1. Consider \( \{W(t), t \geq 0\} \), a standard Brownian motion process.
   (a) Find the conditional distribution of \( W(s) \mid W(t_1) = A, W(t_2) = B \), where \( 0 < t_1 < s < t_2 \).
   (b) Find \( E(W(t_1)W(t_2)W(t_3)) \) where \( 0 < t_1 < t_2 < t_3 \).

2. Let \( \{X(t), t \geq 0\} \) be a Brownian motion with drift coefficient \( \mu \) and variance parameter \( \sigma^2 \).
   Find the joint distribution of \( X(s), X(t) \) where \( 0 < s < t \).

3. Distinguishing between Markov processes and martingales:
   (a) Provide one example of a martingale that is not a Markov process and show why this is the case.
   (b) Provide one example of a Markov process that is not a martingale and show why this is the case.

4. Prove the following result: If \( X_i, i \geq 1 \), are independent and identically distributed (iid) and if \( N \) is a bounded stopping time for \( X_1, X_2, \ldots \) with \( E(N) < \infty \), then
   \[ E \left( \sum_{i=1}^{N} X_i \right) = E(N)E(X) \]
   Hint: consider the process \( Z_n = \sum_{i=1}^{n} (X_i - \mu) \).

5. Let \( \{X(t), t > 0\} \) be standard Brownian motion. Prove that the process \( \{M(t), t > 0\} \) where \( M(t) = \exp(\lambda X(t) - \frac{1}{2} \lambda^2 t) \), is a martingale.

6. Simulate 3 realizations for each of the following processes on the interval \([0, 10]\) on 20 equally spaced points on the interval.
   (a) Simulate 3 realizations of standard Brownian motion.
   (b) Simulate 3 realizations of Brownian motion with variance parameter \( \mu = 0, \sigma^2 = 2 \).
   (c) Simulate 3 realizations of Brownian motion with parameters \( \sigma^2 = 2, \mu = 3 \).
   Overlay the 3 realizations for each process on the same plot. Hence you should submit 3 clearly labeled plots. You do not have to submit your code for this problem but you have to provide pseudocode for each simulation algorithm above.

7. Consider a simple symmetric random walk, \( S_n = \sum_{i=0}^{n} X_i \) where \( X_1, X_2, \ldots \) are iid with \( P(X_i = 1) = 1/2 = P(X_i = -1) \) and define a random time, \( T \in [0, 3] \) at which \( S_n \) takes on its maximum value \( \max\{S_n : 0 \leq n \leq 3\} \). If \( S_n \) takes its maximum value more than once, assume \( T \) is the last such time.
   (a) Show analytically that \( E(X_T) > 0 \) and hence \( E(X_T) \neq E(X_0) \). This therefore results in an “unfair game”, as discussed in class.
   (b) Find the expected value of \( T \) using Monte Carlo. Write pseudocode for the algorithm and report your estimate along with the Monte Carlo sample size and Monte Carlo standard error.