Sample Space and Events

• *Sample space*: The set of all possible outcomes.
  
  1. Roll a die: \(\{1, 2, 3, 4, 5, 6\}\).
  2. Flip a coin twice: \(\{(H, H), (H, T), (T, H), (T, T)\}\).

• *Event*: A subset of the sample space.
  
  1. Roll a die: the outcome is even \(\{2, 4, 6\}\).
  2. Flip a coin twice and the two results are different: \(\{(H, T), (T, H)\}\).
• Set operations

1. Union: \( E \cup F \): an outcome is in \( E \cup F \) if it is either in \( E \) or in \( F \).

2. Intersection: \( E \cap F, EF \): an outcome is in \( EF \) if it is in both \( E \) and \( F \).

3. Mutually exclusive: \( E \) and \( F \) are mutually exclusive if \( EF = \emptyset \).

(a) Roll a die:
   \( E_1 \): outcome is below 3: \( \{1, 2\} \)
   \( E_2 \): outcome is above 4: \( \{5, 6\} \)
   \( E_1 \cap E_2 = \emptyset \).

4. Complement: \( E^c \): outcome that is not in \( E \).
Probabilities

- Sample space $S$, event $E$.
- The probability of event $E$ is a number $P(E)$ assigned to $E$ that satisfies the following conditions:
  1. $0 \leq P(E) \leq 1$.
  2. $P(S) = 1$.
  3. For any sequence of events $E_1$, $E_2$, ..., which are mutually exclusive, that is $E_nE_m = \phi$ for any $n \neq m$, $P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n)$. 
• Compute Probabilities:

1. Roll a fair die, 6 equally likely outcomes: \{1, 2, 3, 4, 5, 6\}
   \[ P(\{1\}) = \frac{1}{6}, \quad P(\{2\}) = \frac{1}{6}, \ldots, \quad P(\{6\}) = \frac{1}{6} \, . \]

2. \( E \): the outcome is even. \( P(E) = ? \):
   \[ P(E) = P(\{2, 4, 6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{2} \, . \]

• Properties:

1. \( P(E) + P(E^c) = 1 \).

2. Any event \( E \) and \( F \) (may not be mutually exclusive):
   \[ P(E \cup F) = P(E) + P(F) - P(EF) \]

Proof:
Let \( A = EF \), \( B = E - A \), \( C = F - A \). Then \( E = B \cup A \), \( F = C \cup A \), \( E \cup F = B \cup C \cup A \). Note \( A, B, C \) are mutually exclusive.

\[
\begin{align*}
P(E \cup F) & = P(B) + P(C) + P(A) \\
& = (P(B) + P(A)) + (P(C) + P(A)) - P(A) \\
& = P(E) + P(F) - P(EF)
\end{align*}
\]
Conditional Probability

• Given one event has occurred, what is the probability that another event occurs? Let $F$ be given, then the conditional probability of $E$ is $P(E \mid F)$.

• Example: Flip a coin twice.
  $S = \{(H, H), (H, T), (T, H), (T, T)\}$.
  $F$: the first flip is $H$.
  $F = \{(H, H), (H, T)\}$, $P(F) = 1/2$.
  $E$: the two flips are not both $H$.
  $E = \{(H, T), (T, T), (T, H)\}$, $P(E) = 3/4$.
  If $F$ occurs, in order for $E$ to occur, the second flip has to be $T$. Since the coin is fair $P(E \mid F) = 1/2$.

• Definition:
  
  $P(E \mid F) = \frac{P(EF)}{P(F)}$.

  Example: the above coin flip setup.

  $P(E \mid F) = \frac{P(EF)}{P(F)} = \frac{P(\{(H, T)\})}{P(F)} = \frac{1/4}{1/2} = 1/2$.
Independent Events

• Definition: Two events $E$ and $F$ are independent if $P(EF) = P(E)P(F)$.

• Equivalently: $P(E|F) = P(E)$, $P(F|E) = P(F)$.

• Distinguish independent and mutually exclusive:
  – $E$ and $F$ mutually exclusive: $EF = \emptyset$, $P(EF) = 0$. $P(E \cup F) = P(E) + P(F)$.

• Extension to $n$ events: $E_1, E_2, ..., E_n$ are independent if for any subset $E_1', E_2', ..., E_r'$, $r \leq n$,

\[ P(E_1'E_2' \cdots E_r') = \prod_{i=1}^{r} P(E_i') \]

• Independent trials: A sequence of experiments with results being either a “success” or a “failure”, and the experiments are independent.
Bayes Formula

• Total probability formula:
  Suppose events $F_1, F_2, ..., F_n$ are mutually exclusive and $\bigcup_{i=1}^{n} F_i = S$. Given any event $E$, we have

  \[ P(E) = \sum_{i=1}^{n} P(F_i | E) = \sum_{i=1}^{n} P(F_i) P(E | F_i) \]

• Bayes formula:
  Suppose events $F_1, F_2, ..., F_n$ are mutually exclusive and $\bigcup_{i=1}^{n} F_i = S$. Given any event $E$:

  \[ P(F_i | E) = \frac{P(F_i) P(E | F_i)}{\sum_{j=1}^{n} P(F_j) P(E | F_j)} \]