Problem set 10

Due Dec. 1, 2014

NOT GRADED 1. For functions $f(x)$ defined on the interval $0 < x < L$, there is a standard orthogonal basis of functions $e_m(x) = e^{ik_m x}$, where $k_m = 2\pi m / L$. From these construct an orthogonal basis of functions proportional to $\cos kx$ and $\sin kx$, for suitable values of $k$; these are often more convenient than the complex exponentials in constructing Fourier series for real-valued functions. What values of $m$ should be used to ensure that the set of functions is a basis? Derive the orthogonality properties. (N.B. Define orthogonality with respect to the inner product $\langle f | g \rangle = \int_0^L f(x)^* g(x) \, dx$.)

NOT GRADED 2. With the same set up as in Prob. 1, derive the values of the coefficients of the expansion of $f$ in terms of the inner products of $f$ with the cos and sin basis functions. Give the explicit values of any normalization constants that are needed.

GRADED 3. With the same set up as in Prob. 1 and using the appropriate cos and sin basis, calculate a Fourier series representation for the following function defined on $0 < x < 2\pi$:

$$f(x) = \begin{cases} x, & 0 < x < \pi, \\ 2\pi - x, & \pi < x < 2\pi. \end{cases}$$

(1)

Certain terms in the series are zero; explain why.

Mathematica: You may use Mathematica (or similar software) to carry out the integrals in this problem. If you do, you should attach a printout to show how the calculation was done.

GRADED 4. Consider the Fourier transform of a function $f$.

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx.$$ (2)

Assume that $f(x)$ is piecewise differentiable, and is continuously differentiable except at isolated points, and falls off more rapidly than $1/x^2$ as $x \to \pm\infty$.

(a) Show that if $f(x)$ has a discontinuity, then $\tilde{f}(k)$ behaves like $1/k$ at large $|k|$, and quantitatively determine the contribution to the large $1/k$ behavior in terms of the size and position of the discontinuity.

(b) If $\tilde{f}$ behaves like $3e^{5ik}/k$ at large $k$, what can you deduce about the discontinuity/ies of $f(x)$?
GRADED 5. In a resonant cavity an electromagnetic oscillation of frequency $\omega_0$ dies out
as
$$A(t) = \begin{cases} A_0 e^{-\omega_0 t/2Q} \sin(\omega_0 t) & \text{if } t > 0, \\ 0 & \text{if } t < 0. \end{cases}$$

(3)

The dimensionless parameter $Q$ is a standard measure of the ratio of stored energy to energy
loss per cycle. Calculate the frequency distribution of the oscillation $|\tilde{A}(\omega)|^2$, where $\tilde{A}(\omega)$ is
the Fourier transform of $A(t)$. How does this distribution behave when the parameter $Q$ is
large? (Suggestion: Sketch a graph to show the qualitative features of $|\tilde{A}(\omega)|^2$ as a function
of $\omega$.)