Problem set 8 (updated, and corrected)

Due Nov. 3, 2014

1. A uniform thin disk has radius $R$ and mass $M$. The mass is uniformly distributed on the disk. You should neglect the thickness of the disk.

Compute its moment of inertia tensor (about the center of the disk) in Cartesian coordinates, with the $z$ axis being the axis of the disk and the $x$ and $y$ axes being in the plane of the disk.

2. A uniform spherical object has radius $R$ and mass $M$. The mass is uniformly distributed inside the sphere. Compute its moment of inertia tensor (about the center of the sphere) in Cartesian coordinates. You should find it is of the form $I^{ij} = I \delta^{ij}$, i.e., that it is proportional to the Kronecker delta.

3. Consider the disk of Prob. 1. But now transform $I^{ij}$ as follows. Let coordinates relative to the disk now be called $(u, v, w)$, with the axis of the disk being in the $w$ direction.

The axis of the disk is rotated with respect to $(x, y, z)$ coordinates by an angle $\theta$ with respect to the $z$ axis, the rotation being in the $x$-$z$ plane. Thus a point $(u, v, w)$ relative to the disk has $(x, y, z) = (u \cos \theta + w \sin \theta, v, -u \sin \theta + w \cos \theta)$.

Find the moment-of-inertia tensor of the disk in $(x, y, z)$ coordinates.

Correction here about the axis of rotation: Suppose now that the disk is rotating about the $z$ axis with constant angular velocity $\omega$, with above orientation being true at time $t = 0$. What is the angular momentum of the disk at $t = 0$?

At a later time $t$, there is a rotation of angle $\omega t$ (around the $z$ axis, of course). Find the matrix that rotates a vector by this amount, acting on coordinates in the $(x, y, z)$ frame. What is the resulting angular momentum at this time?

[N.B. All coordinate systems in this problem are right-handed and Cartesian.]

NOT GRADED 4. Let $C$ be a coordinate system, which is Cartesian and right-handed. Let position coordinates be written $(x^1, x^2, x^3)^T = (x, y, z)^T$.

Let $D$ be a different coordinate system, with position vectors written as $(u^1, u^2, u^3)^T = (u, v, w)^T$, which are given in terms of those in system $C$ by

$$u = x + y, \quad v = \frac{1}{2}y, \quad w = \frac{1}{2}z.$$  \hspace{1cm} (1)

(a) What is the transformation matrix $R$ used to transform contravariant vectors like $u^\alpha = R^\alpha_i x^i$?
(b) For a covariant vector, like the gradient of a scalar function, \( \nabla_i f = \partial f / \partial x^i \), what is the transformation law? I suggest denoting the gradient in system \( D \) by a notation like \( \nabla' \alpha f \), or \( \nabla^D \alpha f \), if you need to disambiguate notation.

(c) The metric tensor in system \( C \) is \( g^C_{ij} = \delta_{ij} \), since the coordinates are Cartesian. Obtain the metric tensor \( g^D_{\alpha\beta} \) in system \( D \).

(d) Find the contravariant metric tensor \( R^{(D)}_{\alpha\beta} \).

(e) Deduce the Laplacian \( \nabla^2 f \), of \( f \) in system \( D \). Express your result as a sum of terms with explicit derivatives, \( \partial^2 f / \partial u^2 \), \( \partial^2 f / \partial u \partial v \), etc.

(f) Find the contravariant version of the derivative \( \nabla^{(D)} \alpha f \). Give explicit formulae for each of its 3 components.

(g) Find the components of the standard antisymmetric tensor \( \epsilon^D_{\alpha\beta\gamma} \) in system \( D \), given that it has its standard value in system \( C \). (N.B. To save a lot of work, use the fact that general linear transformations preserve the symmetry and antisymmetry properties of such tensors.)