Conditional Probability

- Conditional probability: for events E and F:

$$P(E \mid F) = \frac{P(EF)}{P(F)}$$

- Conditional probability mass function (pmf)

$$\begin{aligned} p_{X|Y}(x \mid y) &= P\{X = x \mid Y = y\} \\ &= \frac{P\{X = x, Y = y\}}{P\{Y = y\}} \\ &= \frac{p(x, y)}{p_Y(y)} \end{aligned}$$

  defined for $y : p_Y(y) > 0$.

- Conditional expectation of X given $Y = y$

$$E[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

- If X and Y are independent, then $E[X \mid Y = y] = E[X]$.

Examples

1. Suppose the joint pmf of X and Y is given by $p(1,1) = 0.5$, $p(1,2) = 0.1$, $p(2,1) = 0.1$, $p(2,2) = 0.3$. Find the pmf of X given $Y = 1$.

   Solution:

$$p_{X|Y=1}(1) = p(1,1)/p_Y(1) = 0.5/0.6 = 5/6$$
$$p_{X|Y=1}(2) = p(2,1)/p_Y(1) = 0.1/0.6 = 1/6$$

2. If X and Y are independent Poisson RVs with respective means λ_1 and λ_2, find the conditional pmf of X given $X + Y = n$ and the conditional expected value of X given $X + Y = n$.

   Solution:
   Let $Z = X + Y$. We want to find $p_{X|Z=n}(k)$. For $k = 0, 1, 2, ..., n$

$$\begin{aligned} p_{X|Z=n}(k) &= \frac{P(X = k, Z = n)}{P(Z = n)} \\ &= \frac{P(X = k, X + Y = n)}{P(Z = n)} \\ &= \frac{P(X = k, Y = n - k)}{P(Z = n)} \\ &= \frac{P(X = k)P(Y = n - k)}{P(Z = n)} \end{aligned}$$

We know that Z is Poisson with mean $\lambda_1 + \lambda_2$.

$$\begin{aligned} p_{X|Z=n}(k) &= \frac{P(X=k, Z=n)}{P(Z=n)} \\ &= \frac{P(X=k)P(Y=n-k)}{P(Z=n)} \\ &= \frac{e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!}} \\ &= \binom{n}{k} \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \cdot \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k} \end{aligned}$$

Hence the conditional distribution of X given $X + Y = n$ is a binomial distribution with parameters n and $\frac{\lambda_1}{\lambda_1+\lambda_2}$.

$$E(X|X+Y=n) = \frac{\lambda_1 n}{\lambda_1 + \lambda_2} .$$

3. Consider $n + m$ independent trials, each of which results in a success with probability p. Compute the expected number of successes in the first n trials given that there are k successes in all.

   Solution: Let Y be the number of successes in $n + m$ trials. Let X be the number of successes in the first n trials. Define

$$X_i = \begin{cases} 1 & \text{if the } i\text{th trial is a success} \\ 0 & \text{otherwise} \end{cases}$$

The $X = \sum_{i=1}^{n} X_i$.

$$E(X|Y = k) = E(\sum_{i=1}^{n} X_i|Y = k) = \sum_{i=1}^{n} E(X_i|Y = k)$$

Since the trials are independent $X_i|Y = k$ have the same distribution. Hence

$$E(X_i|Y = k) = P(X_i = 1|Y = k) = P(X_i = 1|Y = k)$$

$$\begin{aligned} P(X_1 = 1|Y = k) &= \frac{P(X_1 = 1, Y = k)}{P(Y = k)} \\ &= \frac{\binom{n+m-1}{k-1} \cdot p^k \cdot (1-p)^{n+m-k}}{\binom{n+m}{k} \cdot p^k \cdot (1-p)^{n+m-k}} \\ &= \frac{k}{n+m} \end{aligned}$$

Hence

$$E(X|Y = k) = \frac{nk}{n+m}$$

Conditional Density

- Conditional probability density function:

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$

  defined for $y : f_Y(y) > 0$.

- $P(X \in R \mid Y = y) = \int_R f_{X|Y}(x \mid y)dx$
- Conditional expectation of X given $Y = y$

$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)dx$$

- For function g, the conditional expectation of $g(X)$

$$E[g(X) \mid Y = y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y)dx$$

Computing Expectation by Conditioning

- Discrete:

$$
\begin{aligned}
E[X] &= \sum_y E[X \mid Y = y] p_Y(y) \\
&= \sum_y \sum_x x p_{X|Y}(x \mid y) p_Y(y)
\end{aligned}
$$

- Continuous:

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} E[X \mid Y = y] f_Y(y) dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) f_Y(y) dx dy
\end{aligned}
$$

- Chain expansion: $E[X] = E_Y[E_{X|Y}(X \mid Y)]$.

- Expectation of the sum of a random number of random variables:
  If $X = \sum_{i=1}^{N} X_i$, N is a random variable independent of X_i's. X_i's have common mean μ. Then $E[X] = E[N]\mu$.

- Example: Suppose that the expected number of accidents per week at an industrial plant is four. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean of 2. Assume also that the number of workers injured in each accident is independent of the number of accidents that occur. What is the expected number of injuries during a week?

- The variance of a random number of random variables:
  $Z = \sum_{i=1}^{N} X_i$, $E(X_i) = \mu$, $Var(X_i) = \sigma^2$,
  $Var(Z) = \sigma^2 E[N] + \mu^2 Var(N)$.

Computing Probability by Conditioning

- Total probability formula: Suppose F_1, F_2, ..., F_n are mutually exclusive and $\cup_{i=1}^{n} F_i = S$

$$P(E) = \sum_{i=1}^{n} P(F_i) P(E \mid F_i)$$

- $p_X(x) = \sum_{y_i} p_{X|Y}(x \mid y_i) p_Y(y_i)$.
- $f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x \mid y) f_Y(y) dy$.