Markov Chain

- *Stochastic process* (discrete time):

$$\{X_1, X_2, ..., \}$$

- *Markov chain*
  - Consider a discrete time stochastic process with discrete space. $X_n \in \{0, 1, 2, ...\}$.
  - Markovian property

$$\begin{aligned} & P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, ..., X_0 = i_0\} \\ = \; & P\{X_{n+1} = j \mid X_n = i\} = P_{i,j} \end{aligned}$$

  - $P_{i,j}$ is the *transition probability*: the probability of making a transition from i to j.
  - *Transition probability matrix*

$$\mathbf{P} = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ P_{i,0} & P_{i,1} & P_{i,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Example

- Suppose whether it will rain tomorrow depends on past weather condition ony through whether it rains today. Consider the stochastic process $\{X_n, n = 1, 2, ...\}$

$$X_n = \begin{cases} 0 & \text{rain on day } n \\ 1 & \text{not rain on day } n \end{cases}$$

$$P(X_{n+1}|X_n, X_{n-1}, ..., X_1) = P(X_{n+1} \mid X_n)$$

- State space $\{0, 1\}$.
- Transition matrix:

$$\begin{pmatrix} P_{0,0} & P_{0,1} \\ P_{1,0} & P_{1,1} \end{pmatrix}$$

- $P_{0,0} = P(\text{tomorrow rain}|\text{today rain}) = \alpha$. Then $P_{0,1} = 1 - \alpha$.
- $P_{1,0} = P(\text{tomorrow rain}|\text{today not rain}) = \beta$. Then $P_{1,1} = 1 - \beta$.
- Transition matrix:

$$\begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$$

Transforming into a Markov Chain

- Suppose whether it will rain tomorrow depends on whether it rained today and yesterday.
- $P(X_{n+1}|X_n, X_{n-1}, ..., X_1) = P(X_{n+1}|X_n, X_{n-1})$. The process is not a first order Markov chain.
- Define Y_n:

$$Y_n = \begin{cases} 0 & X_n = 0, X_{n-1} = 0 \quad RR \\ 1 & X_n = 0, X_{n-1} = 1 \quad NR \\ 2 & X_n = 1, X_{n-1} = 0 \quad RN \\ 3 & X_n = 1, X_{n-1} = 1 \quad NN \end{cases}$$

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$$\begin{aligned} & P(Y_{n+1}|Y_n, Y_{n-1}, ...) \\ = & P(X_{n+1}, X_n|X_n, X_{n-1}, ...) \\ = & P(X_{n+1}, X_n|X_n, X_{n-1}) \\ = & P(Y_{n+1}|Y_n) \end{aligned}$$

- $\{Y_n, n = 1, 2, ...\}$ is a Markov chain.

$$\begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{pmatrix}$$

- $P_{0,1} = P(Y_{n+1} = 1|Y_n = 0) = P(X_{n+1} = 0, X_n = 1|X_n = 0, X_{n-1} = 0) = 0$.
- $P_{0,3} = P(Y_{n+1} = 3|Y_n = 0) = P(X_{n+1} = 1, X_n = 1|X_n = 0, X_{n-1} = 0) = 0$.
- Similarly, $P_{1,1} = P_{1,3} = 0$, $P_{2,0} = P_{2,2} = 0$, $P_{3,0} = P_{3,2} = 0$.
- Transition matrix

$$\begin{pmatrix} P_{0,0} & 0 & P_{0,2} & 0 \\ P_{1,0} & 0 & P_{1,2} & 0 \\ 0 & P_{2,1} & 0 & P_{2,3} \\ 0 & P_{3,1} & 0 & P_{3,3} \end{pmatrix} = \begin{pmatrix} P_{0,0} & 0 & 1 - P_{0,0} & 0 \\ P_{1,0} & 0 & 1 - P_{1,0} & 0 \\ 0 & P_{2,1} & 0 & 1 - P_{2,1} \\ 0 & P_{3,1} & 0 & 1 - P_{3,1} \end{pmatrix}$$

- The Markov chain is specified by $P_{0,0}$, $P_{1,0}$, $P_{2,1}$, $P_{3,1}$.
  1. $P_{0,0} = P(\text{tomorrow will rain}|\text{today rain, yesterday rain})$.
  2. $P_{1,0} = P(\text{tomorrow will rain}|\text{today rain, yesterday not rain})$.
  3. $P_{2,1} = P(\text{tomorrow will rain}|\text{today not rain, yesterday rain})$.
  4. $P_{3,1} = P(\text{tomorrow will rain}|\text{today not rain, yesterday not rain})$.

Chapman-Kolmogorov Equations

- Transition after nth steps:
  $P^n_{i,j} = P(X_{n+m} = j \mid X_m = i)$.
- **Chapman-Kolmogorov Equations**:

$$P^{n+m}_{i,j} = \sum_{k=0}^{\infty} P^n_{i,k} P^m_{k,j}, \quad n, m \geq 0 \text{ for all } i, j.$$

- Proof (by Total probability formula):

$$\begin{aligned}
P^{n+m}_{i,j} &= P(X_{n+m} = j | X_0 = i) \\
&= \sum_{k=0}^{\infty} P(X_{n+m} = j, X_n = k | X_0 = i) \\
&= \sum_{k=0}^{\infty} P(X_n = k | X_0 = i) \cdot \\
&\quad P(X_{n+m} = j | X_n = k, X_0 = i) \\
&= \sum_{k=0}^{\infty} P^n_{i,k} P^m_{k,j}
\end{aligned}$$

- n-step transition matrix:

$$\mathbf{P^{(n)}} = \begin{pmatrix} P^n_{0,0} & P^n_{0,1} & P^n_{0,2} & \cdots \\ P^n_{1,0} & P^n_{1,1} & P^n_{1,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ P^n_{i,0} & P^n_{i,1} & P^n_{i,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- Chapman-Kolmogorov Equations:

$$\mathbf{P^{(n+m)}} = \mathbf{P^{(n)}} \cdot \mathbf{P^{(m)}}, \qquad \mathbf{P^{(n)}} = \mathbf{P^n}.$$

- Weather example:

$$\mathbf{P} = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

Find $P(\text{rain on Tuesday} \mid \text{rain on Sunday})$ and $P(\text{rain on Tuesday and rain on Wednesday} \mid \text{rain on Sunday})$.

Solution:

$$P(\text{rain on Tuesday} \mid \text{rain on Sunday}) = P^2_{0,0}$$

$$\begin{aligned} \mathbf{P^{(2)}} &= \mathbf{P} \cdot \mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \times \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \\ &= \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} \end{aligned}$$

$$P(\text{rain on Tuesday} \mid \text{rain on Sunday}) = 0.61$$

$$
\begin{aligned}
& P(\text{rain on Tuesday and rain on Wednesday} \mid \text{rain on Sunday}) \\
= \ & P(X_n = 0, X_{n+1} = 0 \mid X_{n-2} = 0) \\
= \ & P(X_n = 0 | X_{n-2} = 0) P(X_{n+1} = 0 | X_n = 0, X_{n-2} = 0) \\
= \ & P(X_n = 0 | X_{n-2} = 0) P(X_{n+1} = 0 | X_n = 0) \\
= \ & P^2_{0,0} P_{0,0} \\
= \ & 0.61 \times 0.7 = 0.427
\end{aligned}
$$

Classification of States

- *Accessible*: State j is accessible from state i if $P_{i,j}^n > 0$ for some $n \geq 0$.
  - $i \to j$.
  - Equivalent to: $P(\text{ever enter } j | \text{start in } i) > 0$.

$$\begin{aligned} & P(\text{ever enter } j | \text{start in } i) \\ = \ & P(\cup_{n=0}^{\infty}\{X_n = j\} | X_0 = i) \\ \leq \ & \sum_{n=0}^{\infty} P(X_n = j \mid X_0 = i) \\ = \ & \sum_{n=0}^{\infty} P_{i,j}^n \end{aligned}$$

Hence if $P_{i,j}^n = 0$ for all n, $P(\text{ever enter } j | \text{start in } i) = 0$. On the other hand,

$$\begin{aligned} & P(\text{ever enter } j | \text{start in } i) \\ = \ & P(\cup_{n=0}^{\infty}\{X_n = j\} | X_0 = i) \\ \geq \ & P(\{X_n = j\} | X_0 = i) \ \text{ for any } n \\ = \ & P_{i,j}^n \ . \end{aligned}$$

If $P_{i,j}^n > 0$ for some n, $P(\text{ever enter } j | \text{start in } i) \geq P_{i,j}^n > 0$.

  - Examples

- *Communicate*: State i and j communicate if they are accessible from each other.
  - $i \leftrightarrow j$.
  - Properties:
    1. $P^0_{i,i} = P(X_0 = i | X_0 = i) = 1$. Any state i communicates with itself.
    2. If $i \leftrightarrow j$, then $j \leftrightarrow i$.
    3. If $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

       Proof:

$$
\begin{aligned}
i \leftrightarrow j &\implies P^n_{i,j} > 0 \text{ and } P^{n'}_{j,i} > 0 \\
j \leftrightarrow k &\implies P^m_{j,k} > 0 \text{ and } P^{m'}_{k,j} > 0 \\
P^{n+m}_{i,k} &= \sum_{l=0}^{\infty} P^n_{i,l} P^m_{l,k} \quad \text{Chapman-Kolmogorov Eq.} \\
&> P^n_{i,j} \cdot P^m_{j,k} \\
&> 0
\end{aligned}
$$

       Similarly, we can show $P^{n'+m'}_{k,i} > 0$. Hence $i \leftrightarrow k$.

- *Class*: Two states that communciate are said to be in the same class. A class is a subset of states that communicate with each other.
  - Different classes do NOT overlap.
  - Classes form a partition of states.
- *Irreducible*: A Markov chain is irreducible if there is only one class.
  - Consider the Markov chain with transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

    The MC is irreducible.
  - MC with transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

    Three classes: $\{0, 1\}$, $\{2\}$, $\{3\}$.

Recurrent and Transient States

- f_i: probability that starting in state i, the MC will ever reenter state i.
- *Recurrent*: If $f_i = 1$, state i is recurrent.
  - A recurrent states will be visited infinitely many times by the process starting from i.
- *Transient*: If $f_i < 1$, state i is transient.
  - Starting from i, the MC will be in state i for exactly n times (including the starting state) is

    $$f_i^{n-1}(1 - f_i) \ , \ n = 1, 2, ...$$

    This is a geometric distribution with parameter $1 - f_i$. The expected number of times spent in state i is $1/(1 - f_i)$.

- A state is recurrent *if and only if* the expected number of time periods that the process is in state i, starting from state i, is infinite.

$$\text{Recurrent} \iff E(\text{number of visits to } i | X_0 = i) = \infty$$
$$\text{Transient} \iff E(\text{number of visits to } i | X_0 = i) < \infty$$

- Compute $E(\text{number of visits to } i | X_0 = i)$. Define

$$I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}$$

Then the number of visits to i is $\sum_{n=0}^{\infty} I_n$.

$$\begin{aligned}
& E(\text{number of visits to } i | X_0 = i) \\
= & \sum_{n=0}^{\infty} E(I_n | X_0 = i) \\
= & \sum_{n=0}^{\infty} P(I_n = 1 | X_0 = i) \\
= & \sum_{n=0}^{\infty} P(X_n = i | X_0 = i) \\
= & \sum_{n=0}^{\infty} P_{i,i}^n
\end{aligned}$$

- **Proposition 4.1**: State i is recurrent if $\sum_{n=0}^{\infty} P_{i,i}^{n} = \infty$, and transient if $\sum_{n=0}^{\infty} P_{i,i}^{n} < \infty$.
- **Corollary 4.2**: If state i is recurrent and state i communicates with state j, then state j is recurrent.
- **Corollary 4.3**: A finite state Markov chain cannot have all transient states.
  - For any irreducible and finite-state Markov chain, all states are recurrent.

- Consider a MC with

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

  The MC is irreducible and finite state, hence all states are recurrent.

- Consider a MC with

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

  Three classes: $\{0, 1\}$, $\{2, 3\}$, $\{4\}$. State $0, 1, 2, 3$ are recurrent and state 4 is transient.

Random Walk

- A Markov chain with state space $i = 0, \pm 1, \pm 2, ...$
- Transition probability: $P_{i,i+1} = p = 1 - P_{i,i-1}$.
  - At every step, move either 1 step forward or 1 step backward.
- Example: a gambler either wins a dollar or loses a dollar at every game. X_n is the number of dollars he has when starting the nth game.
- For any $i < j$, $P_{i,j}^{j-i} = p^{j-i} > 0$, $P_{j,i}^{j-i} = (1-p)^{j-i} > 0$. The MC is irreducible.
- Hence, either all the states are transient or all the states are recurrent.

- Under which condition are the states transient or recurrent?
  - Consider State 0.

$$\sum_{n=1}^{\infty} P_{0,0}^{n} = \begin{cases} \infty & \text{recurrent} \\ \text{finite} & \text{transient} \end{cases}$$

  - Only for even m, $P_{0,0}^{m} > 0$.

$$P_{0,0}^{2n} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n!n!}(p(1-p))^n$$
$$n = 1, 2, 3, \ldots$$

  - By Stirling's approximation

$$n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi}$$

  - $P_{0,0}^{2n} \sim \frac{(4p(1-p))^n}{\sqrt{\pi n}}$.
  - When $p = 1/2$, $4p(1-p) = 1$.

$$\sum_{n=0}^{\infty} \frac{(4p(1-p))^n}{\sqrt{\pi n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}} \,,$$

    The summation diverges. Hence, all the states are recurrent.
  - When $p \neq 1/2$, $4p(1-p) < 1$. $\sum_{n=0}^{\infty} \frac{(4p(1-p))^n}{\sqrt{\pi n}}$ converges. All the states are transient.

Limiting Probabilities

- Weather example

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

- $$\mathbf{P}^{(4)} = \mathbf{P}^4 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix}$$

$$\mathbf{P}^{(8)} = \mathbf{P}^{(4)}\mathbf{P}^{(4)} = \begin{pmatrix} 0.572 & 0.428 \\ 0.570 & 0.430 \end{pmatrix}$$

- $\mathbf{P}^{(4)}$ and $\mathbf{P}^{(8)}$ are close. The rows in $\mathbf{P}^{(8)}$ are close.
- Limiting probabilities?

- *Period d*: For state i, if $P^n_{i,i} = 0$ whenever n is not divisible by d and d is the largest integer with this property, d is the period of state i.
  - Period d is the greatest common divisor of all the m such that $P^m_{i,i} > 0$.
- *Aperiodic*: State i is aperiodic if its period is 1.
- *Positive recurrent*: If a state i is recurrent and the expected time until the process returns to state i is finite.
  - If $i \leftrightarrow j$ and i is positive recurrent, then j is positive recurrent.
  - For a finite-state MC, a recurrent state is also positive recurrent.
  - A finite-state irreducible MC contains all positive recurrent states.
- *Ergodic*: A positive recurrent and aperiodic state is an ergodic state.
- A Markov chain is ergodic if all its states are ergodic.

- **Theorem 4.1**: For an irreducible ergodic Markov chain, $\lim_{n\to\infty} P_{i,j}^n$ exists and is independent of i. Let $\pi_j = \lim_{n\to\infty} P_{i,j}^n$, $j \geq 0$, then π_j is the unique nonnegative solution of

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{i,j} \quad j \geq 0$$
$$\sum_{j=0}^{\infty} \pi_j = 1 .$$

- The Weather Example:

$$\mathbf{P} = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$$

The MC is irreducible and ergodic.

$$\begin{aligned} \pi_0 + \pi_1 &= 1 \\ \pi_0 &= \pi_0 P_{0,0} + \pi_1 P_{1,0} = \pi_0 \alpha + \pi_1 \beta \\ \pi_1 &= \pi_0 P_{0,1} + \pi_1 P_{1,1} = \pi_0(1-\alpha) + \pi_1(1-\beta) \end{aligned}$$

Solve the linear equations, $\pi_0 = \frac{\beta}{1+\beta-\alpha}$, $\pi_1 = \frac{1-\alpha}{1+\beta-\alpha}$.

Gambler's Ruin Problem

- At each play, a gambler either wins a unit with probability p or loses a unite with probability $q = 1 - p$. Suppose the gambler starts with i units, what is the probability that the gambler's fortune will reach N before reaching 0 (broke)?
- Solution:
  - Let X_t be the number of units the gambler has at time t. $\{X_t; t = 0, 1, 2, ...\}$ is a Markov chain.
  - Transition probabilities:

$$P_{0,0} = P_{N,N} = 1$$

$$P_{i,i+1} = p, \ P_{i,i-1} = 1 - p = q \ .$$

  - Denote the probability that starting from i units, the gambler's fortune will reach N before reaching 0 by P_i, $i = 0, 1, ..., N$.
  - Condition on the result of the first game and apply the total probability formula:

$$P_i = pP_{i+1} + qP_{i-1} \ .$$

– Changing forms:

$$(p+q)P_i = pP_{i+1} + qP_{i-1}$$
$$q(P_i - P_{i-1}) = p(P_{i+1} - P_i)$$
$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1})$$

– Recursion:

$$\begin{aligned}
P_0 &= 0 \\
P_2 - P_1 &= \frac{q}{p}(P_1 - P_0) = \frac{q}{p}P_1 \\
P_3 - P_2 &= \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1 \\
&\vdots \\
P_i - P_{i-1} &= \frac{q}{p}(P_{i-1} - P_{i-2}) = \left(\frac{q}{p}\right)^{i-1} P_1
\end{aligned}$$

– Add up the equations:

$$\begin{aligned}
P_i &= P_1[1 + \frac{q}{p} + \cdots + (\frac{q}{p})^{i-1}] \\
&= \begin{cases} P_1 \cdot \frac{1-(\frac{q}{p})^i}{1-\frac{q}{p}} & \text{if } \frac{q}{p} \neq 1 \\ iP_1 & \text{if } \frac{q}{p} = 1 \end{cases}
\end{aligned}$$

– We know $P_N = 1$

$$P_N = \begin{cases} P_1 \cdot \frac{1-(\frac{q}{p})^N}{1-\frac{q}{p}} & \text{if } \frac{q}{p} \neq 1 \\ NP_1 & \text{if } \frac{q}{p} = 1 \end{cases}$$

– Hence,

$$P_1 = \begin{cases} \frac{1-\frac{q}{p}}{1-(\frac{q}{p})^N} & \text{if } p \neq 1/2 \\ 1/N & \text{if } p = 1/2 \end{cases}$$

– In summary,

$$P_i = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N} & \text{if } p \neq 1/2 \\ i/N & \text{if } p = 1/2 \end{cases}$$

- Note, if $N \to \infty$,

$$P_i = \begin{cases} 1 - (\frac{q}{p})^i & \text{if } p > 1/2 \\ 0 & \text{if } p \leq 1/2 \end{cases}$$

- When $p > 1/2$, there is a positive probability that the gambler will win infinitely many units.
- When $p \leq 1/2$, the gambler will surely go broke (with probability 1) if not stop at a finite fortune (assuming the opponent is infinitely rich).