Mean Time Spent in Transient States

- Consider a finite state Markov Chain with a set of transient states: $T = \{1, 2, \ldots, t\}$.

- Gambler’s ruin problem:
  states: $\{0, 1, \ldots, N\}$.
  The transient states are $1, 2, \ldots, N - 1$.

  $$T = \{1, 2, \ldots, N - 1\}.$$

- Let the transition probability matrix be $P$.

- A part of $P$ formed by probabilities from transient states to transient states:

  $$P_T = \begin{bmatrix}
P_{11} & P_{12} & \cdots & P_{1t} \\
P_{21} & P_{22} & \cdots & P_{2t} \\
  \vdots & \vdots & & \vdots \\
P_{t1} & P_{t2} & \cdots & P_{tt}
\end{bmatrix}$$
• Examples:

  – Gamblers ruin problem:

\[
P = \begin{bmatrix}
P_{00} & P_{01} & \cdots & P_{0,N-1} & P_{0,N} \\
P_{10} & P_{11} & \cdots & P_{1,N-1} & P_{1,N} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
P_{N0} & P_{N1} & \cdots & P_{N,N-1} & P_{N,N}
\end{bmatrix}
\]

\[
P_T = \begin{bmatrix}
P_{11} & P_{12} & \cdots & P_{1,N-1} \\
P_{21} & P_{22} & \cdots & P_{2,N-1} \\
\vdots & \vdots & \ddots & \vdots \\
P_{N-1,1} & P_{N-1,2} & \cdots & P_{N-1,N-1}
\end{bmatrix}
\]

  – Suppose a four state MC with states \(\{0, 1, 2, 3\}\) has two transient states: 0 and 3. Then:

\[
P = \begin{bmatrix}
P_{00} & P_{01} & P_{02} & P_{03} \\
P_{10} & P_{11} & P_{12} & P_{13} \\
P_{20} & P_{21} & P_{22} & P_{23} \\
P_{30} & P_{31} & P_{32} & P_{33}
\end{bmatrix}
\]

\[
P_T = \begin{bmatrix}
P_{00} & P_{03} \\
P_{30} & P_{33}
\end{bmatrix}
\]
• For transient states $i$ and $j$:
  – $s_{ij}$: expected number of time periods the MC is in state $j$, given that it starts in state $i$.

  – Special case $s_{ii}$: starting from $i$, the number of time periods in $i$.

  – Transient states: $f_i < 1$. Recall that $f_i$ is the probability of ever revisit state $i$ starting from state $i$.

  – Define $f_{ij}$: the probability that the MC ever visits state $j$ given that it starts in $i$.
    Special case: $f_{ii} = f_i$. 

Compute $s_{ij}$ by conditioning:

If $i = j$:

$$s_{ij} = 1 + E\{\text{# visits to } j \text{ after the initial state}\} \quad (1)$$

If $i \neq j$:

$$s_{ij} = E\{\text{# visits to } j \text{ after the initial state}\} \quad (2)$$

Note

$$E\{\text{# visits to } j \text{ after the initial state}\} = \sum_k P\{\text{the first transition is to state } k\} \times E\{\text{# visits to } j \mid \text{starting from } k\}$$

$$= \sum_k P_{ik} s_{kj}$$

If $k$ is recurrent, then $P_{kj}^n = 0$ for all $n$. Hence $s_{kj} = 0$.

To see $P_{kj}^n = 0$ for all $n$, suppose $P_{kj}^n > 0$ for some $n$. Then $P_{jk}^m = 0$ for all $m$, otherwise $k$ and $j$ communicate and this conflicts the fact $j$ is transient. Since $P_{jk}^m = 0$ for all $m$, with a positive probability state $k$ will transit to state $j$ and never come back to state $k$, which conflicts the fact $k$ is recurrent ($k$ should be revisited infinitely many times). Hence $P_{kj}^n > 0$ for some $n$ is not true.
The above equation can be written as:

\[ E\{\# \text{ visits to } j \text{ after the initial state}\} = \sum_{k=1}^{t} P_{ik}s_{kj} \]

To combine Eq. (1) and (2) into a unified form:

\[ \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \]

Hence:

\[ s_{ij} = \delta_{ij} + \sum_{k=1}^{t} P_{ik}s_{kj} \quad (3) \]

for all \( i, j \in \{1, 2, \ldots, t\} \).

Use matrix:

\[
S = \begin{bmatrix}
    s_{11} & s_{12} & \cdots & s_{1t} \\
    s_{21} & s_{22} & \cdots & s_{2t} \\
    \vdots & \vdots & \ddots & \vdots \\
    s_{t1} & s_{t2} & \cdots & s_{tt}
\end{bmatrix}
\]

Eq. (3) is equivalent to

\[ S = I + P_T S \]

\[ S = (I - P_T)^{-1} \]
To obtain $f_{ij}$, the probability of ever transit to $j$ starting in $i$, compute expectation by conditioning:

$$s_{ij} = P\{\text{ever transit to } j \mid \text{start in } i\} \times$$

$$E[\text{time spent in } j \mid \text{start in } i \text{ and transit to } j] +$$

$$P\{\text{never transit to } j \mid \text{start in } i\} \times$$

$$E[\text{time spent in } j \mid \text{start in } i, \text{never transit to } j]$$

$$= f_{ij}(\delta_{ij} + s_{jj}) + (1 - f_{ij})\delta_{ij}$$

$$= \delta_{ij} + f_{ij}s_{jj}$$

Hence

$$f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}} \quad (4)$$

Special case:

$$f_{ii} = \frac{s_{ii} - 1}{s_{ii}}.$$
Example: Gambler’s ruin problem. Suppose \( p = 0.4, N = 7 \). Start with 3. (a) The expected amount of time the gambler has 5 units, or 2 units. (b) The probability that the gambler ever has a fortune of 1.

Solution:
States: \( \{0, 1, 2, 3, 4, 5, 6, 7\} \).
Transient states: \( \{1, 2, 3, 4, 5, 6\} \).

\[
P_T = \begin{bmatrix} 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \\ \end{bmatrix}
\]

Solve \( S = (I - P_T)^{-1} \), we get

\[
s_{3,5} = 0.9228 \quad s_{3,2} = 2.3677
\]

Hence the expected number of times the gambler has 5 units is 0.9228, 2 units is 2.3677.

\[
f_{3,1} = \frac{s_{3,1} - \delta_{3,1}}{s_{1,1}} = \frac{s_{3,1}}{s_{1,1}} = \frac{1.4206}{1.6149} = 0.8797. \quad \text{The probability that the gambler ever has a fortune of 1 is 0.8797.}
\]