Asymptotic Performance of Vector Quantizers with a Perceptual Distortion Measure

Jia Li, Navin Chaddha, and R.M. Gray

E-mail: jiali@isl.stanford.edu, navin@vxtreme.com, rmgray@stanford.edu
Outline

Source mismatch

Examples

Asymptotic analysis
- Variable rate coding
- Fixed rate coding

Distortion

Bounds for asymptotic optimal performance with a perceptual

Preliminaries
Motivation

- What is the performance loss?
- Perceptual distortion measure has to be estimated in real life.
- Source mismatch

Standards for image quality become more demanding.
- MSE does not correlate well with subjective quality
- Probability density function has to be estimated in real life.

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VectorQuantizer

\[
\forall x \in S_i \quad \mathcal{Y} = (x)^\mathcal{Q} 
\]

A partition \( \{ S_i \}_{i=1}^{\mathcal{N}} \) of \( \mathcal{X} \) •

\( \{ S_i \}_{i=1}^{\mathcal{N}} \)

Codebook \( \mathcal{C} \) •

\( \mathcal{X} \subset \mathbb{R}^k \) Euclidean space.

Quantize vector \( x \in \mathbb{R}^k \) to \( x_i \in \mathcal{Y} \), \( i=1, \ldots, \mathcal{N} \) •

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is positive definite almost everywhere.

\[ \kappa = x \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} I = (\kappa)^{uw} B \]

with the \( i \), \( m \) element with the \( m \), \( n \) element as a \( l \) by \( l \) dimensional matrix almost everywhere.

\[ \kappa = x \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} I \]

has continuous partial derivatives of third order almost everywhere.

\[ \kappa = x \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} I \]

\[ \kappa \triangleq \begin{cases} 0 & \text{if} \ k \leq 0 \\ 1 & \text{if} \ k > 0 \end{cases} \]

Regularity constraints on \( \kappa \)

General form of distortion is denoted by function \( (\kappa, x) \)
Why perceptually meaningful?

Examples:


More examples can be found in:

\[(\Lambda - x)(x)B_{\lambda}(\Lambda - x)\text{ and } (\Lambda - x)(x)B_{\lambda}(\Lambda - x)\]

- Output and input weighted quadratic distortion.
- Output and input weighted quadratic distortion.
- Parameters of the LPC model for speech.
- A low log spectral distortion (LSD) for the quantization of

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Approximation for $L(x, y)$.

Given a code book $B = \{ N, \ldots, N \}$ denote $B_i = 1$.

Assumptions for approximation:
- Quantizer is high rate, all $S_i$ small volume and $N$ large.
- As a result of the above assumptions, $(\Lambda, x) \approx (\Lambda, x)^T$ can be approximated by:

$$ (\Lambda - x)B_i(\Lambda - x) \approx (\Lambda, x)^T $$

Assumptions for approximation:

**Approximation for** $L(x, y)$
The volume of the unit sphere in the $k$-dimensional space for the quadratic norm is:

\[
V_i = \frac{\det B_i}{C_k^{k/2}} = \frac{\Lambda}{2^{k/2}}
\]

where \( C_k = \frac{1}{2^{k/2}} \) is the volume of the $k$-dimensional Euclidean norm.

The quadratic norm \( \| \cdot \|_{\mathcal{B}} \) satisfies:

\[
\mathcal{B}_i x = \| x \|
\]

The volume of the unit sphere in the $k$-dimensional space is:

\[
V_i = \frac{\det B_i}{C_k^{k/2}} = \frac{\Lambda}{2^{k/2}}
\]
\[
(S)\Lambda = \Lambda_{\|}(S)\mathcal{U} = ((S)\mathcal{L})\Lambda
\]
\[
\{(S)\mathcal{U} = \| \Lambda - x \| : x\} = \{(S)\mathcal{U} \supset (\Lambda - x)^2B_4(\Lambda - x)^\wedge : x\} = (S)\mathcal{L}
\]
\[
\text{The effective region of } (S)\mathcal{L} \text{ centered at } \Lambda
\]
\[
\frac{1}{\frac{1}{4}(\Lambda/(S)\Lambda)} = \frac{1}{(S)\mathcal{U}}
\]
\[
\text{The effective radius of } (S)\mathcal{U} \text{ of } (S)\Lambda
\]
\[ M_i = \begin{cases} \frac{1}{c_k} w_i t, \\ \frac{1}{1-c_k} v_i = \frac{1}{1-c_k} w_i d, \end{cases} \]

For mathematical convenience:

\[ \Delta p \left( \mathbb{W} \frac{y_i}{\lambda} \right)_i \frac{\mathbb{W} \frac{y_i}{\lambda}}{\lambda} \int \frac{y_i}{1} = (a)^M \]

**Generalized Gish-Pierce Function**
\[ xp(\mathcal{X} - x) \mathcal{B}_q(\mathcal{X} - x) \int ((\mathcal{S})_\Lambda \mathcal{I}^I) \sum_{N} \approx D \]

\[ xp(x) \mathcal{D} \int = \mathcal{D} \]

Assuming \( p(x) \) is sufficiently smoothed, for high rate quantizer \( Q \) denote \( P_i = R(S_i \mathcal{T}(x)) \).

\[ \cdot \quad xp(\mathcal{X} \mathcal{T}(x)) \mathcal{D} \int \sum_{N} = \]

\[ xp((x) \mathcal{D}(x)) \mathcal{T}(x) \mathcal{D} \int = ((x) \mathcal{D}(x)) \mathcal{T} \mathcal{E} = D \]

The performance is measured by the average distortion.

The rate is measured by \( \log N \) where \( N \) is the total number of codewords.

**Fixed Rate Coding**
Lower Bounds for FRC

A crucial inequality:

\[
\cdot \left( \frac{1}{t} (S') \Lambda \right) \frac{1}{N} \sum_{N} W \supseteq \mathcal{D}
\]

\[
\cdot \left( \frac{1}{t} (S') \Lambda \right) W (S') \Lambda = \chi p (\lambda - x) \mathcal{B}_i (\lambda - x) \int
\]

\[
\chi p (\lambda - x) \mathcal{B}_i (\lambda - x) \int \supseteq \chi p (\lambda - x) \mathcal{B}_i (\lambda - x) \int
\]

Lower bound:

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Following Gersho's assumption, we assume that as $N \to \infty$, there is a limiting density $(x) N$ (quantization point density function).

Using Gersho's assumption, we assume that as $N \to \infty$, there is a limiting density $(x) N$ (quantization point density function).

\[ \Lambda_{-1}((x) N) \approx (x) N - \text{has unit integral} \]

\[ N \to \infty \]

\[ \forall N \in \mathbb{R}, x \in S \quad \exists \lim_{N \to \infty} ((x) N) = (x) N \]
Lower Bounds for FRC

Using limiting density function

\[
\frac{xp \frac{z+y}{y} [\frac{y}{x} ((x)G_\gamma \text{det})(x)] \int \frac{z+y}{y} [((x)G_\gamma \text{det})(x)]}{z+y} = (x)^{d_0} \gamma \text{ where}
\]

\[
xp(x)d_{\frac{y}{z}}[((x)G_\gamma \text{det})]\frac{y}{z} - (x)^{d_0} \gamma \int \cdot \frac{y}{z} - N \cdot \frac{y}{z} C \frac{z+y}{y} = (d_0 \gamma) D
\]

The lower bound for the asymptotic distortion is

\[
\cdot \left( \frac{y}{z} - ((\gamma N))^{d_0} \gamma \sum_{N} \int \right) \leq D
\]

Using limiting density function

\[
(x)^{d_0} \gamma
\]
In conclusion:

\[ \text{Approximation holds } (\vartheta^d \vartheta)^T \vartheta \approx (\vartheta^d \vartheta) \vartheta \]

\[
\left[ x p_{\frac{\gamma}{\gamma+1}} \left[ \frac{1}{\gamma} ( (x) (x)^T ) d \right] \int \right] \cdot \frac{\gamma}{\gamma+1} \frac{1}{\gamma} - N \frac{\gamma}{\gamma+1} \frac{1}{\gamma} \leq (\vartheta^d \vartheta)^T \vartheta \leq (\vartheta^d \vartheta) \vartheta
\]
Question: For a given rate, what is the asymptotic average distortion?

The rate is the entropy of the encoded source.

Variable Rate Coding
Recall that the lower bound is:

\[
\cdot \quad xP(x) \int_{-\infty}^{\infty} \left[ (x)^{i \cdot d \cdot O} N \right] \int \\
\times \left[ xP(x) d_{\frac{1}{T}} \left[ (x) B \right] \right] \frac{2 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}} = (x) f \quad \text{defining 
B \text{ by defining}} \quad B: \text{to simplify}
\]

Recall that the lower bound is:

\[
\text{Lower Bound for VRC}
\]
The entropy of the encoded source is approximately

\[ \text{opt} \quad \mathcal{H}(x) = \min_{\tilde{x}} \mathcal{H}(\tilde{x}) \]

i.e.,

\[ \min_{\tilde{x}} \mathcal{H}(\tilde{x}) \]

\( (\text{opt} \circ \mathcal{H}) \) and

\[ \min_{\tilde{x}} \mathcal{H}(\tilde{x}) \]

where \( C \) is a constant.

\[ C \geq \mathcal{H}(x) \delta \log \left( \frac{\mathcal{H}(x)}{I} \right) + (d) \eta \]

Given the constant \( \delta \).

Reduce to optimization problem as follows:

\[ (x) \mathcal{H} = (x) \delta \]

Denote

\[ \left[ (x) \mathcal{H} \right] \delta \log \left( \frac{\mathcal{H}(x)}{x} \right) \quad \mathcal{H} - (d) \eta \approx \mathcal{O} \]

The entropy of the encoded source is approximately

\[ (x) \mathcal{H} \text{ opt} \]

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By Jensen’s inequality, the optimal limiting density is 

\[
(x)_{\text{opt}} \subset (\partial') \subset \partial \\
\text{The lower bound is} \\
\text{Obtain lower bound and} \\
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\]
\[
\begin{align*}
\mathbf{y}^\top \mathbf{\Lambda} \mathbf{y} & \propto (\mathbf{\Lambda})^y \\
\mathbf{y}^\top \mathbf{\Lambda} \mathbf{y} & = ((\mathbf{\Lambda}) \mathbf{B})^\text{det}
\end{align*}
\]

Hence

\[
\begin{align*}
\frac{z^\top \mathbf{\Lambda} \frac{z}{1} = (\mathbf{\Lambda})^{u \cdot w} & \quad u = w \\
0 = (\mathbf{\Lambda})^{u \cdot w} & \quad u \neq w
\end{align*}
\]

\[
\mathbf{I} \frac{z - \mathbf{x}}{1} = (\mathbf{x}) \mathbf{M} \quad \text{where} \quad (\mathbf{\Lambda} - \mathbf{x})(\mathbf{x}) \mathbf{M}_1(\mathbf{\Lambda} - \mathbf{x}) = (\mathbf{\Lambda}, \mathbf{x})^T
\]

\begin{quote}
Example
\end{quote}
performance due to source mismatch.

Apply previous results to quantify the possible change in

\[ f(x)p(x) \]  

As shown before, for fixed rate coding, \( f(x)p(x) \) depends on \( d(x) \).

In real life, \( f(x) \) must be estimated.

Source Mismatch
determining the distortion.

All the sub-vectors $\mathbf{x}$ play an equal and independent role in

\[
\left( \begin{array}{cccc}
\mathbf{x}^{(1)} & \cdots & \mathbf{x}^{(k)} \\
\mathbf{x}^{(1)} & \cdots & \mathbf{x}^{(k)}
\end{array} \right) = \mathbf{x}' \quad \text{where, } \left( \begin{array}{c}
((\mathbf{x})\mathcal{B}) \\
((\mathbf{x})\mathcal{B})
\end{array} \right) \prod_{i=1}^{k} \det = ((\mathbf{x})\mathcal{B})', \text{ and vectors, }

\text{where } \mathbf{x}^{(i)} \text{'s are i.i.d. random vectors,}

\left( \begin{array}{cccc}
\mathbf{x}^{(1)} & \cdots & \mathbf{x}^{(k)} \\
\mathbf{x}^{(1)} & \cdots & \mathbf{x}^{(k)}
\end{array} \right) = \mathbf{x} \quad \text{where, } \left( \begin{array}{c}
((\mathbf{x})\mathcal{B}) \\
((\mathbf{x})\mathcal{B})
\end{array} \right) \prod_{i=1}^{k} \det \text{ is easily generalized to } \mathbf{x} \text{ and random variables.}

\text{Vector } \mathbf{x} \text{ 's are i.i.d. random variables.

Basic Assumptions}
Analyzetheasymptoticcasewhenvectordimension

\[ k \approx 1 \]

Thelossofperformanceismeasuredbytheincreaseof
distortionindB, i.e., \( 10 \log_{10} \frac{D}{D_{opt}} \).

The loss of performance is measured by the increase of
distortion in dB, i.e., \( 10 \log_{10} \frac{D}{D_{opt}} \).

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distortion in dB, i.e., \( 10 \log_{10} \frac{D}{D_{opt}} \).

\[ \left( \frac{d_{opt}}{D} \right) \frac{d_{opt}}{D} \]

Analyzetheasymptoticcasewhenvectordimension

\( k \approx \infty \).
LossofPerformance

Byusingestimatedpdf/pd/x, theoptimalpd/x/x = \( (\vec{x})_{\opt} \)

where

\[ \prod_{\gamma} \left( (\vec{x})_{\opt} \right)^{\frac{1}{\gamma}} = (\vec{x})_{\opt} \]

Asaresultof

\[ (\vec{x})_{\opt} \prod_{\gamma} \left( (\vec{x})_{\opt} \right)^{\frac{1}{\gamma}} = (\vec{x})_{\opt} \]

\( (\vec{x})_{\opt} \prod_{\gamma} \left( (\vec{x})_{\opt} \right)^{\frac{1}{\gamma}} = (\vec{x})_{\opt} \)

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Byusingestimatedpdfpd/x, theoptimalpd/x is

\[ \text{Loss of Performance} \]
The limits for the three terms will be derived.

\[
\begin{align*}
\text{Continued}
\end{align*}
\]
Under conditions stated later, the following three limits exist:
are continuous functions, the above four conditions hold. If the domain of $x$ is a bounded closed set, and $p$ and $q$ are continuous functions, the above four conditions hold.

1. \[ \int_{\mathcal{A}} \frac{(x\mathcal{B} - (x)\mathcal{d})}{\mathcal{S}} \, d\mathcal{E} \]

2. \[ \int_{\mathcal{A}} \frac{(x\mathcal{B} - (x)\mathcal{d})}{\mathcal{S}} \, d\mathcal{E} \]

3. \[ \int_{\mathcal{A}} \frac{(x\mathcal{B} - (x)\mathcal{d})}{\mathcal{S}} \, d\mathcal{E} \]

4. \[ \int_{\mathcal{A}} \frac{(x\mathcal{B} - (x)\mathcal{d})}{\mathcal{S}} \, d\mathcal{E} \]
\[
\infty \leftarrow \infty
\]
which means the effect of \(0\) is washed out when \(k\) is.

\[I \text{ is interesting to notice the limit loss is independent of } B.\]

An example of the relative entropy being an important distance measure on probability density functions.

\[
\mathcal{D}(\pi(x) \| \pi(x)) \approx \frac{1}{\lambda} \log \frac{D}{Q}
\]

The loss in \( DB \) when \( k \) is:

(Final Result)
References

