Exercises for Chapter 3
Conditional Probability and Expectation

1. Suppose that the expected number of accidents per week at an industrial plant is 4. Suppose also that the number of workers injured in each accident are independent RVs with a common mean of 2. Assume also that the number of workers injured in each accident is independent of the number of accidents that occur. What is the expected number of injuries during a week?

2. Random variables $X$ and $Y$ have the joint pdf $f(x, y) = k(x + y)$, $0 < x < 1$, $0 < y < 1$.

   (a) Find $k$
   (b) Find $f_{Y|X}(y|x)$
   (c) Find $E(Y|X = x)$
   (d) Find $E(Y^2|X = x)$

3. $X$ and $Y$ have joint pmf specified below Find the conditional pmf of $X|Y$ and $E(X|Y = 1)$.

<table>
<thead>
<tr>
<th></th>
<th>$Y$</th>
<th>$-1$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td></td>
<td>1/6</td>
<td>0</td>
<td>1/6</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td></td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td>1/6</td>
<td>0</td>
</tr>
</tbody>
</table>

1
4. The joint pdf of $X$ and $Y$ is

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2}ye^{-xy} & x > 0, 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

What is $E[e^{x/2} \mid Y = 1]$?

5. The joint density of $X$ and $Y$ is given by $f_{X,Y}(x, y) = e^{-y}, 0 < x < y, y > 0$. Compute $E[X^2 \mid Y = y]$.

6. Let $U$ be a uniform $[0, 1]$ random variable. Suppose that $n$ trials are to be performed and that condition on $U = u$, these trials will be independent with a common success probability $u$. Compute the mean and variance of the number of successes that occur in these trials.

7. A customer entering a store is served by clerk $i$ with probability $p_i, i = 1, ..., n$. The time taken by clerk $i$ to serve a customer is exponentially distributed random variable with parameter $\alpha_i$. Find $E[T]$ and $Var[T]$ where $T$ is the service time of a customer.

8. Each customer who enters Rebecca’s clothing store will purchase a suit with probability $p$. If the number of customers entering the store is Poisson distributed with mean $\lambda$, what is the probability that Rebecca sells $k$ suit?
9. Consider two coins. One is a fair coin and the other has probability \( \frac{1}{4} \) showing heads. A coin is randomly chosen from the two and is flipped 10 times. Let \( M \) be the number of heads obtained. Find

(a) The pmf of \( M \)
(b) \( E(M) \)

10. Two coins have probability \( \frac{1}{2} \) and \( \frac{1}{4} \) coming up heads respectively. A coin is randomly chosen and flipped until the first head appears. Let \( X \) be the number of flips needs. Find the pmf of \( X \).
11. A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel which takes him to safety after 2 hours of travel. The second door leads to a tunnel which returns him to the mine after 3 hours of travel. The third door leads to a tunnel which returns him to his mine after 5 hours. Assume that the miner is at all times equally likely to choose any one of the doors. What is the expected length of time until the miner reaches safety?

12. A coin, having probability $p$ of landing heads is continually flipped until at least one head and one tail have been flipped.

(a) Find the expected number of flips needed
(b) Find the expected number of flips that land on heads
(c) Find the expected number of flips that land on tails

13. A coin is randomly flipped, the chance of getting a head is $p$. What is the expected number of trials to get $k$ consecutive heads?
14. At a party $n$ men take off their hats. The hats are then mixed up and each man randomly selects one. We say that a match occurs if a man selects his own hat. What is the probability of no matches? What is the probability of exactly $k$ matches? (Textbook)

15. (The Ballot Problem, Textbook) In an election, candidate $A$ receives $n$ votes, and candidate $B$ receives $m$ votes, where $n > m$. Assuming that all ordering are equally likely, show that the probability that $A$ is always ahead in the count of votes is $\frac{\binom{n-m}{m}}{\binom{n+m}{n}}$.

16. (The Best Prize Problem, Textbook) Suppose that we are to be presented with $n$ distinct prizes in sequence. After being presented with a prize we must immediately decide whether to accept it or reject it and consider the next prize. The only information we are given when deciding whether to accept a prize is the relative rank of that prize compared to ones already seen. That is, for instance, when the fifth prize is presented we learn how it compares with the first four prizes already seen. Suppose that once a prize is rejected it is lost, and that our objective is to maximize the probability of obtaining the best prize. Assuming that all $n!$ orderings of the prizes are equally likely, how well can we do?