# FOURIER SERIES

We write $L^2([-\pi,\pi])$ for the set of functions $f : [-\pi,\pi] \to \mathbb{R}$ which are square-integrable, i.e. for which $\int_{-\pi}^{\pi} f(x)^2\,dx$ converges. Given an element of this set, you can think of it as giving a function on all of $\mathbb{R}$ with period $2\pi$ – just repeat the same function from $[\pi, 3\pi]$ that you used on $[-\pi,\pi]$, etc. Conversely, any function with period $2\pi$ can be thought of as a function on $[-\pi,\pi]$.

However you choose to think about it, $L^2([-\pi,\pi])$ is a vector space, we can define an inner product by setting

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

**Theorem 1.** *The functions*

$$\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x), \ldots$$

*are an orthonormal set in* $L^2([-\pi,\pi])$.

*Proof.* This was (more or less) a problem on Homework 10. You checked orthonormality by doing a bunch of integrals, probably using the product-to-sum formulas for sine and cosine more times than you have since you took precalculus. □

But something more is true:

**Theorem* 2.** *The functions*

$$\frac{1}{\sqrt{2}}, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x), \ldots$$

*are an orthonormal* basis *for* $L^2([-\pi,\pi])$.

The content of this theorem is that this list of functions actually spans: this means that any function with period $2\pi$ can be written as a linear combination of the sine and cosine functions appearing in the statement.

I've flagged the theorem with a "$*$" because it isn't really true. Most of the other theorems in these notes should get the same caution. There are a few difficulties here, but the most basic is this: for a set of vectors to be a basis, every element of the space has to be a finite linear combination of these vectors. In this case, we're going to need to take infinite sums.

To do that, we need to figure out how to make sense of infinite sums in a vector space. The value of a series is defined to be the limit of its partial sums, so we need to be able to take limits in our vector space, which in turn means we need to give the vector space a topology. . . this is a road I don't want to go down. Just trust me that with a little more foundational work, this all can be made rigorous.

The upshot is this: given any function $f(x)$ with period $2\pi$, we should be able to write it as a linear combination

$$\begin{aligned} f(x) &= a_0\left(\frac{1}{\sqrt{2}}\right) + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + a_3\cos(3x) + b_3\sin(3x) + \cdots \\ &= a_0\left(\frac{1}{\sqrt{2}}\right) + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx). \end{aligned}$$

The question is how to find the coefficients. But we already know how to do this, at least in the finite-dimensional case. If $V$ is a vector space with an orthonormal basis $e_1, \ldots, e_n$, then for any $v \in V$ we can simply write

$$v = \langle v, e_1\rangle\, e_1 + \cdots + \langle v, e_n\rangle\, e_n.$$

The coefficients are just the inner products. That lets us find the $a_j$'s and $b_k$'s in the expression above.

**Theorem 3.** *Let $f \in L^2([-\pi, \pi])$. Then*

$$f(x) = a_0\left(\frac{1}{\sqrt{2}}\right) + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx),$$

*where*

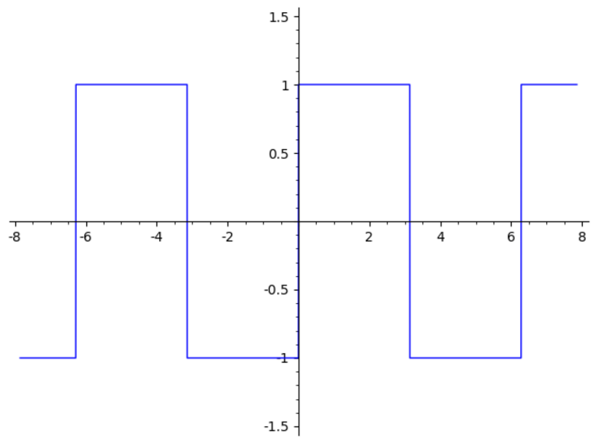
$$\begin{aligned} a_0 &= \left\langle f, \frac{1}{\sqrt{2}}\right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x)\,dx \\ a_n &= \langle f, \cos(nx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ b_n &= \langle f, \sin(nx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx. \end{aligned}$$

If you look at other references, you're likely to see a variety of normalizations. Make sure you're consistent in which formulas you use. The most common difference is that most people will set $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$, and just start with $f(x) = a_0 + \cdots$, to avoid the $\sqrt{2}$ factors.

Let's try this in an example. Consider the sawtooth wave, defined by

$$f(x) = \begin{cases} -1 & \text{if } -\pi \le x < 0, \\ 1 & \text{if } 0 \le x < 2\pi. \end{cases}$$

Here's a graph:




How to find the Fourier series? Well,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x)\,dx = 0,$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0.$$

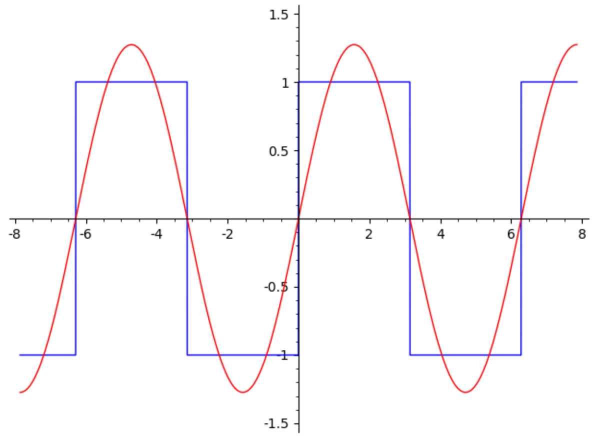
Those two integrals are easy: $f(x)$ is an odd function, which means the first is 0. The second is odd times even (cosine), which is odd. So again the integral is 0. If you didn't remember this, there's no harm in just doing the integral by hand. But you're going to get 0. The more interesting one is the $b_n$:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)\,dx \\ &= \frac{2}{\pi}\left(-\frac{1}{n}\cos(nx)\right)\Big|_0^{\pi} \\ &= \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Our formula is telling us that

$$f(x) = \frac{4}{\pi}\sin(x) + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x) + \frac{4}{7\pi}\sin(7x) + \cdots$$

Can this possibly be true? Well, here are some plots. The first one shows what we get with just the first term $\frac{4}{\pi}\sin(x)$. The later ones include more and more of the trig functions from the infinite sums.


1.5
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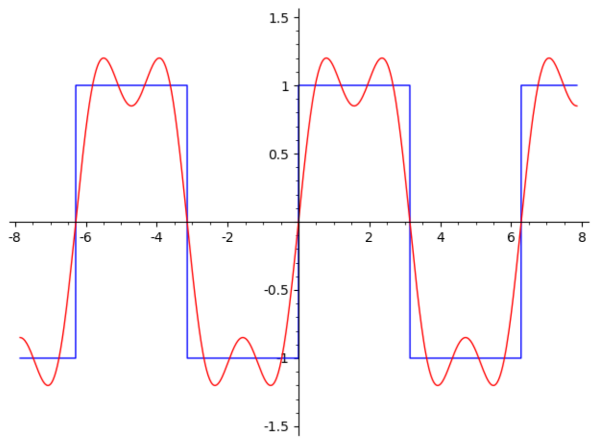
0.5
-8 -6 -4 -2 2 4 6 8
-0.5
-1
-1.5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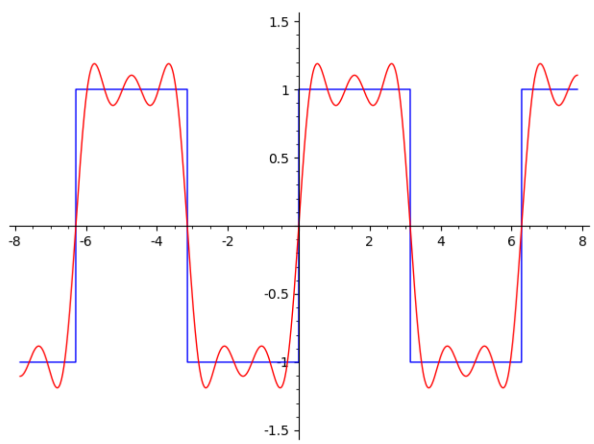
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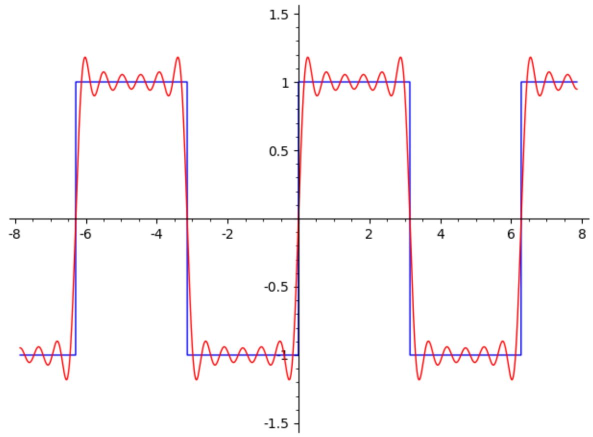
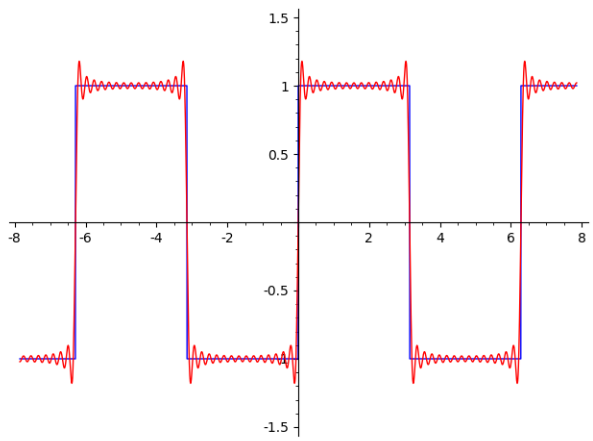
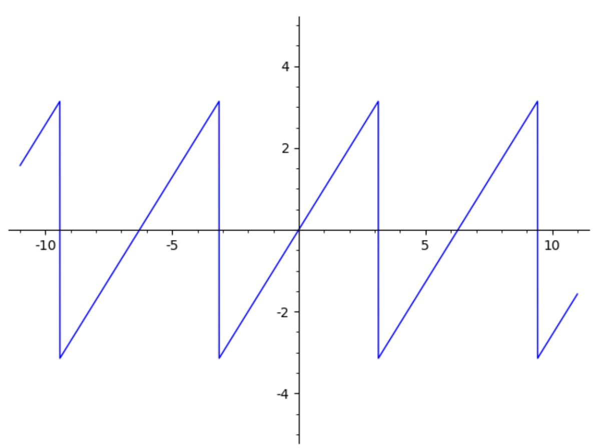



They do seem to be getting closer and closer to $f(x)$, as expected.

Let's try another. This time we'll use the sawtooth wave, defined by $f(x) = x$ between $-\pi$ and $\pi$.



To find the Fourier series, we again have to do some integrals.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f(x)\,dx = 0,$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0.$$

These are for the same reason as before: we're integrating an odd function from $-\pi$ to $\pi$, so we get 0.

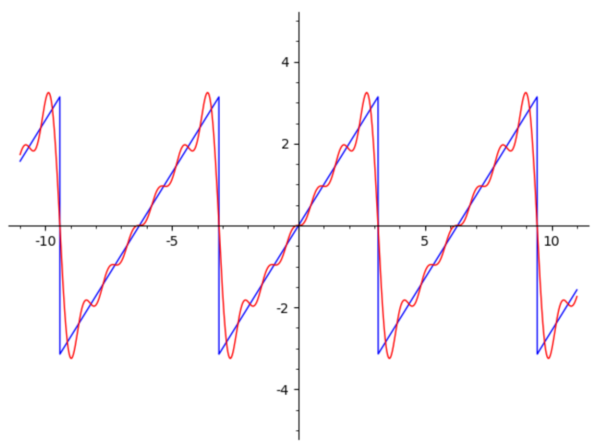
$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \cdots \\ &= \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd,} \\ -\frac{2}{n} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

(I omit the integration – use integration by parts.)

So

$$f(x) = \frac{\sin(x)}{1} - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \cdots.$$

Again, we can check that this works by plotting what we get if we just use a few of the terms. Here's one, including about 10 terms:




Here's another cool trick. We know that if we have an orthonormal basis for an inner product space, and $v = c_1e_1 + \cdots + c_ne_n$, then

$$\|v\|^2 = c_1^2 + \cdots + c_n^2.$$

This works for Fourier series as well:

$$\|f\|^2 = a_0^2 + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2.$$

What does this say for the sawtooth wave? Well,

$$\|f\|^2 = \langle f, f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\frac{2\pi^3}{3} = \frac{2\pi^2}{3}.$$

On the other hand, the sum of the squares of the coefficients is:

$$\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}.$$

The formula is therefore telling us that

$$\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

This remarkable identity is actually correct, and was first worked out by Euler. What you learned about series in calculus class shows that the series converges – use the comparison test with $\int_1^{\infty} \frac{1}{x^2}\,dx$ – but finding the exact value is a much harder problem.