Penn State Astrostatistics MCMC tutorial

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A Bayesian change point model

Consider the following hierarchical changepoint model for the number of occurrences Y_i of some event during time interval i with change point k.

$$\begin{aligned}
Y_i|k,\theta,\lambda &\sim \text{Poisson}(\theta) \text{ for } i = 1,\ldots,k \\
Y_i|k,\theta,\lambda &\sim \text{Poisson}(\lambda) \text{ for } i = k+1,\ldots,n
\end{aligned}$$

Assume the following prior distributions:

$$\begin{aligned}
\theta|b_1 &\sim \text{Gamma}(0.5, b_1) \qquad (\text{pdf}{=}g_1(\theta|b_1)) \\
\lambda|b_2 &\sim \text{Gamma}(0.5, b_2) \qquad (\text{pdf}{=}g_2(\lambda|b_2)) \\
b_1 &\sim \text{IG}(0,1) \qquad (\text{pdf}{=}h_1(b_1)) \\
b_2 &\sim \text{IG}(0,1) \qquad (\text{pdf}{=}h_2(b_2)) \\
k &\sim \text{Uniform}(1,\ldots,n) \qquad (\text{pmf } = u(k))
\end{aligned}$$

k, θ, λ are conditionally independent and b_1, b_2 are independent.
Assume the Gamma density parameterization $\text{Gamma}(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$ and IG (Inverse Gamma) density parameterization $\text{IG}(\alpha,\beta) = \frac{e^{-1/\beta x}}{\Gamma(\alpha)\beta^\alpha x^{\alpha+1}}$

Inference for this model is therefore based on the 5-dimensional **posterior** distribution $f(k,\theta,\lambda,b_1,b_2|\mathbf{Y})$ where $\mathbf{Y}=(Y_1,\ldots,Y_n)$. The posterior distribution is obtained *up to a constant* (that is, the normalizing constant is unknown) by taking the product of all the conditional distributions. Thus we have

$$\begin{aligned}
f(k,\theta,\lambda,b_1,b_2|\mathbf{Y}) &\propto \prod_{i=1}^{k} f_1(Y_i|\theta,\lambda,k) \prod_{i=k+1}^{n} f_2(Y_i|\theta,\lambda,k) \\
&\times g_1(\theta|b_1) g_2(\lambda|b_2) h_1(b_1) h_2(b_2) u(k) \\
&= \prod_{i=1}^{k} \frac{\theta^{Y_i} e^{-\theta}}{Y_i!} \prod_{i=k+1}^{n} \frac{\lambda^{Y_i} e^{-\lambda}}{Y_i!} \\
&\times \frac{1}{\Gamma(0.5) b_1^{0.5}} \theta^{-0.5} e^{-\theta/b_1} \times \frac{1}{\Gamma(0.5) b_2^{0.5}} \lambda^{-0.5} e^{-\lambda/b_2} \\
&\times \frac{e^{-1/b_1}}{b_1} \frac{e^{-1/b_2}}{b_2} \frac{1}{n}
\end{aligned}$$