Formal exponential maps and Atiyah class of dg manifolds

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Atiyah class of holomorphic vector bundle

- X : complex manifold
- E : holomorphic vector bundle over X
- (smooth) connection $\nabla^{1,0} : \Gamma(E) \to \Omega^{1,0}(E)$ of type $(1,0)$:

$$\nabla^{1,0}(f \cdot s) = \partial(f) \cdot s + f \cdot \nabla^{1,0}(s), \quad s \in \Gamma(E), f \in C^\infty(X)$$

- Choose $\mathcal{R} = \nabla^{1,0}\bar{\partial} + \bar{\partial}\nabla^{1,0} : \Gamma(E) \to \Omega^{1,1}(E)$, then

$$\mathcal{R} \in \Omega^{1,1}(\mathrm{End}(E)).$$

Definition (Atiyah, 1957): The Atiyah class α_E of E is the cohomology class

$$\alpha_E = [\mathcal{R}] \in H^1\big(X; \Omega^1_X \otimes \mathrm{End}(E)\big)$$

– Atiyah class α_E is an obstruction to the existence of holomorphic connection on E.

$L_\infty[1]$-algebra

Definition: A graded vector space V is an L_∞ algebra if there is a sequence of maps $q_i : \Lambda^i V \to V$ of degree $2-i$ for $i = 1, 2, \ldots,$ satisfying list of axioms.

Example: A Lie algebra $\mathfrak{g}$ with Lie bracket $[-,-]$ is an L_∞ algebra by $q_1 = 0$, $q_2 = [-,-]$, $q_{\geq 3} = 0$.

Example: A differential graded Lie algebra $\mathfrak{g}$ with differential d and Lie bracket $[-,-]$ is an L_∞ algebra by $q_1 = d$, $q_2 = [-,-]$ and $q_{\geq 3} = 0$.

- W is an $L_\infty[1]$ algebra $\iff$ $W[-1]$ is an L_∞ algebra
- Each map $r_i : S^i(W) \to W$ has degree $+1$

Kapranov's $L_\infty[1]$-algebra

Theorem (Kapranov, 1999): Let X be a Kähler manifold. The Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ admits an $L_\infty[1]$ algebra structure $(\lambda_k)_{k\geq 1}$ where λ_k is the wedge product

$$\Omega^{0,j_1}(T_X^{1,0}) \odot \cdots \odot \Omega^{0,j_k}(T_X^{1,0}) \to \Omega^{0,j_1+\cdots+j_k}(S^k(T_X^{1,0}))$$

composed with

$$R_k : \Omega^{0,\bullet}(S^k(T_X^{1,0})) \to \Omega^{0,\bullet+1}(T_X^{1,0})$$

where $\odot$ is graded symmetric tensor (w.r.t to $j_1, j_2, \cdots$) and

- $R_1 = \bar{\partial}$,
- $R_2 = \mathcal{R} = \mathcal{R}^{\nabla^{1,0}}$ is an Atiyah cocycle,
- $R_{n+1} = d^{\nabla^{1,0}}(R_n)$ for $n \geq 2$

Graded manifolds

Let M be a smooth manifold equipped with structure sheaf $\mathcal{O}_M$.

Definition: A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with base manifold M is a sheaf $\mathcal{A}$ of $\mathbb{Z}$-graded commutative $\mathcal{O}_M$-algebras such that

$$\mathcal{A}(U) \cong \mathcal{O}_M(U) \hat{\otimes} \hat{S}(V^\vee)$$

for sufficiently small open subsets $U \subset M$ and some $\mathbb{Z}$-graded vector space V. In other words, smooth functions on $\mathcal{M}$ are locally formal power series in V with coefficients in $\mathcal{O}_M$.

$$C^\infty(\mathcal{M}) := \mathcal{A}(\mathcal{M})$$

Example: Given a $\mathbb{Z}$-graded vector bundle $E \to M$, $\mathcal{A}(U) = \Gamma(U; \hat{S}(E^\vee))$ defines a $\mathbb{Z}$-graded manifold.

Differential graded manifolds

Definition: A dg manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ endowed with a (homological) vector field $Q \in \mathfrak{X}(\mathcal{M})$ of degree $+1$ such that $[Q, Q] = 2\ Q \circ Q = 0$.

Example: Given a Lie algebra $\mathfrak{g}$, $(\mathfrak{g}[1], Q = d_{\mathrm{CE}})$ is a dg manifold.

- $C^\infty(\mathfrak{g}[1]) = \Lambda^\bullet \mathfrak{g}^\vee$
- $Q = d_{\mathrm{CE}} : \Lambda^\bullet \mathfrak{g}^\vee \to \Lambda^{\bullet+1} \mathfrak{g}^\vee$ — Chevalley–Eilenberg differential

More generally, when $M = \mathrm{pt}$:
$(\mathcal{M}, Q)$ is a dg manifold $\iff$ $(\mathcal{M}, Q)$ is a curved $L_\infty[1]$ algebra.

Example: Given a complex manifold X, $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar{\partial})$ is a dg manifold;

- $C^\infty(T_X^{0,1}[1]) = \Omega^{0,\bullet}(X)$ — Space of anti-holomorphic forms
- $Q = \bar{\partial} : \Omega^{0,\bullet}(X) \to \Omega^{0,\bullet+1}(X)$ — Dolbeault operator

Example: Given a vector bundle $E \to M$ and smooth section s, $(\mathcal{M} = E[-1], Q = i_s)$ is a dg manifold, called derived intersection of s with the zero section.

- $C^\infty(E[-1]) = \Gamma(\Lambda^{-\bullet} E^\vee)$
- $Q = i_s : \Gamma(\Lambda^{-\bullet} E^\vee) \to \Gamma(\Lambda^{-\bullet+1} E^\vee)$ — interior product with s

For instance, if $f \in C^\infty(M)$, then $(T_M^\vee[-1], i_{df})$ is a dg-manifold called derived critical locus of f.

Atiyah class of a differential graded manifold

- Choose a torsion-free connection ∇ on a dg manifold $(\mathcal{M}, Q)$.
- Define a degree $+1$ section $\mathrm{At}^{\nabla} \in \Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}}))$

$$\mathrm{At}^{\nabla}(X, Y) = L_Q(\nabla_X Y) - \nabla_{L_Q X} Y - (-1)^{|X|} \nabla_X (L_Q Y)$$

for homogeneous vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$.

Note: $\mathrm{At}^{\nabla} := L_Q \nabla$ but $\nabla \notin \Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}}))$

Lemma:

- $L_Q \circ L_Q = 0$ and $L_Q \mathrm{At}^{\nabla} = 0$
- $\left[\mathrm{At}^{\nabla}\right] \in H^1\left(\Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})), L_Q\right)$ is independent of the connection ∇.

Definition:

1. $\mathrm{At}^{\nabla} \in \Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}}))$ is an Atiyah cocycle of ∇.
2. The Atiyah class of the dg manifold $(\mathcal{M}, Q)$

$$\alpha_{\mathcal{M}} := \left[\mathrm{At}^{\nabla}\right] \in H^1\left(\Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})), L_Q\right)$$

is the obstruction to existence of a connection on $\mathcal{M}$ compatible with the homological vector field Q.

A connection ∇ on a dg manifold $(\mathcal{M}, Q)$ is said to be *compatible* with the homological vector field if

$$L_Q(\nabla_X Y) = \nabla_{L_Q X} Y + (-1)^{|X|} \nabla_X (L_Q Y) \quad \text{for all } X, Y \in \mathfrak{X}(\mathcal{M}).$$

Example: Let $\mathfrak{g}$ be a finite-dimensional Lie algebra

- $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ is the corresponding dg manifold
- $T_{\mathcal{M}} \cong \mathfrak{g}[1] \times \mathfrak{g}[1]$ implies

$$H^1(\Gamma(S^2(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}), Q) \cong H^0_{\mathrm{CE}}(\mathfrak{g}; \Lambda^2\mathfrak{g}^\vee \otimes \mathfrak{g}) \cong (\Lambda^2\mathfrak{g}^\vee \otimes \mathfrak{g})^{\mathfrak{g}}$$

- $\alpha_{\mathfrak{g}[1]} \leftrightarrow$ the Lie bracket of $\mathfrak{g}$

Example: Let X be a complex manifold.

- $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar\partial)$ is a corresponding dg manifold;
- There exists a quasi-isomorphism

$$(\Gamma(T_{\mathcal{M}}), L_Q) \xrightarrow{\text{q.i.}} (\Omega^{0,\bullet}(T_X^{1,0}), \bar\partial)$$

- There is an isomorphism

$$\begin{aligned} & H^1(\Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})), L_Q) \\ & \qquad \cong H^1(\Omega^{0,\bullet}(\mathrm{Hom}(S^2(T_X^{1,0}), T_X^{1,0}),), \bar\partial) \\ & \qquad\qquad \subset H^1(X, \Omega_X^1 \otimes \mathrm{End}(T_X)) \end{aligned}$$

- $\alpha_{T_X^{0,1}[1]} \leftrightarrow$ the classical Atiyah class of X

Exponential maps arise naturally in the contexts of linearization problems

(1) Lie theory

(2) smooth manifolds

PBW isomorphism in Lie theory

- $\mathfrak{g}$: finite dimensional Lie algebra
- $\exp : \mathfrak{g} \to G$
- $\exp$: local diffeomorphism from nbd of 0 to nbd of 1
- $\exp$ induces an isomorphism on differential operators evaluated at $0 \in \mathfrak{g}$ and $1 \in G$:

$$\mathrm{pbw} = (\exp)_* : S\mathfrak{g} \xrightarrow{\cong} U\mathfrak{g}$$

$$X_1 \odot \cdots \odot X_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

Fact: Poincaré–Birkhoff–Witt map is an isomorphism of coalgebras.

Exponential map on smooth manifolds

- an affine connection ∇ on smooth manifold M
- $\exp^{\nabla} : T_M \to M \times M$ (bundle map) defined by $\exp^{\nabla}(X_m) = (m, \gamma(1))$ where γ is the smooth path in M satisfying $\dot{\gamma}(0) = X_m$ and $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$
  - $\Gamma(S(T_M))$ seen as space of differential operators on T_M, all derivatives in the direction of the fibers, evaluated along the zero section of T_M
  - $\mathcal{D}(M)$ seen as space of differential operators on $M \times M$, all derivatives in the direction of the fibers, evaluated along the diagonal section $M \to M \times M$
- $\mathrm{pbw}^{\nabla} := \exp^{\nabla}_{*} : \Gamma(S(T_M)) \xrightarrow{\cong} \mathcal{D}(M)$ is an isomorphism of left modules over $C^{\infty}(M)$ called Poincaré–Birkhoff–Witt isomorphism.

pbw^{∇} as infinite jet of exp

The Taylor series of the composition

$$T_m M \xrightarrow{\exp} \{m\} \times M \xrightarrow{f} \mathbb{R}$$

at the point $0_m \in T_m M$ is

$$\sum_{J \in \mathbb{N}_0^n} \frac{1}{J!} \big(\mathrm{pbw}^{\nabla}(\partial_x^J) f \big)(m) \otimes y_J \quad \in \hat{S}(T_m^{\vee} M).$$

- $(x_i)_{i \in \{1,\dots,n\}}$ are local coordinates on M
- $(y_j)_{j \in \{1,\dots,n\}}$ induced local frame of $T_M^{\vee}$ regarded as fiberwise linear functions on T_M

Hence pbw^{∇} is the fiberwise infinite jet of the bundle map $\exp : T_M \to M \times M$ along the zero section of $T_M \to M$.

Algebraic characterization of pbw^∇

Theorem (Laurent-Gengoux, Stiénon, Xu, 2014):

$$\mathrm{pbw}^\nabla(f) = f, \quad \forall f \in C^\infty(M);$$
$$\mathrm{pbw}^\nabla(X) = X, \quad \forall X \in \mathfrak{X}(M);$$
$$\mathrm{pbw}^\nabla(X^{n+1}) = X \cdot \mathrm{pbw}^\nabla(X^n) - \mathrm{pbw}^\nabla(\nabla_X X^n), \quad \forall n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$ and $X_0, \dots, X_n \in \mathfrak{X}(M)$,

$$\mathrm{pbw}^\nabla(X_0 \odot \cdots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^{n} \left\{ X_k \cdot \mathrm{pbw}^\nabla(X^{\{k\}}) - \mathrm{pbw}^\nabla\left(\nabla_{X_k}(X^{\{k\}})\right) \right\}$$

where $X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$.

Example:

- G: Lie group, X_i^L: Left invariant vector field
- Choose a connection ∇ such that $\nabla_{X_i^L} X_j^L = 0$. Then,

$$\mathrm{pbw}^{\nabla}(X_1^L \odot \cdots \odot X_n^L) = \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)}^L \cdots X_{\sigma(n)}^L$$

Both $\Gamma(S(T_M))$ and $\mathcal{D}(M)$ are left coalgebras over $R := C^\infty(M)$.

Comultiplication in both $\Gamma(S(T_M))$ and $\mathcal{D}(M)$ by deconcatenation:

$$\begin{aligned}\Delta(X_1 \cdots X_n) = 1 \otimes (X_1 \cdots X_n) \\ + \sum_{\substack{p+q=n \\ p,q \in \mathbb{N}}} \sum_{\sigma \in \mathfrak{S}_p^q} (X_{\sigma(1)} \cdots X_{\sigma(p)}) \otimes (X_{\sigma(p+1)} \cdots X_{\sigma(n)}) \\ + (X_1 \cdots X_n) \otimes 1\end{aligned}$$

for all $X_1, \ldots, X_n \in \mathfrak{X}(\mathcal{M})$.

Proposition: $\mathrm{pbw}^\nabla : \Gamma(S(T_M)) \to \mathcal{D}(M)$ is an isomorphism of coalgebras over $C^\infty(M)$.

- $(\mathrm{pbw}^\nabla)^{-1} : \mathcal{D}(M) \to \Gamma(S(T_M))$ takes a differential operator to its *complete symbol*
- pbw^∇ preserves comultiplication, but does NOT preserve multiplication.
- The algebraic characterization of pbw^∇ does NOT involve any points of M or any geodesic curves of ∇.
- The isomorphism pbw^∇ is a sort of formal exponential map defined inductively.

Exponential map on graded manifolds

$\mathcal{M}$: graded manifold

Theorem (Liao, Stiénon, 2015):

1. The formal exponential map associated to an affine connection ∇ on $\mathcal{M}$ is the morphism of left $C^\infty(\mathcal{M})$-modules
$$\mathrm{pbw}^\nabla : \Gamma(S(T_\mathcal{M})) \to \mathcal{D}(\mathcal{M})$$
inductively defined by a formula.
2. Moreover,
$$\mathrm{pbw}^\nabla : \Gamma(S(T_\mathcal{M})) \to \mathcal{D}(\mathcal{M})$$
is an isomorphism of graded coalgebras over $C^\infty(\mathcal{M})$.

Exponential map on differential graded manifolds

Given a dg manifold $(\mathcal{M}, Q)$, there exists two differential graded coalgebras:

1. $(\Gamma(S(T_{\mathcal{M}})), L_Q)$
2. $(\mathcal{D}(\mathcal{M}), \mathcal{L}_Q := [Q, -])$

Question: When is

$$\mathrm{pbw}^{\nabla} : (\Gamma(S(T_{\mathcal{M}})), L_Q) \to (\mathcal{D}(\mathcal{M}), \mathcal{L}_Q)$$

an isomorphism of differential graded coalgebras?

– If pbw^{∇} is an isomorphism of differential graded coalgebras, then we can consider it as "a formal exponential map" of $(\mathcal{M}, Q)$.

Existence of formal exponential map of dg mfd

Theorem (S, Stiénon, Xu, 2021): The Atiyah class $\alpha_{\mathcal{M}}$ vanishes if and only if there exists a torsion-free connection ∇ such that

$$\mathrm{pbw}^{\nabla} : (\Gamma(S(T_{\mathcal{M}})), L_Q) \to (\mathcal{D}(\mathcal{M}), \mathcal{L}_Q)$$

is an isomorphism of differential graded coalgebras over $C^{\infty}(\mathcal{M})$.

Kapranov $L_\infty[1]$ algebra on dg manifolds

In general, the failure of pbw$^\nabla$ to preserve dg structure is measured by

$$(\mathrm{pbw}^\nabla)^{-1} \circ \mathcal{L}_Q \circ \mathrm{pbw}^\nabla - L_Q = \sum_{k=0}^{\infty} R_k$$

where $R_k \in \Gamma(\mathrm{Hom}(S^k(T_\mathcal{M}), T_\mathcal{M}))$ are sections of degree $+1$.

$R_0 = R_1 = 0, \quad R_2 = -\mathrm{At}^\nabla$

Theorem (S, Stiénon, Xu, 2021):

1. The R_k for $k \geq 2$, together with L_Q induce an $L_\infty[1]$ algebra on the space of vector fields $\mathfrak{X}(\mathcal{M})$.
2. The R_k for $k \geq 2$ are completely determined by Atiyah cocycle At^∇, the curvature R^∇, and their exterior derivatives.
   In particular, if the curvature vanishes (i.e. $R^\nabla = 0$), then

$$R_2 = -\mathrm{At}^\nabla, \quad R_{n+1} = \frac{1}{n+1} d^\nabla R_n \quad \text{for } n \geq 2$$

Example:

- $\mathfrak{g}$: finite-dimensional Lie algebra
- $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ is a dg manifold
- $\mathfrak{X}(\mathcal{M})[-1] = \Lambda\mathfrak{g}^\vee \otimes \mathfrak{g}$ is an L_∞ algebra equipped with

$$L_Q = d_{CE}^{\mathfrak{g}}, \quad R_2 = 1 \otimes [\ ,\ ]_{\mathfrak{g}}, \quad R_{\geqslant 3} = 0$$

– Chevalley-Eilenberg cohomology $H_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{g})$ is a Lie algebra.

Theorem (S. Stiénon, Xu, 2021):

- X: Kähler manifold
- $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar{\partial})$ is a dg manifold
- $\mathfrak{X}(\mathcal{M}) = \mathfrak{X}(T_X^{0,1}[1])$ admits an $L_\infty[1]$ algebra structure
- There is an $L_\infty[1]$ quasi-isomorphism

$$(\mathfrak{X}(T_X^{0,1}[1]), \{R_i\}) \xrightarrow{L_\infty[1] \text{ q.i.}} (\Omega^{0,\bullet}(T_X^{1,0}), \{\lambda_i\})$$

Moreover, our $L_\infty[1]$ algebra structure on $\mathfrak{X}(T_X^{0,1}[1])$ can be transferred to Kapranov's $L_\infty[1]$ algebra structure on $\Omega^{0,\bullet}(T_X^{1,0})$.

Thank you!