Logistic Regression

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Logistic Regression

Preserve linear classification boundaries.

- By the Bayes rule:

$$\hat{G}(x) = \arg\max_k Pr(G = k \mid X = x) .$$

- Decision boundary between class k and l is determined by the equation:

$$Pr(G = k \mid X = x) = Pr(G = l \mid X = x) .$$

- Divide both sides by $Pr(G = l \mid X = x)$ and take log. The above equation is equivalent to

$$\log \frac{Pr(G = k \mid X = x)}{Pr(G = l \mid X = x)} = 0 .$$

- Since we enforce linear boundary, we can assume

$$\log \frac{Pr(G = k \mid X = x)}{Pr(G = l \mid X = x)} = a_0^{(k,l)} + \sum_{j=1}^{p} a_j^{(k,l)} x_j \; .$$

- For logistic regression, there are restrictive relations between $a^{(k,l)}$ for different pairs of (k, l).

Assumptions

$$
\begin{aligned}
\log \frac{Pr(G=1 \mid X=x)}{Pr(G=K \mid X=x)} &= \beta_{10} + \beta_1^T x \\
\log \frac{Pr(G=2 \mid X=x)}{Pr(G=K \mid X=x)} &= \beta_{20} + \beta_2^T x \\
&\vdots \\
\log \frac{Pr(G=K-1 \mid X=x)}{Pr(G=K \mid X=x)} &= \beta_{(K-1)0} + \beta_{K-1}^T x
\end{aligned}
$$

- For any pair (k, l):

$$\log \frac{Pr(G = k \mid X = x)}{Pr(G = l \mid X = x)} = \beta_{k0} - \beta_{l0} + (\beta_k - \beta_l)^T x \ .$$

- Number of parameters: $(K-1)(p+1)$.
- Denote the entire parameter set by

$$\theta = \{\beta_{10}, \beta_1, \beta_{20}, \beta_2, ..., \beta_{(K-1)0}, \beta_{K-1}\} \ .$$

- The log ratio of posterior probabilities are called *log-odds* or *logit transformations*.

- Under the assumptions, the posterior probabilities are given by:

$$Pr(G = k \mid X = x) = \frac{\exp(\beta_{k0} + \beta_k^T x)}{1 + \sum_{l=1}^{K-1} \exp(\beta_{l0} + \beta_l^T x)}$$

$$\text{for } k = 1, ..., K - 1$$

$$Pr(G = K \mid X = x) = \frac{1}{1 + \sum_{l=1}^{K-1} \exp(\beta_{l0} + \beta_l^T x)} .$$

- For $Pr(G = k \mid X = x)$ given above, obviously
  - Sum up to 1: $\sum_{k=1}^{K} Pr(G = k \mid X = x) = 1$.
  - A simple calculation shows that the assumptions are satisfied.

Comparison with LR on Indicators

- Similarities:
  - Both attempt to estimate $Pr(G = k \mid X = x)$.
  - Both have linear classification boundaries.
- Difference:
  - Linear regression on indicator matrix: approximate $Pr(G = k \mid X = x)$ by a linear function of x. $Pr(G = k \mid X = x)$ is not guaranteed to fall between 0 and 1 and to sum up to 1.
  - Logistic regression: $Pr(G = k \mid X = x)$ is a *nonlinear* function of x. It is guaranteed to range from 0 to 1 and to sum up to 1.

Fitting Logistic Regression Models

- Criteria: find parameters that maximize the conditional likelihood of G given X using the training data.
- Denote $p_k(x_i; \theta) = Pr(G = k \mid X = x_i; \theta)$.
- Given the first input x_1, the posterior probability of its class being g_1 is $Pr(G = g_1 \mid X = x_1)$.
- Since samples in the training data set are independent, the posterior probability for the N samples each having class g_i, $i = 1, 2, ..., N$, given their inputs x_1, x_2, ..., x_N is:

$$\prod_{i=1}^{N} Pr(G = g_i \mid X = x_i) \ .$$

- The conditional log-likelihood of the class labels in the training data set is

$$\begin{aligned} L(\theta) &= \sum_{i=1}^{N} \log Pr(G = g_i \mid X = x_i) \\ &= \sum_{i=1}^{N} \log p_{g_i}(x_i; \theta) \ . \end{aligned}$$

Binary Classification

- For binary classification, if $g_i = 1$, denote $y_i = 1$; if $g_i = 2$, denote $y_i = 0$.
- Let $p_1(x;\theta) = p(x;\theta)$, then

$$p_2(x;\theta) = 1 - p_1(x;\theta) = 1 - p(x;\theta)\ .$$

- Since $K = 2$, the parameters $\theta = \{\beta_{10}, \beta_1\}$.
  We denote $\beta = (\beta_{10}, \beta_1)^T$.

- If $y_i = 1$, i.e., $g_i = 1$,

$$\begin{aligned} \log p_{g_i}(x;\beta) &= \log p_1(x;\beta) \\ &= 1 \cdot \log p(x;\beta) \\ &= y_i \log p(x;\beta) \ . \end{aligned}$$

If $y_i = 0$, i.e., $g_i = 2$,

$$\begin{aligned} \log p_{g_i}(x;\beta) &= \log p_2(x;\beta) \\ &= 1 \cdot \log(1 - p(x;\beta)) \\ &= (1 - y_i) \log(1 - p(x;\beta)) \ . \end{aligned}$$

Since either $y_i = 0$ or $1 - y_i = 0$, we have

$$\log p_{g_i}(x;\beta) = y_i \log p(x;\beta) + (1 - y_i) \log(1 - p(x;\beta)) \ .$$

- The conditional likelihood

$$
\begin{aligned}
L(\beta) &= \sum_{i=1}^{N} \log p_{g_i}(x_i;\beta) \\
&= \sum_{i=1}^{N} [y_i \log p(x_i;\beta) + (1-y_i)\log(1-p(x_i;\beta))]
\end{aligned}
$$

- There $p+1$ parameters in $\beta = (\beta_{10}, \beta_1)^T$.
- Assume a column vector form for β:

$$
\beta = \begin{pmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \vdots \\ \beta_{1,p} \end{pmatrix} .
$$

- Here we add the constant term 1 to x to accommodate the intercept.

$$x = \begin{pmatrix} 1 \\ x_{,1} \\ x_{,2} \\ \vdots \\ x_{,p} \end{pmatrix} .$$

- By the assumption of logistic regression model:

$$\begin{aligned} p(x;\beta) &= Pr(G=1 \mid X=x) = \frac{\exp(\beta^T x)}{1+\exp(\beta^T x)} \\ 1-p(x;\beta) &= Pr(G=2 \mid X=x) = \frac{1}{1+\exp(\beta^T x)} \end{aligned}$$

- Substitute the above in $L(\beta)$:

$$L(\beta) = \sum_{i=1}^{N} \left[y_i \beta^T x_i - \log(1+e^{\beta^T x_i}) \right] .$$

- To maximize $L(\beta)$, we set the first order partial derivatives of $L(\beta)$ to zero.

$$
\begin{aligned}
\frac{\partial L(\beta)}{\beta_{1j}} &= \sum_{i=1}^{N} y_i x_{ij} - \sum_{i=1}^{N} \frac{x_{ij} e^{\beta^T x_i}}{1 + e^{\beta^T x_i}} \\
&= \sum_{i=1}^{N} y_i x_{ij} - \sum_{i=1}^{N} p(x;\beta) x_{ij} \\
&= \sum_{i=1}^{N} x_{ij}(y_i - p(x_i;\beta))
\end{aligned}
$$

for all $j = 0, 1, ..., p$.

- In matrix form, we write

$$\frac{\partial L(\beta)}{\partial \beta} = \sum_{i=1}^{N} x_i (y_i - p(x_i; \beta)) \ .$$

- To solve the set of $p+1$ nonlinear equations $\frac{\partial L(\beta)}{\partial \beta_{1j}} = 0$, $j = 0, 1, ..., p$, use the Newton-Raphson algorithm.
- The Newton-Raphson algorithm requires the second-derivatives or Hessian matrix:

$$\frac{\partial^2 L(\beta)}{\partial \beta \partial \beta^T} = -\sum_{i=1}^{N} x_i x_i^T p(x_i; \beta)(1 - p(x_i; \beta)) \ .$$

- The element on the jth row and nth column is (counting from 0):

$$
\begin{aligned}
& \frac{\partial L(\beta)}{\partial \beta_{1j} \partial \beta_{1n}} \\
= & -\sum_{i=1}^{N} \frac{(1+e^{\beta^T x_i}) e^{\beta^T x_i} x_{ij} x_{in} - (e^{\beta^T x_i})^2 x_{ij} x_{in}}{(1+e^{\beta^T x_i})^2} \\
= & -\sum_{i=1}^{N} x_{ij} x_{in} p(x_i;\beta) - x_{ij} x_{in} p(x_i;\beta)^2 \\
= & -\sum_{i=1}^{N} x_{ij} x_{in} p(x_i;\beta)(1 - p(x_i;\beta)) .
\end{aligned}
$$

- Starting with β^{old}, a single Newton-Raphson update is

$$\beta^{new} = \beta^{old} - \left(\frac{\partial^2 L(\beta)}{\partial \beta \partial \beta^T} \right)^{-1} \frac{\partial L(\beta)}{\partial \beta} ,$$

where the derivatives are evaluated at β^{old}.

- The iteration can be expressed compactly in matrix form.
  - Let $\mathbf{y}$ be the column vector of y_i.
  - Let $\mathbf{X}$ be the $N \times (p+1)$ input matrix.
  - Let $\mathbf{p}$ be the N-vector of fitted probabilities with ith element $p(x_i; \beta^{old})$.
  - Let $\mathbf{W}$ be an $N \times N$ diagonal matrix of weights with ith element $p(x_i; \beta^{old})(1 - p(x_i; \beta^{old}))$.
  - Then

$$\begin{aligned} \frac{\partial L(\beta)}{\partial \beta} &= \mathbf{X}^T(\mathbf{y} - \mathbf{p}) \\ \frac{\partial^2 L(\beta)}{\partial \beta \partial \beta^T} &= -\mathbf{X}^T \mathbf{W} \mathbf{X} \, . \end{aligned}$$

- The Newton-Raphson step is

$$\begin{aligned}\beta^{new} &= \beta^{old} + (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T(\mathbf{y}-\mathbf{p}) \\ &= (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}(\mathbf{X}\beta^{old} + \mathbf{W}^{-1}(\mathbf{y}-\mathbf{p})) \\ &= (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}\mathbf{z}\,,\end{aligned}$$

  where $\mathbf{z} \triangleq \mathbf{X}\beta^{old} + \mathbf{W}^{-1}(\mathbf{y}-\mathbf{p})$.
- If $\mathbf{z}$ is viewed as a response and $\mathbf{X}$ is the input matrix, β^{new} is the solution to a weighted least square problem:

$$\beta^{new} \leftarrow \arg\min_{\beta}(\mathbf{z}-\mathbf{X}\beta)^T\mathbf{W}(\mathbf{z}-\mathbf{X}\beta)\ .$$

- Recall that linear regression by least square is to solve

$$\arg\min_{\beta}(\mathbf{z}-\mathbf{X}\beta)^T(\mathbf{z}-\mathbf{X}\beta)\ .$$

- $\mathbf{z}$ is referred to as the *adjusted response*.
- The algorithm is referred to as *iteratively reweighted least squares* or *IRLS*.

Pseudo Code

1. $0 \rightarrow \beta$
2. Compute $\mathbf{y}$ by setting its elements to
$$y_i = \begin{cases} 1 & \text{if } g_i = 1 \\ 0 & \text{if } g_i = 2 \end{cases},$$
$i = 1, 2, ..., N$.
3. Compute $\mathbf{p}$ by setting its elements to
$$p(x_i; \beta) = \frac{e^{\beta^T x_i}}{1 + e^{\beta^T x_i}} \quad i = 1, 2, ..., N.$$
4. Compute the diagonal matrix $\mathbf{W}$. The ith diagonal element is $p(x_i; \beta)(1 - p(x_i; \beta))$, $i = 1, 2, ..., N$.
5. $\mathbf{z} \leftarrow \mathbf{X}\beta + \mathbf{W}^{-1}(\mathbf{y} - \mathbf{p})$.
6. $\beta \leftarrow (\mathbf{X}^T\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}\mathbf{z}$.
7. If the stopping criteria is met, stop; otherwise go back to step 3.

Computational Efficiency

- Since $\mathbf{W}$ is an $N \times N$ diagonal matrix, direct matrix operations with it may be very inefficient.
- A modified pseudo code is provided next.

1. $0 \rightarrow \beta$
2. Compute $\mathbf{y}$ by setting its elements to

$$y_i = \begin{cases} 1 & \text{if } g_i = 1 \\ 0 & \text{if } g_i = 2 \end{cases}, i = 1, 2, ..., N .$$

3. Compute $\mathbf{p}$ by setting its elements to

$$p(x_i; \beta) = \frac{e^{\beta^T x_i}}{1 + e^{\beta^T x_i}} \quad i = 1, 2, ..., N.$$

4. Compute the $N \times (p+1)$ matrix $\tilde{\mathbf{X}}$ by multiplying the ith row of matrix $\mathbf{X}$ by $p(x_i; \beta)(1 - p(x_i; \beta))$, $i = 1, 2, ..., N$:

$$\mathbf{X} = \begin{pmatrix} x_1^T \\ x_2^T \\ \cdots \\ x_N^T \end{pmatrix} \quad \tilde{\mathbf{X}} = \begin{pmatrix} p(x_1; \beta)(1 - p(x_1; \beta))x_1^T \\ p(x_2; \beta)(1 - p(x_2; \beta))x_2^T \\ \cdots \\ p(x_N; \beta)(1 - p(x_N; \beta))x_N^T \end{pmatrix}$$

5. $\beta \leftarrow \beta + (\mathbf{X}^T \tilde{\mathbf{X}})^{-1} \mathbf{X}^T (\mathbf{y} - \mathbf{p})$.
6. If the stopping criteria is met, stop; otherwise go back to step 3.

Example

Diabetes data set

- Input X is two dimensional. X_1 and X_2 are the two principal components of the original 8 variables.
- Class 1: without diabetes; Class 2: with diabetes.
- Applying logistic regression, we obtain

$$\beta = (0.7679, -0.6816, -0.3664)^T .$$

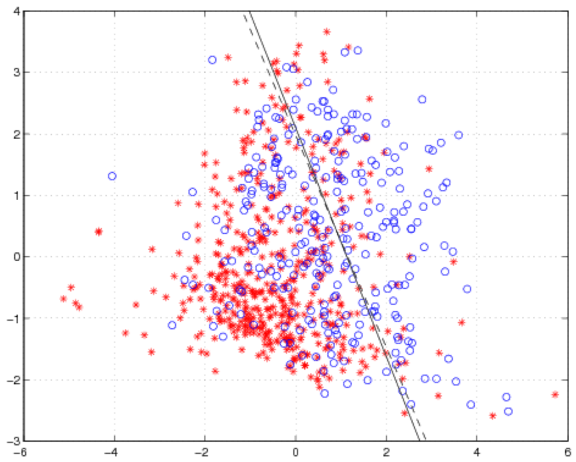
- The posterior probabilities are:

$$\begin{aligned} Pr(G = 1 \mid X = x) &= \frac{e^{0.7679 - 0.6816X_1 - 0.3664X_2}}{1 + e^{0.7679 - 0.6816X_1 - 0.3664X_2}} \\ Pr(G = 2 \mid X = x) &= \frac{1}{1 + e^{0.7679 - 0.6816X_1 - 0.3664X_2}} \end{aligned}$$

- The classification rule is:

$$\hat{G}(x) = \begin{cases} 1 & 0.7679 - 0.6816X_1 - 0.3664X_2 \geq 0 \\ 2 & 0.7679 - 0.6816X_1 - 0.3664X_2 < 0 \end{cases}$$

Solid line: decision boundary obtained by logistic regression. Dash line: decision boundary obtained by LDA.



- Within training data set classification error rate: 28.12%.
- Sensitivity: 45.9%.
- Specificity: 85.8%.

Multiclass Case ($K \geq 3$)

- When $K \geq 3$, β is a (K-1)(p+1)-vector:

$$\beta = \begin{pmatrix} \beta_{10} \\ \beta_1 \\ \beta_{20} \\ \beta_2 \\ \vdots \\ \beta_{(K-1)0} \\ \beta_{K-1} \end{pmatrix} = \begin{pmatrix} \beta_{10} \\ \beta_{11} \\ \vdots \\ \beta_{1p} \\ \beta_{20} \\ \vdots \\ \beta_{2p} \\ \vdots \\ \beta_{(K-1)0} \\ \vdots \\ \beta_{(K-1)p} \end{pmatrix}$$

- Let $\bar{\beta}_l = \begin{pmatrix} \beta_{l0} \\ \beta_l \end{pmatrix}$.
- The likelihood function becomes

$$
\begin{aligned}
L(\beta) &= \sum_{i=1}^{N} \log p_{g_i}(x_i;\beta) \\
&= \sum_{i=1}^{N} \log \left(\frac{e^{\bar{\beta}_{g_i}^T x_i}}{1+\sum_{l=1}^{K-1} e^{\bar{\beta}_l^T x_i}} \right) \\
&= \sum_{i=1}^{N} \left[\bar{\beta}_{g_i}^T x_i - \log \left(1 + \sum_{l=1}^{K-1} e^{\bar{\beta}_l^T x_i} \right) \right]
\end{aligned}
$$

- Note: the indicator function $I(\cdot)$ equals 1 when the argument is true and 0 otherwise.
- First order derivatives:

$$
\begin{aligned}
\frac{\partial L(\beta)}{\partial \beta_{kj}} &= \sum_{i=1}^{N} \left[I(g_i = k) x_{ij} - \frac{e^{\bar{\beta}_k^T x_i} x_{ij}}{1 + \sum_{l=1}^{K-1} e^{\bar{\beta}_l^T x_i}} \right] \\
&= \sum_{i=1}^{N} x_{ij} (I(g_i = k) - p_k(x_i; \beta))
\end{aligned}
$$

- Second order derivatives:

$$
\begin{aligned}
& \frac{\partial^2 L(\beta)}{\partial \beta_{kj} \partial \beta_{mn}} \\
= & \sum_{i=1}^{N} x_{ij} \cdot \frac{1}{(1 + \sum_{l=1}^{K-1} e^{\bar{\beta}_l^T x_i})^2} \cdot \\
& \left[-e^{\bar{\beta}_k^T x_i} I(k = m) x_{in} (1 + \sum_{l=1}^{K-1} e^{\bar{\beta}_l^T x_i}) + e^{\bar{\beta}_k^T x_i} e^{\bar{\beta}_m^T x_i} x_{in} \right] \\
= & \sum_{i=1}^{N} x_{ij} x_{in} (-p_k(x_i; \beta) I(k = m) + p_k(x_i; \beta) p_m(x_i; \beta)) \\
= & -\sum_{i=1}^{N} x_{ij} x_{in} p_k(x_i; \beta) [I(k = m) - p_m(x_i; \beta)] \, .
\end{aligned}
$$

- Matrix form.
  - **y** is the concatenated indicator vector of dimension $N \times (K-1)$.

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_{K-1} \end{pmatrix} \quad \mathbf{y}_k = \begin{pmatrix} I(g_1 = k) \\ I(g_2 = k) \\ \vdots \\ I(g_N = k) \end{pmatrix}$$

$$1 \le k \le K-1$$

  - **p** is the concatenated vector of fitted probabilities of dimension $N \times (K-1)$.

$$\mathbf{p} = \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_{K-1} \end{pmatrix} \quad \mathbf{p}_k = \begin{pmatrix} p_k(x_1;\beta) \\ p_k(x_2;\beta) \\ \vdots \\ p_k(x_N;\beta) \end{pmatrix}$$

$$1 \le k \le K-1$$

- $\tilde{\mathbf{X}}$ is an $N(K-1) \times (p+1)(K-1)$ matrix:

$$\tilde{\mathbf{X}} = \begin{pmatrix} \mathbf{X} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{X} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X} \end{pmatrix}$$

- Matrix $\mathbf{W}$ is an $N(K-1) \times N(K-1)$ square matrix:

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} & \cdots & \mathbf{W}_{1(K-1)} \\ \mathbf{W}_{21} & \mathbf{W}_{22} & \cdots & \mathbf{W}_{2(K-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{W}_{(K-1),1} & \mathbf{W}_{(K-1),2} & \cdots & \mathbf{W}_{(K-1),(K-1)} \end{pmatrix}$$

- Each submatrix $\mathbf{W}_{km}$, $1 \le k, m \le K-1$, is an $N \times N$ diagonal matrix.
- When $k = m$, the ith diagonal element in $\mathbf{W}_{kk}$ is $p_k(x_i; \beta^{old})(1 - p_k(x_i; \beta^{old}))$.
- When $k \neq m$, the ith diagonal element in $\mathbf{W}_{km}$ is $-p_k(x_i; \beta^{old}) p_m(x_i; \beta^{old})$.

- Similarly as with binary classification

$$\frac{\partial L(\beta)}{\partial \beta} = \tilde{\mathbf{X}}^T(\mathbf{y}-\mathbf{p})$$

$$\frac{\partial^2 L(\beta)}{\partial \beta \partial \beta^T} = -\tilde{\mathbf{X}}^T \mathbf{W} \tilde{\mathbf{X}} .$$

- The formula for updating β^{new} in the binary classification case holds for multiclass.

$$\beta^{new} = (\tilde{\mathbf{X}}^T \mathbf{W} \tilde{\mathbf{X}})^{-1} \tilde{\mathbf{X}}^T \mathbf{W} \mathbf{z} ,$$

where $\mathbf{z} \triangleq \tilde{\mathbf{X}}\beta^{old} + \mathbf{W}^{-1}(\mathbf{y}-\mathbf{p})$. Or simply:

$$\beta^{new} = \beta^{old} + (\tilde{\mathbf{X}}^T \mathbf{W} \tilde{\mathbf{X}})^{-1} \tilde{\mathbf{X}}^T (\mathbf{y}-\mathbf{p}) .$$

Computation Issues

- Initialization: one option is to use $\beta = 0$.
- Convergence is not guaranteed, but usually is the case.
- Usually, the log-likelihood increases after each iteration, but overshooting can occur.
- In the rare cases that the log-likelihood decreases, cut step size by half.

Connection with LDA

- Under the model of LDA:

$$
\begin{aligned}
& \log \frac{Pr(G = k \mid X = x)}{Pr(G = K \mid X = x)} \\
= & \log \frac{\pi_k}{\pi_K} - \frac{1}{2}(\mu_k + \mu_K)^T \Sigma^{-1}(\mu_k - \mu_K) \\
& + x^T \Sigma^{-1}(\mu_k - \mu_K) \\
= & a_{k0} + a_k^T x \; .
\end{aligned}
$$

- The model of LDA satisfies the assumption of the linear logistic model.
- The linear logistic model only specifies the conditional distribution $Pr(G = k \mid X = x)$. No assumption is made about $Pr(X)$.

- The LDA model specifies the joint distribution of X and G. $Pr(X)$ is a mixture of Gaussians:

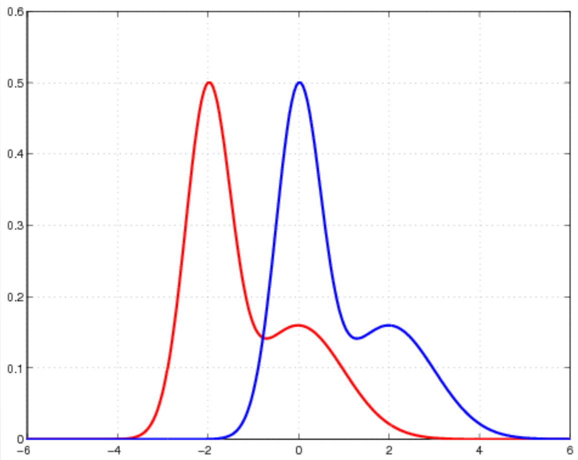
$$Pr(X) = \sum_{k=1}^{K} \pi_k \phi(X; \mu_k, \Sigma) \ .$$

  where ϕ is the Gaussian density function.
- Linear logistic regression maximizes the conditional likelihood of G given X: $Pr(G = k \mid X = x)$.
- LDA maximizes the joint likelihood of G and X: $Pr(X = x, G = k)$.

- If the additional assumption made by LDA is appropriate, LDA tends to estimate the parameters more efficiently by using more information about the data.
- Samples without class labels can be used under the model of LDA.
- LDA is not robust to gross outliers.
- As logistic regression relies on fewer assumptions, it seems to be more robust.
- In practice, logistic regression and LDA often give similar results.

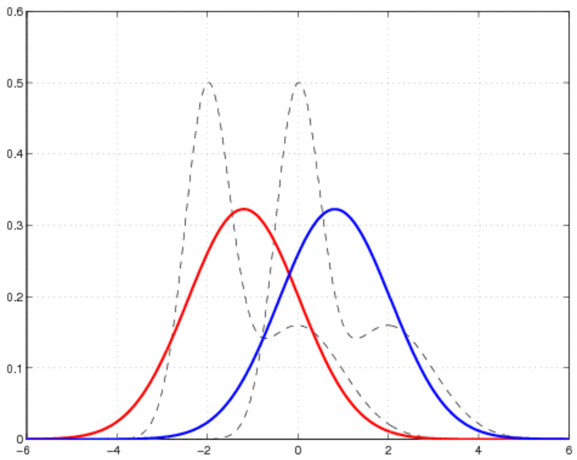
Simulation

- Assume input X is 1-D.
- Two classes have equal priors and the class-conditional densities of X are shifted versions of each other.
- Each conditional density is a mixture of two normals:
  - Class 1 (red): $0.6N(-2, \frac{1}{4}) + 0.4N(0, 1)$.
  - Class 2 (blue): $0.6N(0, \frac{1}{4}) + 0.4N(2, 1)$.
- The class-conditional densities are shown below.


0.6
0.5
0.4
0.3
0.2
0.1
0
−6
−4
−2
0
2
4
6

LDA Result

- Training data set: 2000 samples for each class.
- Test data set: 1000 samples for each class.
- The estimation by LDA: $\hat{\mu}_1 = -1.1948$, $\hat{\mu}_2 = 0.8224$, $\hat{\sigma}^2 = 1.5268$. Boundary value between the two classes is $(\hat{\mu}_1 + \hat{\mu}_2)/2 = -0.1862$.
- The classification error rate on the test data is 0.2315.
- Based on the true distribution, the Bayes (optimal) boundary value between the two classes is -0.7750 and the error rate is 0.1765.


0.6
0.5
0.4
0.3
0.2
0.1
0
−6
−4
−2
0
2
4
6

Logistic Regression Result

- Linear logistic regression obtains

$$\beta = (-0.3288, -1.3275)^T \ .$$

  The boundary value satisfies $-0.3288 - 1.3275X = 0$, hence equals -0.2477.

- The error rate on the test data set is 0.2205.
- The estimated posterior probability is:

$$Pr(G = 1 \mid X = x) = \frac{e^{-0.3288-1.3275x}}{1 + e^{-0.3288-1.3275x}} \ .$$

The estimated posterior probability $Pr(G = 1 \mid X = x)$ and its true value based on the true distribution are compared in the graph below.