Research Statement – Michael Francis

My research activities during my PhD have primarily been in the fields of operator theory and differential geometry. My dissertation centers around groupoids and operator algebras associated to certain singular foliations, building on work of Iakovos Androulidakis and Georges Skandalis. This work was supported by a postgraduate doctoral scholarship (PGS-D) from the Natural Sciences and Engineering Research Council of Canada (NSERC) from 2015-2018 and a teaching assistantship from Pennsylvania State University from 2015-2020.

1 Background information on foliations and holonomy

I begin with an informal primer on foliations and holonomy, highlighting the differences between the singular context, where I have done work, and the regular context.

Definition 1 ([1], Definition 1.1)**.** A (possibly singular) *foliation* $\mathcal{F}$ of a smooth manifold M is a locally finitely-generated $C^\infty(M)$-module of compactly-supported, smooth vector fields on M that is closed under Lie bracket.

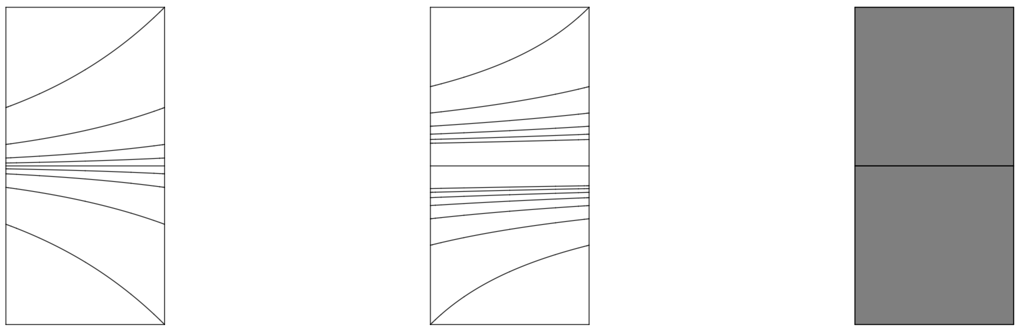
The set of points accessible from a given point using the flows of the vector fields in $\mathcal{F}$ is called a *leaf*. By work of Stefan and Sussmann ([10], [11]), the leaves of $\mathcal{F}$ constitute a partition of M into immersed submanifolds. If all the leaves have the same dimension, the foliation is said to be *regular*. Otherwise, the foliation is *singular*. Besides being a classical topic in geometry, foliations play an important role in classical mechanics and optimal control theory. In the regular setting, the module of vector fields can be recovered from the partition, but this fails in the singular setting. Indeed, considering different modules which determine the same partition is a prominent theme in my work.

Recall that the solution operators of certain PDEs can be usefully represented by smooth integral kernels, e.g. in the case of the heat equation on a Riemannian manifold M. The value of the kernel at $(x, y) \in M \times M$ may be understood as a measure of how much the operator propogates from y to x. Whereas heat flow propogates in all directions, in other important situations one would like to consider operators which only propogate along the leaves of some foliation. A kernel representing such an operator should then be a function on the equivalence relation whose classes are the leaves, and not the whole manifold $M \times M$. This motivates the following problem:

Problem 2. *What is a smooth function on the leaf equivalence relation of a given foliation?*

The problem is complicated by the fact the leaf equivalence relation need not be a submanifold of $M \times M$. The reason, and also the remedy, for this failure of smoothness is an interesting and important phenomenon known as *holonomy*. The *holonomy groupoid* $G(\mathcal{F})$ of a foliation $\mathcal{F}$ tries to desingularize the equivalence relation, somewhat in the spirit of blowups in algebraic geometry. When $G(\mathcal{F})$ is a Lie groupoid (the so-called *almost regular* case), the natural solution to Problem 2 is "a smooth function on $G(\mathcal{F})$". The holonomy groupoid was defined for regular foliations by Winkelnkemper [12] and extended to singular cases by various authors. A very general construction of $G(\mathcal{F})$ was given by Androulidakis and Skandalis in [1]. The use of $G(\mathcal{F})$ in operator theory was pioneered by Connes [3].

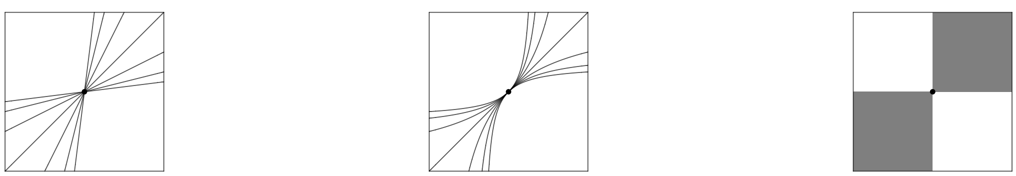
Figure 1 illustrates several foliations of the cylinder $S^1 \times \mathbb{R}$, regarded as having coordinates (x, y), where x is $\mathbb{Z}$-periodic. I have used the notation $\mathcal{F}\{X_1, \ldots, X_n\}$ to denote a foliation generated by a finite set of vector fields $X_1, \ldots, X_n$.



$\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}\}$ $\quad$ $\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}\}$ $\quad$ $\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$.

Figure 1: Leaves of some foliations of $S^1 \times \mathbb{R}$.

The first two foliations in Figure 1 are regular while the third is singular; its leaves are $S^1 \times (0, \infty)$, $S^1 \times \{0\}$ and $S^1 \times (-\infty, 0)$. All three foliations determine nonsmooth equivalence relations. The issue is visible when we restrict attention to the transversal $T = \{0\} \times \mathbb{R}$ passing through $p = (0, 0)$. The resulting subsets of $T \times T$ are depicted in Figure 2.



$\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}\}$ $\quad$ $\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}\}$ $\quad$ $\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$.

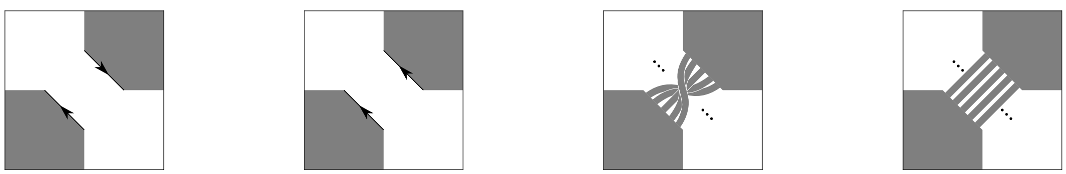
Figure 2: Equivalence relations of some foliations of $S^1 \times \mathbb{R}$, restricted to $T = \{0\} \times \mathbb{R}$.

The holonomy groupoid, however, is smooth for all these foliations. In terms of the pictures above, what occurs is the problematic point (p, p) at the origin gets blown up and replaced with the *holonomy group* H_p. For a regular foliation, given a point p and a transversal T through the leaf of p, H_p may be viewed as the discrete group consisting of all germs of diffeomorphisms of T fixing p which can be obtained using flows of vector fields in $\mathcal{F}$. For both the regular foliations shown above, H_p is infinite cyclic, and it is easy to imagine how such a replacement can resolve the singularity.

A key difference between the regular and singular settings is that, whereas for regular foliations holonomy is purely a discrete phenomenon, for singular foliations one can also have *continuous holonomy.* For the singular foliation $\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$ shown above, H_p is isomorphic to the Lie group $\mathbb{R}$. However, this is just one many foliations of $S^1 \times \mathbb{R}$ whose leaves are $S^1 \times (0,\infty)$, $S^1 \times \{0\}$ and $S^1 \times (-\infty, 0)$. With the exception of some pathological examples, the holonomy group H_p for any such foliation is naturally realized, for some positive integer k, as a one-dimensional subgroup of the group $J^k(\mathbb{R})$ of k-jets of orientation-preserving diffeomorphisms of $\mathbb{R}$ which fix 0. Explicitly, $J^k(\mathbb{R}) = \{a_1 y + a_2 y^2 + \dots a_\ell y^k : a_i \in \mathbb{R}, a_1 \neq 0\}$ under the operation "compose and truncate". Some examples are tabulated below:

$$\begin{array}{rcl}
\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\} & \rightsquigarrow & H_p \cong \{e^t y : t \in \mathbb{R}\} \subseteq J^1(\mathbb{R}) \\
\mathcal{F}\{\frac{d}{dx}, y^2\frac{d}{dy}\} & \rightsquigarrow & H_p \cong \{y + ty^2 : y \in \mathbb{R}\} \subseteq J^2(\mathbb{R}) \\
\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}, y^2\frac{d}{dy}\} & \rightsquigarrow & H_p \cong \{e^n y + ty^2 : n \in \mathbb{Z}, t \in \mathbb{R}\} \subseteq J^2(\mathbb{R}) \\
\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}, y^4\frac{d}{dy}\} & \rightsquigarrow & H_p \cong \{y + ny^2 + n^2y^3 + ty^4 : n \in \mathbb{Z}, t \in \mathbb{R}\} \subseteq J^4(\mathbb{R})
\end{array}$$

Note that H_p is diffeomorphic to $\mathbb{R}$ in the first two cases and $\mathbb{R} \times \mathbb{Z}$ in the second two cases. The precise details of how (p,p) is blown up into a copy of H_p depend also on two natural orderings of group $J^k(\mathbb{R})$, associated to the postive and negative half lines. As Figure 3 shows, the topological possibilities for the blowup space are actually quite rich, especially given how simple the leaf space of these foliations is. The last two surfaces are not homeomorphic, as can be seen by counting the number of topological "ends".



$\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$ $\quad$ $\mathcal{F}\{\frac{d}{dx}, y^2\frac{d}{dy}\}$ $\quad$ $\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}, y^2\frac{d}{dy}\}$ $\quad$ $\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}, y^4\frac{d}{dy}\}$

Figure 3: Holonomy groupoids of some singular foliations of $S^1 \times \mathbb{R}$, restricted to $T = \{0\} \times \mathbb{R}$.

2 Doctoral research

My doctoral research produced three articles in my area of concentration ([8], [5], [7]) and one collaborative paper in graph theory [9]. I summarize the main thrust of each work below.

2.1 The smooth algebra of a one-dimensional singular foliation

Given any singular foliation $\mathcal{F}$ of a smooth manifold M, it was shown in [1] how to construct a holonomy groupoid $G(\mathcal{F})$, a smooth convolution algebra $\mathcal{A}(\mathcal{F})$ and a C*-algebra[1] $C^*(\mathcal{F})$.

[1] Actually, multiple C*-completions can be considered, including a reduced and a maximal version. For the examples considered here, all the standard completions agree.

In the article [8], I consider a specific family of singular foliations of the real line and obtain a complete classification of their smooth convolution algebras and C*-algebras. The main findings may be summarized as follows.

Theorem 3 ([8], Theorems 3,4,5)**.** *For each positive integer k, let $\mathcal{F}_{\mathbb{R}}^k$ denote the singular foliation of the real line singly-generated by $y^k \frac{d}{dy}$.*

1. *The smooth convolution algebras of the $\mathcal{F}_{\mathbb{R}}^k$ are pairwise nonisomorphic.*
2. *The C*-algebras of the $\mathcal{F}_{\mathbb{R}}^k$ are of two isomorphism types that are determined by the parity of k.*
3. *The C*-algebras of the $\mathcal{F}_{\mathbb{R}}^k$ are represented in a natural way on $L^2(\mathbb{R})$. The images of these representations are pairwise distinct.*

This demonstrates the principle that there can be information stored in the smooth algebra which is washed away when one passes to the C*-algebra.

2.2 On certain singular foliations with finitely many leaves

In a forthcoming article [5], I define and analyze a class of singular foliations which I call *transversely order-k foliations*. These are foliations which have exactly one singular leaf L of codimension one around which the transverse structure is modeled on the one-dimensional foliation of $\mathcal{F}_{\mathbb{R}}^k$ in the theorem above.

Unlike in the context of regular foliations, a loop in L does not determine a holonomy transformation in the usual sense of a diffeomorphism germ on a transversal. I show, however, that one does have a well-defined holonomy mapping at the level of $(k-1)$-jets. In this way, I assign an invariant to a transversely-order k foliation taking the form of a homomorphism (well-defined up to conjugation) from $\pi_1(L)$ into $J^{k-1}(\mathbb{R})$, the group of $(k-1)$-jets of diffeomorphisms of $\mathbb{R}$ fixing the origin.

Theorem 4 ([5])**.** *The restriction of a transversely order k foliation to a small neighbourhood of its singular leaf L is uniquely determined by the above invariant. Moreover, the possible values of this invariant are exhausted by transversely order k foliations.*

I furthermore analyze the holonomy groupoid and C*-algebra of transversely order k foliations and obtain the following.

Theorem 5 ([5])**.** *The holonomy groupoid of a transversely order k foliation is a Hausdorff Lie groupoid whose restriction to L is isomorphic to the gauge groupoid of a certain principal bundle. The structure group of this principal bundle is solvable; it is a one-dimensional subgroup $E \subseteq J^k(\mathbb{R})$ determined by the holonomy invariant above. The foliation C*-algebra is isomorphic to an extension of $C^*(E) \otimes \mathbb{K}$ by either $\mathbb{K}$ or by $\mathbb{K} \oplus \mathbb{K}$. Here, $\mathbb{K}$ denotes the algebra of compact operators on separable Hilbert space.*

2.3 A Dixmier-Malliavin theorem for Lie groupoids

A famous theorem of Dixmier and Malliavin ([4], 3.1 Théorème) states that every smooth, compactly-supported function on a Lie group can be expressed as a finite sum in which each term is the convolution (with respect to Haar measure) of two such functions. This result has applications to the representation theory of real reductive groups.

To complete the proof of the main theorem in [8], I needed to know that every element of the smooth convolution algebra of the foliation $\mathcal{F}^k_{\mathbb{R}}$ could be expressed as a finite sum of convolution products. In other words, a version Lie groupoid version of the Dixmier-Malliavin theorem was needed. In the article [7], I took up the general form of this problem and extended Dixmier-Malliavin's result to the setting of arbitrary Lie groupoids.

Theorem 6 (F, 2018)**.** *Let G be a Lie groupoid with a smooth Haar system and form the smooth convolution algebra $C^\infty_c(G)$. Then, every $f \in C^\infty_c(G)$ can be expressed as $g_1 * h_1 + \ldots + g_n * h_n$ for some positive integer n and $g_1, h_1, \ldots, g_n, h_n \in C^\infty_c(G)$.*

In the same article I obtained results on the multiplication structure of certain ideals in $C^\infty_c(G)$ arising from functions vanishing to given order along a given invariant submanifold of the unit space. These results on ideals are only interesting after one has generalized to the groupoid setting. In the group case, the unit space consists of a single point and these ideals do not arise at all.

2.4 Graph theory

My mathematical interests are quite varied and I enjoy interacting with researchers from other areas. A collaboration with Professors Kieka Mynhardt and Jane Wodlinger resulted in the article [9] in the field of graph theory. The main result of this article is stated below. A *minimum decycling set* in a graph G (finite, with no loops or multiple edges) is a set of vertices which breaks every cycle of G and has as few vertices as possible.

Theorem 7 (Theorem 1.1, [9])**.** *With the exception of the complete graph on $r+1$ vertices. every finite graph G with maximum degree r has a minimum decycling set S whose induced subgraph $G[S]$ does not contain any $(r-2)$-regular subgraph.*

This result has several corollaries, including Brooks' theorem, and the statement that (except for the complete graph on 4 vertices), every graph of maximum degree 3 admits a minimum decycling set which is also an independent set.

3 Future research

In this section I indicate several problems relating to singular foliations which I am either working on or intend to work on in the future.

3.1 Characterizing the smooth convolution of singular foliations induced by a Lie groupoids

Let $G \rightrightarrows M$ be a Lie groupoid and let $\mathcal{F}$ be the singular foliation of M induced by G. It is known that the smooth convolution algebra $\mathcal{A}(\mathcal{F})$ of Androulidakis and Skandalis fits into an exact sequence

$$0 \to I \to C_c^\infty(G) \to \mathcal{A}(\mathcal{F}) \to 0$$

(fixing a smooth Haar system on G to make sense of convolution). However, the description of the ideal I is very indirect and involves quantification over an infinite number of relators taking the form of "roof diagrams" (Section 4.3, [1]). I am interested in describing the ideal I in more explicit terms. A useful test case for this problem is the singular foliation of $\mathbb{R}^2$ associated to the action of $\mathrm{SL}(2, \mathbb{R})$. Consider the quadratic mapping $Z : \mathbb{R}^2 \to \mathfrak{sl}(2, \mathbb{R})$ defined by

$$Z(x, y) = \begin{bmatrix} xy & -x^2 \\ y^2 & -xy \end{bmatrix}.$$

One may show that Z defines a bi-invariant vector field on the transformation groupoid $\mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})$ that Z generates the singular foliation of $\mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})$ consisting of vector fields which are vertical with respect to both the source and target projections. Using this vector field, I was able to obtain the following.

Theorem 8 (F, 2019). *Let $\mathcal{F}$ be the singular foliation of $\mathbb{R}^2$ generated by the natural action of $\mathrm{SL}(2, \mathbb{R})$. Then, an element $f \in C_c^\infty(\mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R}))$ belongs to the kernel of the natural quotient map $C_c^\infty(\mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})) \to \mathcal{A}(\mathcal{F})$ if and only if $f = Zg$ for $g \in C_c^\infty(\mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R}))$.*

The proof of the above result is related to the "Moser trick" of symplectic geometry. I am working on generalizing my arguments in order to obtain a concrete model for the smooth convolution algebra for singular foliations induced by other Lie groupoids.

Problem 9. *Let $G \rightrightarrows M$ be a Lie groupoid, satisfying some reasonable hypotheses, and let $\mathcal{F}$ the singular foliation induced on M. Describe the kernel of the quotient mapping $C_c^\infty(G) \to \mathcal{A}(\mathcal{F})$ in terms of the singular foliation of G consisting of vector fields which are vertical with respect to both the source and target projections.*

3.2 Grassmannian manifolds as singular foliations

Unlike in the setting of regular foliations, there can exist interesting singular foliations having only a finite number of leaves. I am particularly interested in studying Grassmannian manifolds and their cell structures from the perspective of singular foliations. For example, consider the canonical cell structure of the real projective plane:

$$\mathbb{RP}^n = \mathbb{R}^0 \cup \mathbb{R}^1 \cup \mathbb{R}^2 \cup \ldots \cup \mathbb{R}^n.$$

To view this as a singular foliation, one must furthermore specify a module of vector fields inducing the partition. A simple choice is the module $\mathcal{F}$ determined by the action of the n-dimensional Lie group $G \subseteq GL(n+1, \mathbb{R})$ consisting of matrices of the form:

$$\begin{bmatrix} 1 & a_1 & a_2 & a_3 & \dots & a_n \\ 0 & 1 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & 0 & 1 & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

The orbits of the natural action of G on $\mathbb{RP}^n$ are exactly the parition under consideration. The group G is commutative and, indeed, isomorphic to $\mathbb{R}^n$. The action of G on the top-dimensional leaf is free and transitive and this can be used to show that the holonomy groupoid of the singular foliation associated to this action is just the transformation groupoid $\mathbb{RP}^n \rtimes G$. Correspondingly, the foliation C*-algebra $C^*(\mathcal{F})$ is isomorphic to the crossed product C*-algebra $C(\mathbb{RP}^n) \rtimes G$. The K-theory of $C^*(\mathcal{F})$ can therefore be computed using Connes' Thom isomorphism [2]. One obtains $K_i(C^*(\mathcal{F})) \cong K^{i+n}(\mathbb{RP}^n)$.

In general, Grassmanians and other spaces admitting a natural finite stratications promise to be a rich source of examples for further study.

4 Master's research

My master's thesis [6] gives a self-contained account of the following result of Connes:

Theorem 10 (Connes, [2])**.** *Let A be a C*-algebra endowed with an action α of $\mathbb{R}$ and an α-invariant trace τ. The invariance assumption leads to a canonical dual trace $\widehat{\tau}$ on the crossed product $A \rtimes_\alpha \mathbb{R}$. Then, $\widehat{\tau}_* \phi^1_\alpha[u] = \frac{1}{2\pi i}\tau(\delta(u)u^*)$ holds, where $\widehat{\tau}_*$ is the induced map $K_0(A \rtimes_\alpha \mathbb{R}) \to \mathbb{R}$, ϕ^1_α is the Connes-Thom isomorphism $K_1(A) \to K_0(A \rtimes_\alpha \mathbb{R})$, $\delta = \frac{d}{dt}|_{t=0}$ and u is a (suitably chosen) unitary element of the unitization of A.*

One novelty of this thesis is its scrupulous avoidance of von Neumann algebraic methods in favour of C*-algebraic ones in its treatment of unbounded traces. Another contribution is a "modern" proof of following quantum mechanical theorem:

Theorem 11 (Bargmann-Wigner, c. 1960)**.** *Given a strongly continuous 1-parameter group α_t of $*$-automorphisms of the algebra of compact operators on a separable Hilbert space, there exists a strongly continuous 1-parameter unitary group U_t of unitaries such that $\alpha_t = \mathrm{Ad}(U_t)$ for all $t \in \mathbb{R}$.*

The usual method of proof is to first implement α_t by a measurable family of unitaries, then correct that family to a 1-parameter group using a measurable, circle-valued cocycle and, finally, appeal to an automatic continuity result of von Neumann. I gave a new proof of the above theorem based on Connes' result that, for any projection e, one can explicitly define a continuous unitary cocycle u_t such that $\mathrm{Ad}(u_t) \circ \alpha_t$ leaves e invariant.

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