the lag length is reduced to the significant lags only, at the 5% significance level, using a series of Wald tests. In the latter case, however, we have not corrected the critical values for pretest bias [cf. Hall (1994)]. On the other hand, the results of Ng and Perron (1995) indicate that this is not a serious problem.

The next four tests are Bierens'(1993) unit root tests on the basis of higher order sample autocorrelations. The first two of them test the unit root against stationarity, and the last two have as alternative linear trend stationarity. These four tests depend on parameters $\mu > 0$, $\alpha > 0$, and $0 < \delta < 1$, and the lag length is: $k = 1 + [\alpha n^{\delta\mu/(3\mu+2)}]$. We have used the values $\mu = 2$, $\alpha = 5$ and $\delta = .5$

The last six tests are the Bierens-Guo (1994) Cauchy tests of the null hypothesis of (linear trend) stationarity against the unit root (with drift) hypothesis. The null distribution is that of the absolute value of a standard Cauchy variate. Cauchy test #4 also employs a Newey-West type variance

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[3] Figure 1 is not included in this draft paper.

estimator. We have used the same truncation parameter as for the Phillips and Phillips-Perron tests.

We see from Table 5 that the test results for the first difference of the LNDEF(100) and LNCPI(100) are consistent in that all the tests favor the (trend) stationarity hypothesis. This is not the case for the first difference of LNCPI(129); the (D)HOAC tests do not reject the unit root hypothesis, while all the other tests favor the trend stationarity hypothesis. The results for the levels of LNDEF(100). LNCPI(129) and LNCPI(100) are slightly inconsistent, because for all three series the test HOAC(1,1) rejects the unit root hypothesis at any significance level, and for LNDEF(100) and LNCPI(100) the test C(6) accepts the trend stationarity hypothesis at the 10% significance level, while the other tests favor the unit root hypothesis. An explanation for these conflicting results might be that the time series involved are nonlinear trend stationary, with a more complicated nonlinear trend than the broken linear trend considered by Perron (1989).

6.2 The GNP deflator

We have conducted the nonlinear ADF test on the basis of auxiliary regression (3) with lag length $p = 1$ and the Chebishev time polynomial order $m = 10$. The choice of p is based on the Akaike (1973) criterion under the unit root with constant drift hypothesis, starting from an AR(10) model with intercept for Δz_t. In view of the results by Ng and Perron (1995) we probably could have determined p also by the Akaike criterion or by a sequence of t-tests on the basis of model (3), but due to the presence of the nonlinear trend in model (3) that needs to be proved. Therefore we have based the choice of p on the null model. Ng and Perron (1995) show that sequential t-testing in order to determine p causes less size distortion than the Akaike criterion, but the latter is more convenient for simulation of the actual size of the test. The test results are presented in Table 6.

<Insert Table 6 about here>

If we would believe that the asymptotic critical values are close to the actual critical values, then we reject the unit root hypothesis, but it is not clear which alternative, linear or nonlinear trend stationarity, is favored. The χ^2 tests $T_1(m)$ and $T_2(m)$ point in the direction of linear trend stationarity, but the last, model-free, test indicates nonlinear trend stationarity. A possible cause of this inconsistency is size distortion. In order to check this, we have estimated the values of the null distribution functions, F, in the realized values of the tests, on the basis of 1000 replications of a Gaussian AR(10) process for Δz_t with parameters and error variance equal to the estimated AR(10) null model, where the order p of the ADF regression is determined by the Akaike (1973) criterion. Thus the simulation is conducted in the same way as we have selected the ADF model, in order to check the effect of pre-testing. The results are presented in Table 7.

<Insert Table 7 about here>

The results in Table 7 show evidence of substantial size distortion. However, the F-test and the model-free test still reject the unit root hypothesis at the 10% significance level. Therefore, the previous conclusions on the basis of asymptotic critical values still stand: LNDEF(100) is not a genuine unit root process with constant drift. The contradictory results of the χ^2 tests $T_1(m)$ and $T_2(m)$ may be due to the possible smoothness of the nonlinear trend $f(t)$ involved, by which $f(t) - f(t-1)$ is relatively small. See Section 4.3.

In order to check the smoothness of the nonlinear time trend, we have estimated the trend model

$$z_t = \sum_{k=0}^{m} \theta_k P_{k,n}^{*}(t) + u_t \qquad \textbf{(14)}$$

for $m = 10$ by least squares, and plotted its fit in Figure 2. We see that the nonlinear trend is indeed quite smooth, which may be the cause of the low power of the tests $T_1(m)$ and $T_2(m)$.

<Insert Figure 2 about here>[4]

6.3 The consumer price index from 1860 to 1988

We have conducted the tests in the same way as for LNDEF(100), with $m = 10$ and p determined by the Akaike criterion, starting from an AR(10) model for the first differences. The Akaike

[4] Figure 2 is not included in this draft paper.

criterion now indicated p = 5. None of the tests rejected the unit root hypothesis. This result is in tune with those of Perron (1989). However, the graphs of the two series over the period 1889-1988 are almost carbon copies (see Figure 1), so why would LNCPI(129) be a unit root process and LNDEF(100) a nonlinear trend stationary process? A possible reason may be that in order to capture the nonlinear trend in the CPI over the period 1860-1888 (which seems to be due to the civil war and its aftermath), we need a higher order of the Chebishev polynomial than 10. Therefore, we have increased the order of the detrended Chebishev time polynomial to m = 20. The test results involved, with asymptotic critical values, are presented in Table 8.

<Insert Table 8 about here>

All tests but the t-test now reject the unit root hypothesis in favor of the nonlinear trend stationarity hypothesis. However, also now there is substantial size distortion, as the results in Table 9 show.

<Insert Table 9 about here>

Nevertheless, the two χ^2 tests $T_1(m)$ and $T_2(m)$ still reject the unit root hypothesis at the 5% significance level, which provides evidence against the unit root hypothesis. In view of our previous discussion, the nonlinear trend $f(t)$ probably now varies sufficiently. Therefore, we may conclude with some reluctancy that LNCPI(129) looks more like a nonlinear trend stationary process than a unit root with drift process.

Again, we have estimated a model of the type (14) for m = 20 by least squares, in order to see how much the nonlinear trend contributes to LNCPI(129). The results are presented in Figure 3.

<Insert Figure 3 about here>[5]

We see that the trend function has more variation than for LNDEF(100), due to the large bump in the trend in the period 1860-1988.

5 Figure 3 is not included in this draft paper.

6.4 The consumer price index from 1888 to 1988

The mixed results for LNCPI(129), in particular the failure of the other tests to reject the unit root hypothesis, are still puzzling, though, in comparison with the results for LNDEF(100). Could it be that the nonlinear trend in CPI over the period 1860-1988 is still not captured well enough by a 20-th order Chebishev polynomial? Since increasing the order further is not feasible with the present tables of critical values, we shall now analyze the shorter series LNCPI(100), in the same way as for LNDEF(100). Thus we specify $m = 10$, and an initial lag length $p = 10$; the Akaike criterion indicates $p = 2$. The asymptotic test results in Table 10 are now consistent with those for LNDEF(100). The same applies to the small sample test results in Table 11.

<Insert Tables 10 and 11 about here>

Also the plot of the trend model (14) for $m = 10$, in Figure 4, is very similar to the one for LNDEF(100).

<Insert Figure 4 about here>[6]

7. Is the interest rate a unit root process?

Another time series for which Perron (1989) could not reject the unit root hypothesis is the nominal interest rate (NINT). However, NINT is a positive variable, and therefore cannot be a unit root process without drift because these processes will take with positive probability negative values regardless of the initial values, due to the fact that for a Wiener process W and arbitrary η, $P(\inf_{0\leq x\leq 1} W(x) + \eta < 0) > 0$. Moreover, if it would be a unit root process with positive drift it would converge to infinity, which is not realistic. Again we suspect that the results of Perron (1989) are due to nonlinear trend stationarity.

First, we test the unit root hypothesis against (trend) stationarity and vice versa, for NINT as well as for the real interest rate (RINT). See Table 12.

<Insert Table 12 about here>

The results for NINT are mixed; the tests HOAC(1,1), HOAC(2,2) and DHOAC(2,2) reject the unit

[6] Figure 4 is not included in this draft paper.

root hypothesis at the 5% significance level and the tests C(3), C(4) and C(6) accept the stationarity hypothesis at the 10% level, whereas all the other tests point towards the unit root hypothesis. On the other hand, all the test results for RINT, except the ADF(1) test, indicate stationarity. But if RINT is stationary and the CPI is nonlinear trend stationary, then NINT, being the sum of RINT and the percentage change of CPI, must be nonlinear trend stationary as well. As a double check, we have therefore applied our new tests to z_t = NINT, in the same way as for LNDEF(100). Thus again we have chosen $m = 10$. The Akaike criterion on the basis of an initial AR(10) model for Δz_t indicated $p = 2$. The asymptotic and small sample test results are presented in Tables 13 and 14.

<Insert Tables 13 and 14 about here>

The tests $A(m)$, $T_1(m)$ and $T_2(m)$ reject the unit root hypothesis at the 5% level on the basis of the asymptotic critical values, but the picture of the small sample tests is different; now only the *t*-test and the model-free test reject the null. Nevertheless, the conclusion is justified that NINT is a nonlinear trend stationary process. Finally, the fit of the nonlinear trend model (14) with $m = 10$ is plotted in Figure 5.

<Insert Figure 5 about here>[7]

8. Concluding remarks

The empirical application, in particular the results based on asymptotic critical values compared with those on the bases of simulated *p*-values, show substantial size distortion of all tests. This size distortion does not only depend on the order m of the Chebishev time polynomial involved, but also on the AR polynomial of the stochastic part of the time series. Also, the power properties of the tests are quite different, which suggests that each of them picks up different aspects of the alternative.

Thus, for time series of modest size like the ones we have analyzed the asymptotic critical values in Tables 1-4 appear of limited use. It is therefore advisable to base the final decision regarding the presence of nonlinear trends on simulated *p*-values the way we did. Moreover, although the tests are not independent, we recommend to conduct them all: don't bet on one horse only!

[7] Figure 5 is not included in this draft paper.

In order to conduct the tests on the basis of the regression (3) one has to specify the lag length p and the order m of the detrended Chebishev polynomial. Instead of specifying p one could also set $p = 0$ and adopt a Phillips-Perron (1988) approach by estimating the long-run variance σ^2 nonparametrically, using a Newey-West (1987) type variance estimator. But then the problem of specifying p is replaced by the problem of specifying the truncation width involved. Alternatively one could adopt a similar approach as in Said and Dickey (1984) and Ng and Perron (1995) by specifying p adaptively as a function of the sample size n. These extensions seem pretty straightforward.

A more difficult problem is to provide general guidelines for specifying the order m of the detrended Chebishev polynomials. Although the specification of m seems less critical than the specification of the order of the polynomial trend in the Ouliaris, Park and Phillips (1989) approach, it is clear that the power as well as the actual size of our tests depend on m. The size problem can be solved by simulation, as we did in the empirical applications, but given m the power of our tests depends on the unknown nonlinear trend under the alternative. If this nonlinear trend is more nonlinear than the Chebishev polynomial approximation with a fixed order m can handle, the power of our test might be low. One could think of determining m adaptively by (say) conducting a sequence of Wald tests, but a problem (although not unsurmountable) is that under the null hypothesis the limiting distribution of the OLS estimate $\hat{\theta}$ is nonstandard. Cf. Theorem 1. Another approach is to let m converge to infinity with the sample size n at some controled rate, like the truncation width of the Newey-West (1987) estimator. The keys for this generalization are the Lemmas A.3 and A.5 in the appendix, which need to be generalized to

$$(1/\sqrt{n})\sum_{t=1}^{n} u_{t-j}\cos(m_n\pi(t-0.5)/n) \; - \; \sigma\{(-1)^{m_n}W(1) \; + \; m_n\pi\int_0^1 sin(m_n\pi x)W(x)dx\} \; \Rightarrow \; 0$$

and

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} \cos(m_n\pi(t \; - \; 0.5)/n)S_n((t-j)/n) \; - \; \sigma\int_0^1 \cos(m_n\pi x)W(x)dx \; \Rightarrow \; 0,$$

respectively, for fixed j and a sequence m_n of natural numbers converging to infinity with n, where $S_n(x)$ = $\sum_{t=1}^{[nx]} u_t$ if $x \in [n^{-1},1]$, $S_n(x) = 0$ if $x \in [0,n^{-1})$. Whether such a sequence m_n exists, and if so how it depends on n, is an open question.

Finally, some words of warning. If our tests reject the unit root hypothesis this should not be automatically interpreted as evidence in favor of the nonlinear trend stationarity hypothesis. As shown by Hassler and Wolters (1995) and others, the ADF and Phillips-Perron (1988) tests of the unit root with drift hypothesis may also have some power against other alternatives than linear trend stationarity, for example the alternative that the process is fractionally integrated. The same may apply to our tests.

Appendix: Proofs

Proof of Lemma 1: See, e.g., Hamming (1973).

Proof of Lemma 2: We can write

$$2\sum_{t=1}^{n} t\cos(2k\pi(t-0.5)/n) = g_n(2k\pi/n) + g_n(-2k\pi/n),$$

where

$$g_n(x) = e^{-0.5ix}\sum_{t=1}^{n} te^{ixt} = \frac{1}{i}e^{-0.5ix}\frac{d}{dx}\sum_{t=1}^{n}\left(e^{ix}\right)^t = \frac{1}{i}e^{-0.5ix}\frac{d}{dx}\left(e^{ix}\frac{1-e^{inx}}{1-e^{ix}}\right)$$

$$= e^{0.5ix}\left(\frac{1-e^{inx}}{\left(e^{-0.5ix}-e^{0.5ix}\right)^2} + \frac{e^{-0.5ix}\left(1-(n+1)e^{inx}\right)}{e^{-0.5ix}-e^{0.5ix}}\right)$$

$$= \frac{\cos(0.5x) + i(2n+1)\sin(0.5x)e^{inx}}{-4\sin^2(0.5x)},$$

hence,

$$g_n(x) + g_n(-x) = \frac{(2n+1)\sin(0.5x)\sin(nx) - \cos(0.5x)(1 - \cos(nx))}{2\sin^2(0.5x)}.$$

Plugging in $x = 2k\pi$, the lemma follows. Q.E.D.

Proof of Lemma 3: Again we can write

$$2\sum_{t=1}^{n} t\cos[(2k-1)\pi(t-0.5)/n] = g_n((2k-1)\pi/n) + g_n(-(2k-1)\pi/n)$$

$$= -\frac{\cos[0.5(2k-1)\pi/n]}{\sin^2[0.5(2k-1)\pi/n]}.$$

Q.E.D.

Proof of Theorem 1: Under H_0 as well as under linear trend stationarity model (3) is equivalent to

$$u_t = \alpha S_n((t-1)/n) + \phi^T U_{t-1} + \theta^T P_{t,n}^{(m)} + \varepsilon_t, \qquad \textbf{(A.1)}$$

where

$$U_{t-1} = (u_{t-1},...,u_{t-p})^T, \ \phi = (\phi_1,...,\phi_p)^T,$$
$$S_n(x) = 0 \ if \ x \in [0,n^{-1}); \ S_n(x) = \sum_{t=1}^{[xn]} u_t \ if \ x \in [n^{-1},1],$$

and the parameters are the same as in model (3), except the first two components of θ. The same applies to the OLS estimators of these parameters.

In order to derive the limiting distribution of the OLS estimators of the parameters of model (A.1) under H_0, we shall first derive the limiting distribution of weighted means of u_t and their partial sums $S_n(t/n)$, where the weights are the detrended Chebishev time polynomials:

Theorem A.1. Let $X_{k,j,n} = (1/\sqrt{n})\Sigma_{t=1}^n u_{t-j} P_{k,n}^*(t), \quad X_{j,n}^{(m)} = (X_{0,j,n}, X_{1,j,n},...,X_{m,j,n})^T$. *Then for fixed* $m > 1$ *and* j, $X_{j,n}^{(m)} \Rightarrow \sigma X^{(m)} \sim N_{m+1}(0, \sigma^2 I_{m+1})$, *where* $\sigma = \sigma_\varepsilon/\phi(1)$.

Theorem A.2. Let $Y_{k,j,n} = (1/(n\sqrt{n}))\Sigma_{t=1}^n S_n((t-j)/n) P_{k,n}^*(t), \quad Y_{j,n}^{(m)} = (Y_{0,j,n}, Y_{1,j,n},...,Y_{m,j,n})^T$. *Then for fixed* $m > 0$ *and* j, $Y_{j,n}^{(m)} \Rightarrow \sigma Y^{(m)}$.

Proof of Theorem A.1: Note that we can write

$$u_t = \phi(L)^{-1}\varepsilon_t = \phi(1)^{-1}\varepsilon_t + \frac{\phi(L)^{-1} - \phi(1)^{-1}}{1 - L}(\varepsilon_t - \varepsilon_{t-1})$$

$$= \phi(1)^{-1}\varepsilon_t + \psi(L)(\varepsilon_t - \varepsilon_{t-1}) = \phi(1)^{-1}\varepsilon_t + v_t - v_{t-1},$$

say, where $v_t = \psi(L)\varepsilon_t$. Denoting

$$W_n(x) = 0 \ \mathit{if} \ x \in [0, n^{-1})$$

$$= (1/\sqrt{n})\sum_{t=1}^{[xn]}(\varepsilon_t/\sigma_\varepsilon) \ \mathit{if} \ x \in [n^{-1}, 1],$$

it follows now that

$$S_n(x)/\sqrt{n} = \sigma W_n(x) + \frac{v_{[xn]} - v_0}{\sqrt{n}},$$

where $\sigma^2 = \sigma_\varepsilon^2/\phi(1)^2$ is the long-run variance of u_t:

$$\sigma^2 = \lim_{n\to\infty} E\left(1/\sqrt{n})\sum_{t=1}^{n} u_t\right)^2.$$

Moreover, under Assumption 1, $W_n \Rightarrow W$, and consequently $S_n/\sqrt{n} \Rightarrow \sigma W$, where W is a standard Wiener process. [Cf. Billingsley (1968)]. Furthermore, we have:

Lemma A.1. Let F be a differentiable real function on [0,1] *with derivative f. Then*

$$\sum_{t=1}^{n} F(t/n)u_t = F(1)S_n(1) - \int_0^1 f(x)S_n(x)dx.$$

Proof of Lemma A.1: Bierens (1994, Lemma 9.6.3, p.200).

Lemma A.2. For fixed j, $(1/\sqrt{n})\sum_{t=1}^{n} u_{t-j}(t/n) \Rightarrow \sigma\{W(1) - \int_0^1 W(x)dx\}$.

Proof of Lemma A.2: Apply Lemma A.1 with $F(x) = x$ and u_t replaced by u_{t-j}. Since

$$\sum_{t=1}^{[nx]} u_{t-j} = S_n(x) + \sum_{t=-j+1}^{0} u_t - \sum_{t=[nx]-j+1}^{[nx]} u_t = S_n(x) + O_p(j),$$

where by stationarity of u_t the $O_p(j)$ part is uniform in x, the lemma follows from the continuous mapping theorem.

Lemma A.3. For fixed k and j,

$$(1/\sqrt{n})\sum_{t=1}^{n} u_{t-j}\cos(k\pi(t-0.5)/n) \Rightarrow \sigma\{(-1)^k W(1) + k\pi\int_0^1 sin(k\pi x)W(x)dx\}.$$

Proof of Lemma A.3: Similarly to Lemma A.2, with $F(x) = \cos(k\pi(x-0.5)/n)$.
Theorem A.1 follows easily from these lemmas. Q.E.D.

Proof of Theorem A.2: It follows straightforwardly from the weak convergence result $S_n/\sqrt{n} \Rightarrow \sigma W$ and the continuous mapping theorem that

Lemma A.4. For fixed m ≥ 0 *and j,* $\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} (t/n)^m S_n((t-j)/n) \Rightarrow \sigma\int_0^1 x^m W(x)dx$.

Lemma A.5. For fixed $k \geq 0$ and j,

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^{n}\cos(k\pi(t - 0.5)/n)S_n((t-j)/n) \Rightarrow \sigma\int_0^1\cos(k\pi x)W(x)dx\,.$$

Using these lemmas, Theorem A.2 follows similarly to Theorem A.1. Q.E.D.

In order to complete the proof of Theorem 1, we need the following lemmas:

Lemma A.6. The following weak convergence results hold jointly:

$$(1/n)\sum_{t=1}^{n} u_t S_n((t-1)/n) \Rightarrow \tfrac{1}{2}[\sigma^2 W(1)^2 - E(u_1^2)],$$

$$(1/n)\sum_{t=1}^{n} S_n((t-1)/n)^2 \Rightarrow \sigma^2\int_0^1 W(x)^2 dx\,.$$

Lemma A.7. $(1/n)\sum_{t=1}^{n} \varepsilon_t S_n((t-1)/n) \Rightarrow \tfrac{1}{2}\sigma_\varepsilon^2\phi(1)^{-1}[W(1)^2 - 1]$.

Lemma A.8. For fixed $j \geq 1$,

$$(1/n)\sum_{t=1}^{n} u_{t-j} S_n((t-1)/n) \Rightarrow \sum_{i=0}^{j-1}\mathrm{cov}(u_1, u_{1-i}) + \tfrac{1}{2}[\sigma^2 W(1)^2 - E(u_1^2)]\,.$$

These lemmas follow easily from results in Phillips (1987).

Now estimating model (A.1) by OLS, with $\hat{\alpha}$, $\hat{\phi}$ and $\hat{\theta}$ the OLS estimators of α, ϕ and θ, respectively, and substituting (A.1) into the normal equations yield

$$(1/n)\sum_{t=1}^{n} \varepsilon_t S_n((t-1)/n) = n\,\hat{\alpha}\,(1/n^2)\sum_{t=1}^{n} S_n((t-1)/n)^2$$

$$+ (1/(n\sqrt{n}))\sum_{t=1} S_n((t-1)/n)U_{t-1}^T\sqrt{n}(\hat{\phi} - \phi)$$

$$+ (1/(n\sqrt{n}))\sum_{t=1}^{n} S_n((t-1)/n)P_{t,n}^{(m)\,T}\sqrt{n}(\hat{\theta} - \theta),$$

$$(1/\sqrt{n})\sum_{t=1}^{n} \varepsilon_t U_{t-1} = n\,\hat{\alpha}(1/(n\sqrt{n}))\sum_{t=1}^{n} S_n((t-1)/n)U_{t-1}$$

$$+ (1/n)\sum_{t=1}^{n} U_{t-1}U_{t-1}^T\sqrt{n}(\hat{\phi} - \phi)$$

$$+ (1/n)\sum_{t=1}^{n} U_{t-1}P_{t,n}^{(m)\,T}\sqrt{n}(\hat{\theta} - \theta),$$

$$(1/\sqrt{n})\sum_{t=1}^{n} \varepsilon_t P_{t,n}^{(m)} = n\,\hat{\alpha}(1/(n\sqrt{n}))\sum_{t=1}^{n} S_n((t-1)/n)P_{t,n}^{(m)}$$

$$+ (1/n)\sum_{t=1}^{n} P_{t,n}^{(m)}U_{t-1}^T\sqrt{n}(\hat{\phi} - \phi) + \sqrt{n}(\hat{\theta} - \theta).$$

Using the fact that by Lemma A.8,

$$(1/(n\sqrt{n}))\sum_{t=1}^{n} S_n((t-1)/n)U_{t-1} = O_p(1/\sqrt{n}), \qquad \textbf{(A.20)}$$

and that by Theorem A.1,

$$(1/n)\sum_{t=1}^{n} U_{t-1}P_{t,n}^{(m)\,T} = O_p(1/\sqrt{n}), \qquad \textbf{(A.21)}$$

we can simplify these normal equations to:

$$(1/n)\sum_{t=1}^{n} \varepsilon_t S_n((t-1)/n) \;=\; n\,\hat{\alpha}\,(1/n^2)\sum_{t=1}^{n} S_n((t-1)/n)^2$$

$$+\;(1/(n\sqrt{n}))\sum_{t=1}^{n} S_n((t-1)/n)P_{t,n}^{(m)\,T}\sqrt{n}(\hat{\theta} - \theta) + O_p(1/\sqrt{n}), \qquad \textbf{(A.22)}$$

$$(1/\sqrt{n})\sum_{t=1}^{n} \varepsilon_t U_{t-1} \;=\; (1/n)\sum_{t=1}^{n} U_{t-1}U_{t-1}^T\sqrt{n}(\hat{\phi} - \phi) + O_p(1/\sqrt{n}), \qquad \textbf{(A.23)}$$

$$(1/\sqrt{n})\sum_{t=1}^{n} \varepsilon_t P_{t,n}^{(m)} \;=\; n\,\hat{\alpha}(1/(n\sqrt{n}))\sum_{t=1}^{n} S_n((t-1)/n)P_{t,n}^{(m)}$$

$$+\;\sqrt{n}(\hat{\theta} - \theta) + O_p(1/\sqrt{n}). \qquad \textbf{(A.24)}$$

Substituting (A.6) in (A.4) yields

$$(1/n)\sum_{t=1}^{n} \varepsilon_t S_n((t-1)/n) - [(1/(n\sqrt{n}))\sum_{t=1}^{n} S_n((t-1)/n)P_{t,n}^{(m)}]^T[(1/\sqrt{n})\sum_{t=1}^{n} \varepsilon_t P_{t,n}^{(m)}]$$

$$= n\,\hat{\alpha}\left[(1/n^2)\sum_{t=1}^{n} S_n((t-1)/n)^2 - \|(1/(n\sqrt{n}))\sum_{t=1}^{n} S_n((t-1)/n)P_{t,n}^{(m)\,T}\|^2\right] \qquad \textbf{(A.25)}$$

$$+\;O_p(1/\sqrt{n}).$$

Since similarly to Theorem A.1,

$$(1/\sqrt{n})\sum_{t=1}^{n} \varepsilon_t P_{t,n}^{(m)} \Rightarrow \sigma_\varepsilon X^{(m)}, \qquad \textbf{(A.26)}$$

it follows from (A.7), Theorems A.1-2 and Lemmas A.7-8 that

$$n\hat{\alpha} \Rightarrow \frac{\frac{1}{2}\phi(1)^{-1}\sigma_\varepsilon^2[W(1)^2 - 1] - \sigma\sigma_\varepsilon Y^{(m)\,T}X^{(m)}}{\sigma^2\left[\int_0^1 W(x)^2dx - Y^{(m)\,T}Y^{(m)}\right]}.$$

Hence, substituting $\sigma = \sigma_\varepsilon/\phi(1)$ in (A.9), the first part of Theorem 1 follows. Moreover, the second part of Theorem 1 follows from (A.5). Furthermore, the third part of Theorem 1 follows from (A.6), (A.8), Theorem A.2 and the first part of Theorem 1.

Finally, observe from model (A.1) that

$$\sum_{t=1}^{n}\hat{\varepsilon}_{m,t}^2 - \sum_{t=1}^{n}\varepsilon_t^2 = -2n\hat{\alpha}(1/n)\sum_{t=1}^{n}S_n((t-1)/n)\varepsilon_t - 2\sqrt{n}(\hat{\phi}-\phi)^T(1/\sqrt{n})\sum_{t=1}^{n}\varepsilon_t U_{t-1}$$

$$-2\sqrt{n}(\hat{\theta}-\theta)^T(1/\sqrt{n})\sum_{t=1}^{n}\varepsilon_t P_{t,n}^{(m)} + 2n\hat{\alpha}(1/(n\sqrt{n}))\sum_{t=1}^{n}U_{t-1}S_n((t-1)/n)$$

$$+ 2n\hat{\alpha}\sqrt{n}(\hat{\theta}-\theta)^T(1/(n\sqrt{n}))\sum_{t=1}^{n}S_n((t-1)/n)P_{t,n}^{(m)}$$

$$+ 2\sqrt{n}(\hat{\phi}-\phi)^T(1/n)\sum_{t=1}^{n}U_{t-1}P_{t,n}^{(m)\,T}\sqrt{n}(\hat{\theta}-\theta) + n^2\hat{\alpha}^2(1/n^2)\sum_{t=1}^{n}S_n((t-1)/n)^2$$

$$+ \sqrt{n}(\hat{\phi}-\phi)^T(1/n)\sum_{t=1}^{n}U_{t-1}U_{t-1}^T\sqrt{n}(\hat{\phi}-\phi) + \sqrt{n}(\hat{\theta}-\theta)^T\sqrt{n}(\hat{\theta}-\theta).$$

It follows from the proof of the second part of Theorem 1 that

$$\sqrt{n}(\hat{\phi}-\phi)^T(1/n)\sum_{t=1}^{n}U_{t-1}U_{t-1}^T\sqrt{n}(\hat{\phi}-\phi) - 2\sqrt{n}(\hat{\phi}-\phi)^T(1/\sqrt{n})\varepsilon_t U_{t-1}$$

$$= -\sqrt{n}(\hat{\phi}-\phi)^T(1/n)\sum_{t=1}^{n}U_{t-1}U_{t-1}^T\sqrt{n}(\hat{\phi}-\phi) + o_p(1) \Rightarrow -\sigma_\varepsilon^2 b_p^T b_p.$$

Using (A.2), (A.3), (A.8), Theorem A.2, Lemma A.8 and the first three parts of Theorem 1, part 5 of Theorem 1 follows. Part 4 follows easily from the other parts of Theorem 1, and the fact that the *t*-test is equivalent to the F-test of the null hypothesis $\alpha = 0$, i.e.,

$$\hat{F}_{1,n-p-m-1} = \frac{\sum_{t=1}^{n} \tilde{\varepsilon}_{m,t}^2 - \sum_{t=1}^{n} \hat{\varepsilon}_{m,t}^2}{\left(\sum_{t=1}^{n} \hat{\varepsilon}_{m,t}^2\right)/(n-p-m-1)}.$$

The rest of Theorem 1 follows easily by setting $a_m = t_m = 0$. Q.E.D.

Proof of Theorem 2: Follows straightforwardly from Theorem 1. Q.E.D.

Proof of Theorem 3: Since under H_0 the last m components of θ in model (3) are zero, it follows from Theorem 1 that

$$\sqrt{n}\hat{\theta}^{(i,m)} \Rightarrow \sigma_{\varepsilon}[X^{(i,m)} - a_m Y^{(i,m)}], \quad i = 1,2,\ldots,m. \tag{A.9}$$

and from Theorem A.2 that

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} z_t P_{t,n}^{(2,m)} \Rightarrow \sigma Y^{(2,m)}.$$

Hence, using Theorem 1 and (A.9), it follows that under H_0,

$$\sqrt{n}\hat{\alpha}\frac{1}{n}\sum_{t=1}^{n} z_{t-1}P_{t,n}^{(i,m)} + \sqrt{n}\hat{\theta}^{(i,m)} \Rightarrow \sigma_{\varepsilon}X^{(i,m)} \sim N_{m-i+1}(0,\sigma_{\varepsilon}^2 I_{m-i+1}),\ i = 1,2,\ldots,m.$$

Q.E.D.

Proof of Theorem 4: It follows from (2) that under H_1^L,

$$\theta^T = \left(\lambda_0 + \lambda_1(n+1)/2, \quad \lambda_1\sqrt{(n^2-1)/12}, \quad 0, \ 0, \ldots, 0\right).$$

Note that, since by the stationarity assumption all the roots of $\phi(L)$ lie outside the unit circle, $\phi(1)$ is always positive, hence $\alpha^* < 0$. Denoting

$$\theta^{(i,m)} = (\theta_i, \ldots.., \theta_m)^T$$

it is now easy to verify from these results and Theorem A.1 that under H_1^L,

$$\sqrt{n}(\hat{\alpha} - \alpha^*) = O_p(1), \ where \ \alpha^* < 0,$$

$$\sqrt{n}(\hat{\theta}^{(i,m)} - \theta^{(i,m)}) \Rightarrow \sigma_\varepsilon X^{(i,m)}, \ i = 0, 1, \ldots., m,$$

$$\sqrt{n}[(1/n)\sum_{t=1}^{n} z_t P_{t,n}^{(i,m)} - \theta^{(i,m)}] \Rightarrow -\frac{\sigma_\varepsilon}{\alpha^*} X^{(i,m)},$$

$$\text{plim}_{n\to\infty}\hat{\theta}^{(1,m)}/n = (\lambda_1/\sqrt{12}, \ 0, 0, \ldots., 0)^T,$$

and

$$\text{plim}_{n\to\infty} s^2 = \sigma_\varepsilon^2.$$

Q.E.D.

Proof of Theorem 5: This theorem follows straightforwardly from (9) through (12), which will be proved below:

Proof of (9): Since

$$n[\cos(k\pi(t + 1 - 0.5)/n) - \cos((k\pi(t - 0.5)/n)]$$
$$= \frac{\cos(k\pi/n) - 1}{1/n}\cos(k\pi(t - 0.5)/n)$$
$$- \frac{\sin(k\pi/n)}{1/n}\sin(k\pi(t - .5)/n)$$
$$= -k\pi\sin(k\pi(t - 0.5)/n) + o(1),$$

uniformly in $t = 1,\ldots,n$, it follows similarly to Theorem A.1 that

$$\sqrt{n}\sum_{t=1}^{n} u_t\left(\cos\left(k\pi(t-0.5)/n\right) - \cos\left(k\pi(t+1-0.5)/n\right)\right)$$

$$= k\pi(1/\sqrt{n})\sum_{t=1}^{n} u_t \sin\left(k\pi(t-0.5)/n\right) + o_p(1)$$

$$\Rightarrow -\sigma(k\pi)^2\int_0^1 \cos(k\pi x)W(x)dx\,,$$

hence

$$\sum_{t=1}^{n} u_t\left(P_{k,n}^*(t) - P_{k,n}^*(t+1)\right) = O_p(1/\sqrt{n}).$$

It follows now from

$$\begin{pmatrix}\hat{\xi}_1\\ \hat{\xi}_2\end{pmatrix} = \begin{pmatrix}\sum_{k=1}^{m} P_{k,n}^*(n+1)^2 & \sum_{k=1}^{m} P_{k,n}^*(n+1)P_{k,n}^*(1)\\ \sum_{k=1}^{m} P_{k,n}^*(n+1)P_{k,n}^*(1) & \sum_{k=1}^{m} P_{k,n}^*(1)^2\end{pmatrix}^{-1} \times \begin{pmatrix}\sum_{k=1}^{m} P_{k,n}^*(n+1)\sum_{t=1}^{n}\Delta z_t P_{k,n}^*(t)\\ \sum_{k=1}^{m} P_{k,n}^*(1)\sum_{t=1}^{n}\Delta z_t P_{k,n}^*(t)\end{pmatrix}$$

that

$$\begin{pmatrix}\hat{\xi}_1\\ \hat{\xi}_2\end{pmatrix} = \begin{pmatrix}u_n\\ -u_0\end{pmatrix} + O_p(1/\sqrt{n}),$$

hence (9) follows.

Proof of (10): It follows from Theorem A.1 that

$$(1/\sqrt{n})\sum_{t=1}^{n} \Delta z_t P_{t,n}^{(1,m)} \Rightarrow \sigma X^{(1,m)}.$$

Moreover, it follows from (2) that

$$\lim_{n\to\infty} P_{1,n}^{*}(1) = -\lim_{n\to\infty} P_{1,n}^{*}(n+1) = -2\sqrt{3},$$

$$\lim_{n\to\infty} P_{2k-1,n}^{*}(1) = \lim_{n\to\infty} P_{2k-1,n}^{*}(n+1) = \sqrt{2}, \ (k > 1),$$

$$\lim_{n\to\infty} P_{2k,n}^{*}(1) = -\lim_{n\to\infty} P_{2k,n}^{*}(n+1) = \sqrt{2}\,\frac{1 - \alpha_k - \sum_{j=1}^{k-1} \beta_k - \gamma_k}{c_k}.$$

$$(k \geq 1)$$

Thus, denoting

$$\varphi_{1,m} = \lim_{n\to\infty} P_{n+1,n}^{(1,m)}, \quad \varphi_{2,m} = \lim_{n\to\infty} P_{1,n}^{(1,m)},$$

we have

$$\frac{1}{\sqrt{n}}\begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \varphi_{1,m}^T\varphi_{1,m} & \varphi_{1,m}^T\varphi_{2,m} \\ \varphi_{2,m}^T\varphi_{1,m} & \varphi_{2,m}^T\varphi_{2,m} \end{pmatrix}^{-1} \begin{pmatrix} \varphi_{1,m}^T \\ \varphi_{2,m}^T \end{pmatrix} \sigma X^{(1,m)}$$

hence (10) follows, with

$$\Sigma_m = I_m - (\varphi_{1,m}, \varphi_{2,m}) \begin{pmatrix} \varphi_{1,m}^T\varphi_{1,m} & \varphi_{1,m}^T\varphi_{2,m} \\ \varphi_{2,m}^T\varphi_{1,m} & \varphi_{2,m}^T\varphi_{2,m} \end{pmatrix}^{-1} \begin{pmatrix} \varphi_{1,m}^T \\ \varphi_{2,m}^T \end{pmatrix}.$$

Note that the matrix Σ_m is idempotent, with rank $m-2$.

Proof of (11) *and* (12): These results follow easily from Theorems A.1-2. Q.E.D.

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Table 1: *Fractiles of a_m:* $P(a_m \leq a) = p$

p:	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99
m:					a						
1	-29.7	-25.4	-22.0	-18.3	-13.5	-9.0	-5.9	-3.8	-2.7	-1.8	-1.0
2	-37.0	-31.6	-27.2	-23.0	-17.0	-11.6	-7.1	-4.1	-2.6	-1.4	-0.0
3	-46.2	-40.4	-35.6	-30.9	-24.5	-18.3	-13.6	-10.2	-8.5	-7.3	-6.1
4	-52.2	-46.5	-41.6	-36.3	-28.8	-21.8	-16.2	-11.7	-9.3	-7.4	-5.4
5	-61.1	-53.8	-48.7	-43.4	-35.6	-28.1	-21.9	-17.5	-15.1	-13.5	-11.8
6	-66.9	-60.1	-54.7	-49.1	-40.4	-32.3	-24.9	-19.5	-16.8	-14.4	-12.0
7	-74.8	-67.5	-61.8	-55.8	-46.9	-38.0	-30.8	-25.1	-22.3	-20.0	-17.3
8	-80.6	-73.5	-67.9	-61.7	-52.0	-42.4	-34.4	-27.9	-24.5	-21.9	-18.7
9	-88.9	-80.2	-74.4	-67.7	-58.2	-48.3	-39.9	-33.1	-29.8	-27.0	-24.0
10	-94.2	-87.0	-80.3	-73.7	-63.3	-52.8	-43.8	-36.6	-32.6	-29.6	-25.7
11	-101.8	-93.9	-87.1	-80.0	-69.3	-58.7	-48.9	-41.6	-37.6	-34.4	-30.9
12	-109.9	-100.5	-93.7	-85.7	-74.4	-63.2	-53.2	-45.0	-40.7	-37.3	-33.0
13	-117.0	-107.3	-100.0	-92.1	-80.3	-68.7	-58.4	-50.1	-45.6	-42.0	-38.1
14	-124.2	-113.8	-106.0	-98.0	-85.9	-73.5	-62.4	-53.7	-48.7	-44.8	-40.7
15	-130.2	-120.3	-112.4	-104.0	-91.8	-79.3	-67.6	-58.5	-53.8	-49.5	-45.4
16	-136.9	-127.2	-118.9	-109.7	-97.2	-83.8	-71.9	-62.2	-57.1	-52.4	-47.6
17	-144.9	-134.5	-125.5	-116.3	-102.6	-89.3	-77.2	-67.4	-61.8	-57.4	-51.8
18	-151.2	-140.9	-132.2	-122.8	-108.2	-94.5	-81.5	-71.1	-65.4	-60.5	-54.5
19	-157.5	-148.2	-139.9	-129.6	-114.3	-100.1	-87.1	-76.3	-70.5	-65.3	-59.8
20	-164.1	-153.9	-145.7	-135.6	-120.0	-105.0	-91.4	-80.5	-74.2	-69.0	-62.6

Table 2: *Fractiles of t_m: $P(t_m \leq t) = p$*

p:	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99
m:						t					
1	-4.20	-3.80	-3.52	-3.21	-2.69	-2.19	-1.69	-1.24	-0.93	-0.67	-0.32
2	-4.61	-4.28	-3.97	-3.64	-3.09	-2.49	-1.85	-1.20	-0.82	-0.47	-0.03
3	-5.01	-4.68	-4.43	-4.09	-3.60	-3.08	-2.59	-2.18	-1.94	-1.72	-1.47
4	-5.47	-5.08	-4.80	-4.49	-3.95	-3.38	-2.81	-2.26	-1.91	-1.59	-1.25
5	-5.82	-5.43	-5.16	-4.83	-4.32	-3.80	-3.32	-2.88	-2.63	-2.43	-2.19
6	-6.19	-5.80	-5.49	-5.16	-4.64	-4.09	-3.53	-3.03	-2.69	-2.42	-2.02
7	-6.47	-6.09	-5.78	-5.46	-4.95	-4.42	-3.92	-3.48	-3.21	-3.00	-2.72
8	-6.85	-6.41	-6.08	-5.74	-5.24	-4.67	-4.12	-3.65	-3.33	-3.05	-2.70
9	-7.10	-6.68	-6.35	-6.01	-5.49	-4.96	-4.45	-4.00	-3.76	-3.54	-3.20
10	-7.42	-6.98	-6.67	-6.29	-5.75	-5.20	-4.66	-4.17	-3.86	-3.58	-3.25
11	-7.74	-7.24	-6.89	-6.56	-6.02	-5.46	-4.95	-4.49	-4.22	-3.98	-3.65
12	-8.00	-7.51	-7.15	-6.82	-6.26	-5.70	-5.15	-4.64	-4.35	-4.05	-3.70
13	-8.24	-7.81	-7.42	-7.03	-6.50	-5.94	-5.41	-4.92	-4.63	-4.40	-4.09
14	-8.48	-8.08	-7.71	-7.30	-6.75	-6.16	-5.58	-5.08	-4.76	-4.50	-4.13
15	-8.75	-8.28	-7.89	-7.52	-6.96	-6.39	-5.83	-5.33	-5.03	-4.76	-4.41
16	-9.02	-8.51	-8.13	-7.74	-7.18	-6.58	-6.01	-5.46	-5.14	-4.84	-4.43
17	-9.27	-8.71	-8.35	-7.95	-7.38	-6.80	-6.25	-5.72	-5.40	-5.09	-4.74
18	-9.48	-8.95	-8.57	-8.16	-7.59	-7.00	-6.43	-5.88	-5.53	-5.21	-4.74
19	-9.70	-9.18	-8.77	-8.38	-7.82	-7.22	-6.64	-6.10	-5.78	-5.49	-5.13
20	-9.93	-9.40	-9.00	-8.60	-8.01	-7.41	-6.82	-6.24	-5.89	-5.60	-5.20

Table 3: *Fractiles of* F_m: $P(F_m \leq F) = p$

p:	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99
m:						F					
1	0.70	0.89	1.08	1.36	1.95	2.89	4.09	5.47	6.49	7.58	8.70
2	0.71	0.90	1.08	1.36	1.91	2.73	3.75	4.88	5.68	6.51	7.37
3	1.13	1.35	1.56	1.85	2.37	3.11	4.01	5.01	5.69	6.36	7.22
4	1.17	1.39	1.60	1.86	2.37	3.06	3.88	4.78	5.38	5.91	6.68
5	1.40	1.64	1.83	2.08	2.58	3.22	3.98	4.80	5.34	5.83	6.60
6	1.44	1.67	1.87	2.11	2.58	3.19	3.92	4.69	5.17	5.68	6.40
7	1.56	1.80	1.99	2.24	2.69	3.28	3.96	4.70	5.17	5.70	6.30
8	1.63	1.83	2.02	2.28	2.70	3.27	3.94	4.62	5.12	5.64	6.28
9	1.70	1.91	2.12	2.34	2.78	3.35	3.98	4.66	5.14	5.61	6.28
10	1.73	1.95	2.15	2.36	2.80	3.34	3.97	4.60	5.06	5.53	6.16
11	1.78	2.03	2.21	2.44	2.87	3.40	3.99	4.60	5.05	5.50	6.07
12	1.82	2.06	2.24	2.48	2.89	3.41	3.98	4.59	5.02	5.45	5.96
13	1.86	2.09	2.28	2.52	2.93	3.43	4.01	4.63	5.04	5.44	6.02
14	1.90	2.11	2.31	2.55	2.94	3.44	4.00	4.60	5.02	5.42	6.03
15	1.95	2.18	2.36	2.58	2.99	3.47	4.03	4.63	5.01	5.41	5.95
16	1.97	2.20	2.39	2.61	3.00	3.48	4.03	4.60	5.00	5.37	5.92
17	2.00	2.25	2.43	2.65	3.05	3.51	4.05	4.62	5.01	5.38	5.90
18	2.04	2.27	2.45	2.67	3.06	3.52	4.06	4.62	4.99	5.35	5.90
19	2.09	2.32	2.49	2.71	3.10	3.55	4.07	4.60	4.97	5.34	5.90
20	2.08	2.31	2.52	2.73	3.10	3.56	4.07	4.60	4.98	5.34	5.84

Table 4: *Fractiltes of* T_m: $P(T_m \leq T) = p$

p:	0.01	0.025	0.05	0.1	0.25	0.5	0.75	0.9	0.95	0.975	0.99
m:						T					
3	0.1	0.3	0.5	1.1	3.3	11.7	38.8	110.0	184.5	281.4	451.6
4	0.5	0.9	1.7	3.3	10.1	33.2	111.8	297.5	532.9	797.6	1245.1
5	0.9	1.6	2.6	4.4	11.4	35.1	121.0	313.9	528.0	817.7	1262.1
6	1.5	2.7	4.3	7.2	18.8	58.9	203.3	532.2	918.9	1420.7	2244.7
7	2.2	3.3	5.1	8.1	20.0	61.4	220.0	581.8	999.4	1473.1	2264.0
8	3.1	4.7	7.0	11.3	27.3	87.0	317.6	843.3	1473.1	2261.4	3363.0
9	3.6	5.4	7.6	11.9	28.5	90.0	335.0	919.3	1535.9	2404.1	3729.5
10	4.4	6.7	9.5	15.0	35.7	115.2	438.4	1208.7	2114.5	3204.3	4980.0
11	5.1	7.1	10.1	15.6	37.0	117.3	454.4	1276.1	2243.4	3335.6	5508.1
12	5.9	8.6	12.2	18.9	44.4	141.5	557.7	1609.4	2807.0	4276.0	6856.6
13	6.5	9.2	12.7	19.8	45.4	146.3	579.5	1711.4	2980.1	4529.7	7479.6
14	7.4	10.6	15.0	23.0	53.0	168.5	688.9	2066.4	3621.5	5583.2	9048.4
15	7.7	11.0	15.3	23.6	54.1	171.7	711.9	2211.1	3861.7	5869.7	9632.4
16	8.6	12.7	17.6	26.9	61.3	196.9	823.0	2610.4	4522.0	7158.8	11615.9
17	9.3	13.0	17.9	27.5	62.7	197.7	837.4	2743.9	4726.5	7478.6	11846.1
18	10.2	14.6	20.0	30.5	70.3	223.9	948.1	3197.4	5528.9	8789.6	13822.1
19	10.4	14.9	20.4	31.0	70.8	225.3	970.4	3296.6	5665.5	8948.8	13879.7
20	11.4	16.4	22.6	34.2	78.5	249.5	1070.6	3761.2	6500.9	10150.1	15989.2

Remark: Table 4 is incorrect! See below for the correct version.

Table 4: *Corrected quantiles of the T-Tilde test*: $p = P(\tilde{T} \le T)$

p	0.010	0.025	0.050	0.100	0.250	0.500	0.750	0.900	0.950	0.975	0.990
m					T						
3	0.01	0.04	0.15	0.58	3.62	15.71	48.78	103.75	155.08	209.93	284.91
4	1.07	2.68	5.36	10.80	27.53	59.70	103.93	159.87	214.42	282.55	390.48
5	6.08	11.29	18.60	30.15	65.07	129.69	234.56	379.62	497.70	620.71	782.89
6	21.56	32.67	47.80	71.16	124.26	209.21	322.36	480.96	602.21	734.87	924.56
7	39.11	57.84	80.73	114.81	192.37	320.27	508.41	748.54	915.73	1087.85	1319.85
8	73.70	109.04	143.69	190.79	293.17	444.19	653.47	904.25	1085.57	1293.19	1523.59
9	110.93	149.86	195.59	256.55	388.99	601.55	877.40	1211.77	1469.01	1724.69	2072.58
10	157.62	223.45	280.57	359.51	516.41	755.40	1064.98	1408.65	1660.07	1930.47	2267.04
11	218.67	291.15	359.89	455.50	658.00	962.58	1362.23	1802.19	2122.98	2408.68	2804.26
12	310.71	391.49	481.32	596.44	824.46	1143.02	1560.35	2054.16	2383.19	2718.76	3157.06
13	382.70	482.13	587.83	725.89	995.85	1389.86	1886.60	2468.23	2873.84	3303.38	3823.09
14	505.88	618.95	732.07	893.25	1201.66	1623.76	2179.03	2789.25	3223.40	3586.08	4115.12
15	615.53	743.90	881.36	1060.23	1408.48	1908.20	2535.01	3223.73	3702.54	4139.92	4811.65
16	719.69	866.19	1021.36	1221.82	1595.50	2153.62	2821.49	3558.54	4049.82	4532.37	5146.51
17	854.40	1016.91	1191.48	1417.48	1858.41	2475.33	3222.22	4074.68	4694.55	5219.46	5827.69
18	993.78	1210.53	1399.42	1638.15	2106.64	2779.70	3589.90	4506.56	5088.56	5692.19	6446.98
19	1162.85	1360.53	1581.29	1852.69	2378.25	3123.31	4020.76	5012.44	5688.82	6352.98	7068.35
20	1321.74	1557.10	1789.70	2111.24	2694.86	3495.29	4423.32	5485.28	6200.62	6877.68	7778.51

Table 5: *Unit root and trend stationarity test results for the price levels*

Test	Test statistics								
	LNDEF(100)		LNCPI(129)		LNCPI(100)		Critical regions		H_0 H_1
		Δz_t	z_t	Δz_t	z_t	Δz_t	5%	10%	
P		-50.89	0.69	-44.67	0.65	-31.89	< -8.00	< -5.70	UR ST
PP(1)	z_t	-52.76	2.29	-47.17	1.75	-34.57		<-11.20	UR ST
PP(2)	0.68	-52.76	-0.79	-40.53	-3.88	-34.90	<-14.00	<-18.10	UR TS
ADF(1)	1.43	-3.09	-0.93	-4.12	-0.84	-3.22	<-21.50	< -3.21	UR TS
ADF(2)	-5.12	-6.00	-1.01	-4.12	-1.73	-5.57	< -3.52	< -3.21	UR TS
HOAC(1,1)	-1.23	-9801	-16641	-11.06	-10000	-9801	< -3.52	<-11.20	UR ST
HOAC(2,2)	-1.59	-9801	-6.15	-12.99	-4.44	-9801	<-14.00	<-13.10	UR ST
DHOAC(1,1)	-10000	-9801	-1.68	-14.80	-3.21	-9801	<-15.70	<-17.10	UR TS
DHOAC(2,2)	-3.85	-9601	-8.49	-16.11	-11.12	-9801	<-20.60	<-18.90	UR TS
C(1)	-3.24	0.08	51.24	1.47	59.80	0.21	<-22.40	> 6.31	ST UR
C(2)	-9.71	0.08	71.10	1.47	79.33	0.21	> 12.71	> 6.31	ST UR
C(3)	62.54	0.09	54.85	1.55	107.09	0.25	> 12.71	> 6.31	ST UR
C(4)	89.62	0.09	25.64	1.58	41.46	0.26	> 12.71	> 6.31	ST UR
C(5)	125.46	0.04	16.69	0.42	47.92	0.14	> 12.71	> 6.31	TS UR
C(6)	51.56	0.26	20.67	1.53	8.74	0.66	> 12.71	> 6.31	TS UR
	36.05						> 12.71		
	7.46								

P = Phillips test; PP(i) = Phillips-Perron test type i; ADF(i) = Augmented Dickey-Fuller test, type (i); (D)HOAC(i,i) = Bierens'(Detrended)Higher Order Autocorrelation Test type i.i; C(i) = Bierens-Guo's Cauchy tests, type i. UR = Unit root; ST = Stationarity; TS = Trend stationarity. The large negative values of the (D)HOAC tests are equal to $-n^2$. Cf. Bierens (1993).

Table 6: *Tests of the unit root with drift hypothesis against nonlinear trend stationarity for LNDEF*(100): *p = 1, m = 10*

		Fractiles of the asymptotic null distribution			
Test type	*Teststatictics*	0.05	0.10	0.90	0.95
$\hat{t}(m)$	-6.48	-6.67	-6.29	-4.17	-3.86
$\hat{A}(m)$	-91.6	-80.30	-73.70	-36.60	-32.60
$\hat{F}(m)$	6.25	2.15	2.36	4.60	5.06
$\hat{T}_1(m)$	0.75	3.94	4.87	16.00	18.30
$\hat{T}_2(m)$	0.75	3.33	4.17	14.70	16.90
$\tilde{T}(m)$	3364.2	9.5	15.0	1208.7	2114.5

Table 7: *Small sample pretest null distribution function F for LNDEF*(100)

$\hat{t}(m)$:	F(-6.48)	=	0.118
$\hat{A}(m)$:	F(-91.6)	=	0.264
$\hat{F}(m)$:	F(6.25)	=	0.903
$\hat{T}_1(m)$:	F(0.75)	=	0.141
$\hat{T}_2(m)$:	F(0.75)	=	0.160
$\tilde{T}(m)$:	F(3364.2)	=	0.933

Table 8: *Tests of the unit root with drift hypothesis against nonlinear trend stationarity for LNCPI*(129): *p = 5, m = 20*

		Fractiles of the asymptotic null distribution			
Test type	*Test statistics*	0.05	0.10	0.90	0.95
$\hat{t}(m)$	-8.58	-9.00	-8.60	-6.25	-5.89
$\hat{A}(m)$	111.8	-145.70	-135.60	-80.50	-74.20
$\hat{F}(m)$	9.20	2.52	2.73	4.60	4.98
$\hat{T}_1(m)$	2151.87	10.90	12.40	20.40	31.40
$\hat{T}_2(m)$	1322.48	10.10	11.70	22.20	30.10
$\tilde{T}(m)$	8724.1	22.6	34.2	3761.2	6500.9

Table 9: *Small sample pretest null distribution function F for LNCPI*(129)

$\hat{t}(m)$:	F(-8.58) = 0.380
$\hat{A}(m)$:	F(111.83) = 0.648
$\hat{F}(m)$:	F(9.20) = 0.883
$\hat{T}_1(m)$:	F(2151.87) = 0.968
$\hat{T}_2(m)$	F(1322.48) = 0.953
$\tilde{T}(m)$:	F(8724.12) = 0.291

Table 10: *Tests of the unit root with drift hypothesis against nonlinear trend stationarity for LNCPI*(100): *p = 2, m = 10*

		Fractiles of the asymptotic null distribution			
Test type	*Test statistics*	0.05	0.10	0.90	0.95
$\hat{t}(m)$	-6.59	-6.67	-6.29	-4.17	-3.86
$\hat{A}(m)$	-125.8	-80.30	-73.70	-36.60	-32.60
$\hat{F}(m)$	7.42	2.15	2.36	4.60	5.06
$\hat{T}_1(m)$	3.25	3.94	4.87	16.00	18.30
$\hat{T}_2(m)$	2.77	3.33	4.17	14.70	16.90
$\tilde{T}(m)$	3954.2	9.5	15.0	1208.7	2114.5

Table 11: *Small sample pretest null distribution function F for LNCPI*(100)

$\hat{t}(m)$:	F(-6.59) = 0.080
$\hat{A}(m)$:	F(-125.82) = 0.240
$\hat{F}(m)$:	F(7.42) = 0.945
$\hat{T}_1(m)$:	F(3.25) = 0.524
$\hat{T}_2(m)$:	F(2.77) = 0.510
$\tilde{T}(m)$:	F(3954.21) = 0.947

Table 12: *Unit root and trend stationarity test results for the interest rates*

Test	Test statistics		Critical regions		H_o H_1
	NINT	RINT	5%	10%	
	z_t	z_t			
P	0.7	-44.06	< -8.00	< -5.70	UR ST
PP(1)	9-1.52	-43.74	<-14.00	<-11.20	UR ST
PP(2)	-4.39	-43.94	<-21.50	<-18.10	UR TS
ADF(1)	-0.86	-2.72	< -3.52	< -3.21	UR TS
ADF(2)	-0.86	-5.29	< -3.52	< -3.21	UR TS
HOAC(1,1)	-7921	-7921	<-14.00	<-11.20	UR ST
HOAC(2,2)	-16.32	-7921	<-15.70	<-13.10	UR ST
DHOAC(1,	-1.80	-7921	<-20.60	<-17.10	UR TS
1)	-27.37	-7921	<-22.40	<-18.90	UR TS
DHOAC(2,	112.42	13.73	> 12.71	> 6.31	ST UR
2)	86.44	13.73	> 12.71	> 6.31	ST UR
C(1)	5.12	13.74	> 12.71	> 6.31	ST UR
C(2)	5.03	13.74	> 12.71	> 6.31	ST UR
C(3)	24.49	0.36	> 12.71	> 6.31	TS UR
C(4)	4.64	0.73	> 12.71	> 6.31	TS UR
C(5)					
C(6)					

Table 13: *Tests of the unit root with drift hypothesis against nonlinear trend stationarity for NINT: p = 2, m = 10*

		Fractiles of the asymptotic null distribution			
Test type	*Teststatistics*	0.05	0.10	0.90	0.95
$\hat{t}(m)$	-3.99	-6.67	-6.29	-4.17	-3.86
$\hat{A}(m)$	-84.5	-80.30	-73.70	-36.60	-32.60
$\hat{F}(m)$	2.87	2.15	2.36	4.60	5.06
$\hat{T}_1(m)$	1.73	3.94	4.87	16.00	18.30
$\hat{T}_2(m)$	1.68	3.33	4.17	14.70	16.90
$\tilde{T}(m)$	703.4	9.5	15.0	1208.7	2114.5

Table 14: *Small sample pretest null distribution function F for NINT*

$\hat{t}(m)$:	F(-3.99) = 0.978
$\hat{A}(m)$:	F(-84.54) = 0.232
$\hat{F}(m)$:	F(2.87) = 0.118
$\hat{T}_1(m)$:	F(1.73) = 0.794
$\hat{T}_2(m)$:	F(1.68) = 0.802
$\tilde{T}(m)$:	F(703.41) = 0.996