

## On hyperelliptic modular curves over function fields

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**Abstract.** We prove that there are only finitely many modular curves of  $\mathcal{D}$ -elliptic sheaves over  $\mathbb{F}_q(T)$  which are hyperelliptic. In odd characteristic we give a complete classification of such curves.

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**1. Introduction.** The hyperelliptic classical modular curves and the hyperelliptic Shimura curves over  $\mathbb{Q}$  have been classified and extensively studied, cf. [7], [8], [9]. One of the reasons for the interest in hyperelliptic modular curves is that the existence of a degree-2 map to the projective line makes such curves amenable for explicit calculations.

Let  $\mathbb{F}_q$  be the finite field with  $q$  elements,  $A = \mathbb{F}_q[T]$  be the ring of polynomials in  $T$  with  $\mathbb{F}_q$ -coefficients, and  $F = \mathbb{F}_q(T)$  be the fraction field of  $A$ . To each ideal  $\mathfrak{n} \triangleleft A$  there is an associated Drinfeld modular curve  $X_0(\mathfrak{n})$  over  $F$ , which classifies rank-2 Drinfeld  $A$ -modules with a cyclic  $\mathfrak{n}$ -torsion subgroup. These curves are the analogues of classical modular curves  $X_0(N)$ . Schweizer [13] determined all  $\mathfrak{n}$  for which  $X_0(\mathfrak{n})$  is hyperelliptic, thus proving the analogue of a result of Ogg [8]. The function field analogue of Shimura curves was introduced by Laumon, Rapoport and Stuhler in [3]. These curves are moduli spaces of certain objects, called  $\mathcal{D}$ -elliptic sheaves, which generalize the notion of Drinfeld modules. The purpose of the present paper is to determine those modular curves of  $\mathcal{D}$ -elliptic sheaves which are hyperelliptic (Theorem 4.1). This is the analogue of a result of Michon and Ogg [7], [9].

**2. Notation.** Let  $C := \mathbb{P}_{\mathbb{F}_q}^1$  be the projective line over  $\mathbb{F}_q$ .  $F = \mathbb{F}_q(T)$  is the field of rational functions on  $C$ . The set of closed points on  $C$  (equivalently, places of  $F$ ) is denoted by  $|C|$ . For each  $x \in |C|$ , we denote by  $\mathcal{O}_x$  and  $F_x$  the completions of  $\mathcal{O}_{C,x}$  and  $F$  at  $x$ , respectively. The residue field of  $\mathcal{O}_x$  is denoted by  $\mathbb{F}_x$ , the cardinality of  $\mathbb{F}_x$  is denoted by  $q_x$ , the degree  $m$  extension of  $\mathbb{F}_x$  is denoted by  $\mathbb{F}_x^{(m)}$ , and  $\deg(x) := \dim_{\mathbb{F}_q}(\mathbb{F}_x)$ .  $A = \mathbb{F}_q[T]$  is the subring of  $F$  consisting of functions which are regular away from  $\infty := 1/T$ . The completion of an algebraic closure of  $F_\infty$  is denoted  $\mathbb{C}_\infty$ . For each  $x \in |C| - \infty$ , we denote by  $\mathfrak{p}_x \triangleleft A$  the corresponding prime ideal of  $A$ . For  $a \in A$ , let  $\deg(a)$  denote the degree of  $a$  as a polynomial in  $T$ .

Given a ring  $H$ , we denote by  $H^\times$  the group of its units.

Let  $D$  be a quaternion algebra over  $F$ , i.e., a 4-dimensional central simple algebra over  $F$ . For  $x \in |C|$ , we let  $D_x := D \otimes_F F_x$ . We assume throughout the paper that  $D$  is split at  $\infty$ , i.e.,  $D_\infty \cong \mathbb{M}_2(F_\infty)$ . ( $\mathbb{M}_2$  is the ring of  $2 \times 2$  matrices.) Let  $R$  be the set of places where  $D$  is ramified. It is a well-known fact that the cardinality of  $R$  is even, and conversely, for any choice of a finite set  $R \subset |C|$  of even cardinality there is a unique quaternion algebra ramified exactly at the places in  $R$ ; see [15, p. 74]. If  $R \neq \emptyset$ , then  $D$  is a division algebra; if  $R = \emptyset$ , then  $D \cong \mathbb{M}_2(F)$ . The ideal  $\mathfrak{r} := \prod_{x \in R} \mathfrak{p}_x$  is the *discriminant* of  $D$ . Let  $D^\times$  be the algebraic group over  $F$  defined by  $D^\times(B) = (D \otimes_F B)^\times$  for any  $F$ -algebra  $B$ ; this is the multiplicative group of  $D$ . Let  $\alpha \mapsto \alpha'$  denote the canonical involution of  $D$ ; thus  $(\alpha\beta)' = \beta'\alpha'$ . The *reduced trace* of  $\alpha$  is  $\text{Tr}(\alpha) = \alpha + \alpha'$  and the *reduced norm* of  $\alpha$  is  $\text{Nr}(\alpha) = \alpha\alpha'$ .

**3. Preliminaries.** From now on we assume that  $D$  is fixed and is a division algebra. In particular,  $R \neq \emptyset$ . Fix a maximal  $A$ -order  $\Lambda$  in  $D$ . Denote  $\Gamma := \Lambda^\times$ . Since  $D$  is split at  $\infty$ , it satisfies the so-called *Eichler condition* relative to  $A$ , cf. [11, (34.3)]. This implies that, up to conjugation,  $\Lambda$  is the unique maximal  $A$ -order in  $D$ , i.e., any other maximal  $A$ -order in  $D$  is of the form  $\alpha\Lambda\alpha^{-1}$  for some  $\alpha \in D^\times(F)$ , cf. [15, Cor. III. 5.7].

Let  $\Omega$  denote the Drinfeld upper half-plane over  $F_\infty$ . As a set  $\Omega = \mathbb{C}_\infty - F_\infty$ , but  $\Omega$  also has a natural structure of a rigid-analytic space, cf. [14]. The group  $\Gamma$  can be considered as a subgroup of  $\text{GL}_2(F_\infty)$  via the embedding

$$\Gamma \subset D^\times(F) \subset D^\times(F_\infty) \cong \text{GL}_2(F_\infty).$$

Hence  $\Gamma$  acts on  $\Omega$  via linear fractional transformations. Denote the image of  $\Gamma$  in  $\text{PGL}_2(F_\infty)$  by  $\bar{\Gamma}$ .  $\bar{\Gamma}$  is a finitely generated, discrete and cocompact subgroup of  $\text{PGL}_2(F_\infty)$ , so the quotient  $\Omega/\bar{\Gamma}$  is the underlying rigid-analytic space of a smooth projective geometrically irreducible curve  $X^R$  over  $F_\infty$ ; see [14, Thm. 3.3]. In fact,  $X^R$  has a canonical model over  $F$ , since it is a coarse moduli scheme of  $\mathcal{D}$ -elliptic sheaves over  $F$  with pole  $\infty$ , where  $\mathcal{D}$  is a maximal  $\mathcal{O}_C$ -order in  $D$ , cf. [1, Ch. 4] and [10]. Using this moduli-theoretic interpretation, one can show that the curve  $X^R$  has good reduction at every  $o \in |C| - R - \infty$ ; see [10, Cor. 3.2]. We will denote the genus of  $X^R$  by  $g(X^R)$  and the reduction of  $X^R$  at  $o$  by  $X_o^R$ .

For a non-empty finite subset  $S \subset |C|$ , let

$$\wp(S) = \begin{cases} 0, & \text{if some place in } S \text{ has even degree;} \\ 1, & \text{otherwise.} \end{cases}$$

**Theorem 3.1.**

$$g(X^R) = 1 + \frac{1}{q^2 - 1} \prod_{x \in R} (q_x - 1) - \frac{q}{q + 1} \cdot 2^{\#R-1} \cdot \wp(R).$$

*Proof.* [10, Theorem 5.4]. □

**Theorem 3.2.** For  $o \in |C| - R - \infty$ , we have

$$\#X_o^R(\mathbb{F}_o^{(2)}) \geq \frac{1}{q^2 - 1} \prod_{x \in R \cup o} (q_x - 1) + \frac{q}{q + 1} \cdot 2^{\#R} \cdot \wp(R \cup o).$$

*Proof.* [10, Corollary 4.8]. □

Let  $X$  be a curve over a field  $K$ . (From now on a curve is always assumed to be smooth, projective and geometrically irreducible.) Denote an algebraic closure of  $K$  by  $\bar{K}$ , and let  $\text{Aut}(X)$  denote the group of  $\bar{K}$ -automorphisms of  $X$ . Denote  $X_{\bar{K}} := X \otimes_K \bar{K}$ .

**Theorem 3.3.** Let  $X$  be a curve over  $K$  of genus greater or equal to 2. The following conditions are equivalent:

1. There exists a  $K$ -morphism  $w : X \rightarrow X'$  of degree 2, where  $X'$  is a curve of genus 0.
2. There exists a  $\bar{K}$ -morphism  $w : X_{\bar{K}} \rightarrow \mathbb{P}_{\bar{K}}^1$  of degree 2.
3.  $\text{Aut}(X)$  contains an involution  $\sigma$  such that the quotient  $X_{\bar{K}}/\sigma$  is isomorphic to  $\mathbb{P}_{\bar{K}}^1$ .

*Proof.* See [5, Theorem 5.5]. □

**Definition 3.4.** A curve  $X$  satisfying the conditions of Theorem 3.3 is called *hyperelliptic*. The involution  $\sigma$  is uniquely determined and is called the *canonical involution* of  $X$ , cf. [5, p. 98]. The uniqueness of  $\sigma$  implies that  $\sigma$  is in the center of  $\text{Aut}(X)$ .

**Lemma 3.5.** Suppose  $K$  is a field of characteristic not equal to 2. If  $X$  is hyperelliptic, then  $\text{Aut}(X)$  does not contain a subgroup isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^4$ .

*Proof.* We can assume  $K$  is algebraically closed. The canonical involution  $\sigma$  is in the center of  $\text{Aut}(X)$ , so if  $(\mathbb{Z}/2\mathbb{Z})^4 \subset \text{Aut}(X)$ , then

$$(\mathbb{Z}/2\mathbb{Z})^3 \subset \text{Aut}(X/\sigma) \cong \text{Aut}(\mathbb{P}_{\bar{K}}^1) \cong \text{PGL}_2(K).$$

But  $\text{PGL}_2(K)$  does not contain a subgroup isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^3$  if the characteristic is not 2, cf. [4, p. 33]. □

**4. Main result.**

**Theorem 4.1.**

1. For a fixed  $q$  there are only finitely many  $R$  such that  $X^R$  is hyperelliptic.
2. If  $q$  is odd, then  $X^R$  is hyperelliptic if and only if  $R = \{x, y\}$  and

$$\{\deg(x), \deg(y)\} = \{1, 2\}.$$

**Remark 4.2.** As we mentioned in the introduction, the hyperelliptic Drinfeld modular curves  $X_0(\mathfrak{n})$  over  $F$  were classified by Schweizer [13]. When  $q > 2$  the answer is similar to Theorem 4.1:  $X_0(\mathfrak{n})$  is hyperelliptic if and only if  $\mathfrak{n}$  is generated by a polynomial of degree 3.

Most of the key ideas in the proof of this theorem go back to Ogg [8], [9]. There are three main tools used in the proof. Two of them are the genus formula for  $X^R$  (Theorem 3.1) and the estimate for the number of rational points on a reduction of  $X^R$  (Theorem 3.2). The third one is the study of certain involutions on  $X^R$ , which are the analogues of Atkin-Lehner involutions on Shimura curves. These involutions will be discussed later in this section.

When  $q$  is a power of 2, to classify those (finitely many)  $R$  for which  $X^R$  is hyperelliptic one can follow the same strategy as for odd  $q$ . Nevertheless, the argument will be technically much more complicated. In fact, almost all preliminary results for the proof of Part (2) of Theorem 4.1 crucially use the assumption that the characteristic is not 2.

*Proof of part (1) of Theorem 4.1.* Suppose  $X^R$  is hyperelliptic. Fix some  $o \in |C| - R - \infty$ , and consider  $X_o^R$ . By [5, Prop. 5.14],  $X_o^R$  is also a hyperelliptic curve. Over a finite field a curve of genus 0 has a rational point, hence is isomorphic to the projective line. In particular, there is a degree-2 morphism  $X_o^R \rightarrow \mathbb{P}_{\mathbb{F}_o}^1$  over  $\mathbb{F}_o$ . This implies

$$\#X_o^R(\mathbb{F}_o^{(2)}) \leq 2\#\mathbb{P}_{\mathbb{F}_o}^1(\mathbb{F}_o^{(2)}) = 2(q_o^2 + 1).$$

Using Theorem 3.2, we get

$$(4.1) \quad \prod_{x \in R \cup o} (q_x - 1) \leq 2(q_o^2 + 1)(q^2 - 1).$$

Choose  $o$  of minimal possible degree. One easily checks that (4.1) can hold only for finitely many  $R$  with such a choice of  $o$ . □

From now on we assume that the characteristic of  $F$  is odd.

**Definition 4.3.** The *normalizer* of  $\Lambda$  (respectively,  $\Gamma$ ) in  $D$  is

$$N(\Lambda) := \{g \in D^\times(F) \mid g\Lambda g^{-1} = \Lambda\},$$

respectively,  $N(\Gamma) := \{g \in D^\times(F) \mid g\Gamma g^{-1} = \Gamma\}$ . Clearly,  $N(\Lambda) \subseteq N(\Gamma)$ .

For a place  $x \in |C| - \infty$ , we define the local versions

$$N(\Lambda_x) := \{g \in D^\times(F_x) \mid g\Lambda_x g^{-1} = \Lambda_x\},$$

where  $\Lambda_x = \Lambda \otimes_A \mathcal{O}_x$ , and  $N(\Lambda_x^\times) := \{g \in D^\times(F_x) \mid g\Lambda_x^\times g^{-1} = \Lambda_x^\times\}$ .

**Proposition 4.4.**

1.  $N(\Lambda) = \{g \in D^\times(F) \mid g \in N(\Lambda_x), x \in |C| - \infty\}$ .
2.  $N(\Gamma) = \{g \in D^\times(F) \mid g \in N(\Lambda_x^\times), x \in |C| - \infty\}$ .
3.  $N(\Lambda) = N(\Gamma)$ .
4. If  $0 \neq g \in \Lambda$  and  $\text{Nr}(g)$  divides  $\mathfrak{r}$ , then  $g \in N(\Lambda)$ .
5.  $N(\Lambda)/F^\times\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^{\#R}$ .
6. There exists a set of elements  $\{\pi_x \in \Lambda\}_{x \in R}$  with  $(\text{Nr}(\pi_x)) = \mathfrak{p}_x$ . Any such set generates  $N(\Lambda)/F^\times\Gamma$ .

*Proof.* From the local-global correspondence for orders,  $g\Lambda g^{-1} = \Lambda$  if and only if  $g\Lambda_x g^{-1} = (g\Lambda g^{-1})_x = \Lambda_x$  for all  $x \in |C| - \infty$ . This proves (1). A similar argument also proves (2).

If  $x \notin R$ , then  $\Lambda_x$  is conjugate to  $\mathbb{M}_2(\mathcal{O}_x)$ . Since the normalizer of  $\mathbb{M}_2(\mathcal{O}_x)$  in  $D_x \cong \mathbb{M}_2(F_x)$  is  $F_x^\times \text{GL}_2(\mathcal{O}_x)$ , we conclude  $N(\Lambda_x) = F_x^\times \Lambda_x^\times$ . Since  $\text{GL}_2(\mathcal{O}_x)$  contains an  $\mathcal{O}_x$ -basis of  $\mathbb{M}_2(\mathcal{O}_x)$ , if  $g \in D_x$  normalizes  $\text{GL}_2(\mathcal{O}_x)$ , then it also normalizes  $\mathbb{M}_2(\mathcal{O}_x)$ . Hence  $N(\Lambda_x^\times) = N(\Lambda_x) = F_x^\times \Lambda_x^\times$ . Next, if  $x \in R$ , then the uniqueness of the maximal  $\mathcal{O}_x$ -order in  $D_x$  for  $x \in R$  implies  $N(\Lambda_x) = D^\times(F_x)$ . Since  $N(\Lambda_x) \subseteq N(\Lambda_x^\times) \subseteq D^\times(F_x)$ ,  $N(\Lambda_x^\times) = D^\times(F_x)$ . Overall, we conclude that  $N(\Lambda_x) = N(\Lambda_x^\times)$  for all  $x \in |C| - \infty$ . Therefore, (3) follows from (1) and (2).

Suppose  $g \in \Lambda$  and  $(\text{Nr}(g))$  divides  $\mathfrak{r}$ . Then  $g \in \Lambda_x^\times$  for all  $x \in |C| - R - \infty$ , in particular, for such  $x$ ,  $g \in N(\Lambda_x)$ . On the other hand, since  $N(\Lambda_x) = D^\times(F_x)$  for  $x \in R$ ,  $g \in N(\Lambda_x)$  for  $x \in R$ . Overall,  $g \in N(\Lambda_x)$  for all  $x \in |C| - \infty$ , and (4) follows from (1).

Let  $x \in |C| - \infty$ . Consider  $N(\Lambda_x)/F_x^\times \Lambda_x^\times$ . As we discussed above, this quotient is trivial if  $x \notin R$ . On the other hand, when  $x \in R$ , the composition  $\text{ord}_x \circ \text{Nr}$  induces an isomorphism  $D^\times(F_x)/F_x^\times \Lambda_x^\times \cong \mathbb{Z}/2\mathbb{Z}$ . Hence, for  $x \in R$ ,  $N(\Lambda_x)/F_x^\times \Lambda_x^\times \cong \mathbb{Z}/2\mathbb{Z}$ . By considering  $g \in N(\Lambda)$  as an element of  $N(\Lambda_x)$ ,  $x \in |C| - \infty$ , we obtain a natural homomorphism

$$N(\Lambda)/F^\times\Gamma \rightarrow \prod_{x \in |C| - \infty} N(\Lambda_x)/F_x^\times \Lambda_x^\times \cong \prod_{x \in R} (\mathbb{Z}/2\mathbb{Z}).$$

By (37.25) and (37.32) in [11], this homomorphism is surjective and has trivial kernel since  $\text{Pic}(A) = 1$ . This proves (5).

The existence of the elements  $\pi_x$  in (6) follows from [11, (34.8)]. The second claim of (6) follows from the fact that the isomorphism in (5) is induced by  $g \mapsto \prod_{x \in R} \text{ord}_x \circ \text{Nr}(g)$ . □

**Definition 4.5.** Let  $\mathfrak{a} \triangleleft A$  be an ideal dividing  $\mathfrak{r}$ , and let  $\mu \in \Lambda$  be such that  $(\text{Nr}(\mu)) = \mathfrak{a}$ . By Proposition 4.4, such an element exists and  $\mu \in N(\Gamma)$ . We can consider  $\mu$  as an element of  $D^\times(F_\infty)$ , so  $\mu$  acts on  $\Omega$ . Due to the fact that it normalizes  $\Gamma$ ,  $\mu$  induces an automorphism  $w_{\mathfrak{a}}$  of  $X^R$ , cf. [2, VII.1]. The ideal  $(\mu)$  is a two-sided integral ideal of  $\Lambda$  which is uniquely characterized by the fact that  $\text{Nr}((\mu)) = \mathfrak{a}$ , cf. [15, p. 86]. Hence  $w_{\mathfrak{a}}$  depends only on  $\mathfrak{a}$ , not on a particular choice of  $\mu$ . Moreover, since  $\mu^2 \in F^\times \Gamma$ ,  $w_{\mathfrak{a}}^2 = 1$ . We call  $w_{\mathfrak{a}}$  the *Atkin-Lehner involution* associated to  $\mathfrak{a}$ . By Proposition 4.4, the Atkin-Lehner involutions form a subgroup  $W = N(\Gamma)/F^\times \Gamma$  of  $\text{Aut}(X^R)$  isomorphic to  $(\mathbb{Z}/2\mathbb{Z})^{\#R}$ . If  $\mathfrak{a}$  and  $\mathfrak{b}$  are divisors of  $\mathfrak{r}$ , and  $\mathfrak{c} = \text{g.c.d.}(\mathfrak{a}, \mathfrak{b})$ , then  $w_{\mathfrak{a}}w_{\mathfrak{b}} = w_{\mathfrak{ab}/\mathfrak{c}^2}$ .

The next theorem is the analogue of [12, Theorem 2].

**Theorem 4.6.** *If  $R$  contains a place of even degree, then  $\text{Aut}(X^R) = W$ .*

*Proof.* Let  $\tilde{F} \neq F$  be a finite field extension of  $F$ . From the theory of quaternion algebras it is well-known that  $\tilde{F}$  embeds into  $D$  as an  $F$ -subalgebra if and only if  $[F : F] = 2$  and no place in  $R$  splits in  $\tilde{F}$ .

Let  $\gamma \in \Gamma$  be a torsion element of order  $n$ :  $\gamma^n = 1$ . Let  $p$  be the characteristic of  $F$ . If  $p|n$ , then  $\gamma^{n/p} \neq 1$  but  $(\gamma^{n/p} - 1)^p = 0$ . This is not possible, since  $D$  is a division algebra. We conclude that the  $\mathbb{F}_q$ -algebra  $\mathbb{F}_q(\gamma)$  is a finite field extension  $\mathbb{F}_q^{(m)}$  of  $\mathbb{F}_q$ . Consider the subfield  $F(\gamma)$  of  $D$ . Since  $\gamma$  is algebraic over  $\mathbb{F}_q$ , we have  $[\mathbb{F}_q(\gamma) : \mathbb{F}_q] = [F(\gamma) : F]$ , so  $m = 1$  or  $2$ . Thus, either  $\gamma \in \mathbb{F}_q^\times$  or  $\mathbb{F}_{q^2}F$  embeds into  $D$ . A place  $o \in |C|$  of even degree splits in  $\mathbb{F}_{q^2}F$ . Therefore, if  $R$  contains a place of even degree then necessarily  $\gamma \in \mathbb{F}_q^\times$ . This implies that the image  $\bar{\Gamma}$  of  $\Gamma$  in  $\text{PGL}_2(F_\infty)$ , which is a finitely generated and discrete subgroup, is also torsion-free. Hence  $\bar{\Gamma}$  is the Schottky group of  $X^R$ ; see [14]. Let  $N_{\text{PGL}_2(F_\infty)}(\bar{\Gamma})$  denote the normalizer of  $\bar{\Gamma}$  in  $\text{PGL}_2(F_\infty)$ . From the theory of Mumford curves [2, p. 216], one knows that  $\text{Aut}(X^R) \cong N_{\text{PGL}_2(F_\infty)}(\bar{\Gamma})/\bar{\Gamma}$ .

Next, we claim that the  $F$ -vector space spanned by  $\Gamma$  is  $D$ . If  $\gamma \in \Gamma$  is a non-torsion element, then  $F(\gamma)$  is a quadratic extension of  $F$ . Suppose  $\gamma_1, \gamma_2 \in \Gamma$  are such that  $F(\gamma_1) \neq F(\gamma_2)$ . Then one easily checks that  $\{1, \gamma_1, \gamma_2, \gamma_1\gamma_2\}$  are linearly independent over  $F$ , hence form a basis of  $D$ . If such  $\gamma_1$  and  $\gamma_2$  do not exist, then  $\Gamma$  is an abelian group of rank at most 1. Such a group cannot have a cocompact image in  $\text{PGL}_2(F_\infty)$ , which is a contradiction.

The previous paragraph implies that if  $g \in N_{\text{GL}_2(F_\infty)}(\Gamma)$ , then  $g$  will actually normalize  $D^\times(F)$ . By the Skolem-Noether theorem,  $g$  induces an inner automorphism of  $D$  so that  $g \in F_\infty^\times D^\times(F)$ . Therefore,  $N_{\text{GL}_2(F_\infty)}(\Gamma) = F_\infty^\times N(\Gamma)$  and

$$\text{Aut}(X^R) \cong N_{\text{PGL}_2(F_\infty)}(\bar{\Gamma})/\bar{\Gamma} \cong N(\Gamma)/F^\times \Gamma = W.$$

□

**Proposition 4.7.** *If  $X^R$  is hyperelliptic, then  $R = \{x, y\}$  and  $\{\deg(x), \deg(y)\}$  is one of the following pairs*

- $\{1, 2\}, \{1, 3\}, \{2, 2\}$  if  $q = 3$ ;
- $\{1, 2\}$  if  $q \geq 5$ .

*Proof.* We know that

$$W \cong (\mathbb{Z}/2\mathbb{Z})^{\#R} \subset \text{Aut}(X^R),$$

so if  $X^R$  is hyperelliptic, then Lemma 3.5 forces  $\#R = 2$ . Hence we can assume  $R = \{x, y\}$  from now on. Now we apply the argument in the proof of Part (1) of Theorem 4.1. Note that we can choose  $o$  of degree 1, since the number of degree 1 places in  $|C| - \infty$  is  $q > 2$ . Then the following inequality must hold:

$$(q_x - 1)(q_y - 1) + 4q \cdot \wp(R) \leq 2(q^2 + 1)(q + 1).$$

One easily checks that this is possible if and only if either

$$\{\deg(x), \deg(y)\} = \{1, 1\}, \{1, 2\}$$

and  $q$  is arbitrary, or  $q = 3$  and

$$\{\deg(x), \deg(y)\} = \{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 2\}.$$

In  $\{1, 1\}$  case,  $X^R$  has genus zero by Theorem 3.1, so it is not hyperelliptic.  $\square$

To determine which of the curves listed in Proposition 4.7 are actually hyperelliptic, we need to compute the number of fixed points of Atkin-Lehner involutions acting on  $X^R$ .

**Definition 4.8** (cf. [15]). Let  $L$  be a quadratic extension of  $F$ , and let  $\mathfrak{R}$  be an  $A$ -order in  $L$  (e.g., the integral closure of  $A$  in  $L$ ). Suppose  $\varphi : L \hookrightarrow D$  is an embedding. One says that  $\varphi$  is an *optimal embedding* of  $\mathfrak{R}$  into  $\Lambda$  if  $\varphi(L) \cap \Lambda = \varphi(\mathfrak{R})$ . Two optimal embeddings  $\varphi$  and  $\psi$  of  $\mathfrak{R}$  into  $\Lambda$  are *equivalent* if there is  $\gamma \in \Gamma$  such that  $\psi = \gamma\varphi\gamma^{-1}$ . Denote by  $\Theta(\mathfrak{R}, \Lambda)$  the number of inequivalent optimal embeddings of  $\mathfrak{R}$  into  $\Lambda$ . By a result of Eichler [15, pp. 92–94], we have

$$(4.2) \quad \Theta(\mathfrak{R}, \Lambda) = h(\mathfrak{R}) \prod_{x \in R} \left( 1 - \left( \frac{L}{x} \right) \right),$$

where  $h(\mathfrak{R})$  is the ideal class number of  $\mathfrak{R}$  and  $\left( \frac{L}{x} \right)$  is the *Artin-Legendre symbol*:

$$\left( \frac{L}{x} \right) = \begin{cases} 1, & \text{if } x \text{ splits in } L; \\ -1, & \text{if } x \text{ is inert in } L; \\ 0, & \text{if } x \text{ ramifies in } L. \end{cases}$$

**Definition 4.9.** A quadratic extension  $L$  of  $F$  is *imaginary* if  $\infty$  does not split in  $L$ .

**Lemma 4.10.** Let  $f \in A$  be a polynomial of degree  $d$  and let  $c \in \mathbb{F}_q^\times$  be its leading coefficient, i.e.,  $f = cT^d + c_{d-1}T^{d-1} + \dots$ . Then  $L := F(\sqrt{f})$  is imaginary if and only if one of the following holds:

- (1)  $d$  is odd;
- (2)  $d$  is even and  $c$  is not a square in  $\mathbb{F}_q^\times$ .

In the first case  $\infty$  ramifies in  $L/F$ , and in the second case it remains inert. The integral closure of  $A$  in  $L$  is  $A[\sqrt{f}]$ , which is also the subring of  $L$  consisting of functions regular away from  $\widetilde{\infty}$  - the unique place over  $\infty$ .

*Proof.* See [6, p. 187]. □

**Notation 4.11.** Let  $\mathfrak{a}|\mathfrak{r}$  and  $f \in A$  be the monic generator of the ideal  $\mathfrak{a}$ . Let  $\kappa \in \mathbb{F}_q^\times$  be a fixed element which is not a square in  $\mathbb{F}_q^\times$ . Denote  $\mathfrak{R}_\mathfrak{a} := A[\sqrt{\kappa f}]$  and  $\mathfrak{R}'_\mathfrak{a} := A[\sqrt{f}]$ , and their fields of fractions by  $K_\mathfrak{a} = F(\sqrt{\kappa f})$  and  $K'_\mathfrak{a} := F(\sqrt{f})$ . Denote by  $\text{Fix}(w_\mathfrak{a})$  the set of fixed points of  $w_\mathfrak{a}$  acting on  $X^R$ .

**Proposition 4.12.** *With above notation,*

$$\#\text{Fix}(w_\mathfrak{a}) = \begin{cases} \Theta(\mathfrak{R}_\mathfrak{a}, \Lambda), & \text{if } \deg(f) \text{ is even;} \\ \Theta(\mathfrak{R}_\mathfrak{a}, \Lambda) + \Theta(\mathfrak{R}'_\mathfrak{a}, \Lambda), & \text{if } \deg(f) \text{ is odd.} \end{cases}$$

*Proof.* (Cf. [9, §2].) We will show that  $\text{Fix}(w_\mathfrak{a})$  is in one-to-one correspondence with the inequivalent optimal embeddings of  $\mathfrak{R}_\mathfrak{a}$  (and  $\mathfrak{R}'_\mathfrak{a}$ ) into  $\Lambda$  when  $\deg(f)$  is even (resp. odd).

Let  $\varphi$  be an optimal embedding of  $\mathfrak{R}_\mathfrak{a}$  into  $\Lambda$ . Then  $\mu := \varphi(\sqrt{\kappa f}) \in \Lambda$ . Note that  $\mu' = -\mu$ , since the canonical involution of  $D$  restricted to  $F(\mu)$  induces the non-trivial automorphism of this field over  $F$ . In particular,  $\text{Tr}(\mu) = 0$  and  $\text{Nr}(\mu) = -\kappa f$ . Therefore, the action of  $\mu$  on  $\Omega$  induces the action of  $w_\mathfrak{a}$  on  $X^R$ .

We claim that  $\mu$  has a fixed point in  $\Omega$ . Let  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be the matrix representation of  $\mu$  under the embedding  $\Lambda \hookrightarrow D(F_\infty) \cong \mathbb{M}_2(F_\infty)$ . The fixed points of  $\mu$  correspond to the solutions of

$$(4.3) \quad cz^2 + (d - a)z - b = 0.$$

Since the matrix of  $\mu$  has trace 0 and determinant  $-\kappa f$ , the discriminant of this quadratic equation is  $4\kappa f$ . Hence its solutions generate an imaginary quadratic extension of  $F$ , in particular, they are in  $\mathbb{C}_\infty - F_\infty$ . The image of such a solution on  $X^R$  gives a fixed point of  $w_\mathfrak{a}$ . The same argument also applies to an optimal embedding of  $\mathfrak{R}'_\mathfrak{a}$  when  $\deg(f)$  is odd.

Now let  $P \in X^R(\mathbb{C}_\infty)$  be a fixed point of  $w_\mathfrak{a}$ . This means that there exists  $z \in \Omega$  and  $\mu \in \Lambda$  with  $(\text{Nr}(\mu)) = \mathfrak{a}$ , such that  $\mu z = \gamma z$  for some  $\gamma \in \Gamma$ . By replacing  $\mu$  by  $\gamma^{-1}\mu$ , we can assume  $\mu z = z$ . Using a matrix representation of  $\mu$  and (4.3), this easily implies that the field extension  $F(\mu)$  is necessarily imaginary and  $[F_\infty(z) : F_\infty] = 2$ . Next, by Proposition 4.4,  $\mu$  generates a two sided  $\Lambda$ -ideal  $I(\mathfrak{a}) = \mu\Lambda = \Lambda\mu$ . By [15, p. 86], any  $\tilde{\mu}$  with  $(\text{Nr}(\tilde{\mu})) = \mathfrak{a}$  generates the same two-sided ideal  $I(\mathfrak{a})$ . In particular,  $\mu'$  generates  $I(\mathfrak{a})$ , so  $\mu' = c\mu$  for some  $c \in F(\mu) \cap \Gamma = \mathbb{F}_q^\times$ . (In fact,  $c = -1$ , since  $\mu + \mu' \in F$ .) Hence  $\mu^2 = sf$ , for some  $s \in \mathbb{F}_q^\times$  such that  $F(\sqrt{sf})$  is imaginary. Therefore,  $A[\mu]$  is isomorphic to  $\mathfrak{R}_\mathfrak{a}$  (or  $\mathfrak{R}'_\mathfrak{a}$  when  $\deg(f)$  is odd), and we obtain two optimal embedding of the corresponding ring into  $\Lambda$  given by  $\mu$  and  $\mu'$ .



The existence of the rigid-analytic uniformization  $\pi : \Omega \rightarrow X^R(\mathbb{C}_\infty)$  is a consequence of a stronger result which shows that the formal schemes  $\hat{\Omega}/\Gamma$  and  $\hat{X}^R$  represent the same functor over  $\mathrm{Spf}(\mathcal{O}_\infty)$ , where  $\hat{\Omega}$  is Mumford’s formal scheme with generic fibre  $\Omega$  and  $\hat{X}^R$  is the formal completion of  $X^R$  along its closed fibre at  $\infty$ ; see [1, Chapter 4]. This stronger statement implies that  $\pi$  is  $\mathrm{Gal}(F_\infty^{\mathrm{sep}}/F_\infty)$ -equivariant, in the sense that if  $z \in F_\infty^{\mathrm{sep}}$  and  $g \in \mathrm{Gal}(F_\infty^{\mathrm{sep}}/F_\infty)$ , then  $\pi(gz) = g\pi(z)$ .

So far to a fixed point  $P$  of  $w_a$  we have associated  $z \in \Omega$  in a quadratic extension of  $F_\infty$  and two optimal embeddings of  $\mathfrak{R}_a$  (or  $\mathfrak{R}'_a$ ) into  $\Lambda$ , corresponding to  $\mu$  and  $\mu'$ , and we showed that any optimal embedding arises in this manner. The points in the orbit  $\Gamma z$  produce equivalent embeddings since if  $z$  corresponds to  $\mu$  and  $\mu'$ , then  $\gamma z$  corresponds to  $\gamma\mu\gamma^{-1}$  and  $\gamma\mu'\gamma^{-1}$ . Let  $\tau$  be the generator of  $\mathrm{Gal}(K_a/F)$ . Note that  $\mu' = \mu \circ \tau$ . Denote the equivalence class of an optimal embedding  $\mu$  by  $[\mu]$ . With this notation, to  $P$  there are two associated equivalence classes of embedding  $\{[\mu], [\mu \circ \tau]\}$ . The auxiliary point  $z$  in this construction gives an embedding  $K_a \hookrightarrow \mathbb{C}_\infty$ , and allows us to consider  $\tau$  as an element of  $\mathrm{Gal}(F_\infty^{\mathrm{sep}}/F_\infty)$ . By the  $\mathrm{Gal}(F_\infty^{\mathrm{sep}}/F_\infty)$ -equivariance of  $\pi$ , if  $P = \pi(z)$  then  $\tau(P) = \pi(\tau z)$ . Since  $w_a$  is defined over  $F_\infty$ ,  $\tau(P)$  is also a fixed point of  $w_a$ . Note that the optimal embeddings corresponding to  $\tau(z)$  are  $\mu \circ \tau$  and  $\mu \circ \tau \circ \tau = \mu$ . Therefore, the equivalence classes of embeddings corresponding to  $\tau(P)$  are also  $\{[\mu], [\mu \circ \tau]\}$ . Finally, it is easy to see that  $[\mu]$  and  $[\mu \circ \tau]$  appear only among the embeddings associated to  $P$  and  $\tau(P)$ . Overall, we get a correspondence between  $\mathrm{Fix}(w_a)$  and the equivalence classes of optimal embeddings of  $\mathfrak{R}_a$  (and  $\mathfrak{R}'_a$ ) into  $\Lambda$ :

$$\{P, \tau P\} \leftrightarrow \{[\mu], [\mu \circ \tau]\}.$$

To deduce the proposition it remains to show that  $P = \tau(P)$  if and only if  $[\mu] = [\mu \circ \tau]$ . Now,  $P = \tau(P)$  if and only if  $\tau(z) = \gamma(z)$  for some  $\gamma \in \Gamma$ . But  $\tau(z)$  is a fixed point of  $\mu'$  and  $\gamma(z)$  is a fixed point of  $\gamma\mu\gamma^{-1}$ . Therefore,  $P = \tau(P)$  if and only if either  $\gamma\mu\gamma^{-1} = \mu'$  or  $\gamma\mu\gamma^{-1} = \mu$ . The first case is equivalent to  $[\mu \circ \tau] = [\mu]$ . The second case  $\gamma\mu\gamma^{-1} = \mu$  implies  $\gamma \in F(\mu) \subset D$ . Since  $\gamma$  is a unit over  $A$  and  $F(\mu)$  is imaginary with field of constants  $\mathbb{F}_q$ , we get  $\gamma \in \mathbb{F}_q^\times$ . This leads to a contradiction since  $\gamma(z) = z$  but  $\tau(z) \neq z$ .  $\square$

Let  $a \in A$ . The field  $F(\sqrt{a})$  is the function field of the curve  $C'$  over  $\mathbb{F}_q$  given by  $y^2 = a$ . A genus formula for such a curve is well-known (cf. [6, p. 332]):

$$g(C') = \left\lfloor \frac{\deg(a) - 1}{2} \right\rfloor.$$

Let  $J_{C'}$  be the Jacobian variety of  $C'$ . The following fact is also well-known (cf. [6, Chapter VIII]):

**Lemma 4.13.** *Suppose  $F(\sqrt{a})$  is imaginary. Then*

$$h(A[\sqrt{a}]) = \begin{cases} \#J_{C'}(\mathbb{F}_q), & \text{if } \deg(a) \text{ is odd;} \\ 2\#J_{C'}(\mathbb{F}_q), & \text{if } \deg(a) \text{ is even.} \end{cases}$$

To simplify the notation, for  $x, y \in R$  denote  $w_x := w_{\mathfrak{p}_x}$ ,  $w_y := w_{\mathfrak{p}_y}$ ,  $w_{xy} := w_x w_y$ . Let  $f_x$  and  $f_y$  be the monic generators of  $\mathfrak{p}_x$  and  $\mathfrak{p}_y$ , respectively.

**Lemma 4.14.** *Suppose  $R = \{x, y\}$ ,  $\deg(x) = 1$  and  $\deg(y) = 2$ . Then one of the involutions  $w_x, w_y$  has 4 fixed points and the other one 0, and  $w_{xy}$  has  $2q + 2$  fixed points.*

*Proof.* From (4.2), Proposition 4.12 and Lemma 4.13, we have

$$\begin{aligned} \#\text{Fix}(w_x) &= 1 \cdot \left(1 - \left(\frac{F(\sqrt{f_x})}{y}\right)\right) + 1 \cdot \left(1 - \left(\frac{F(\sqrt{\kappa f_x})}{y}\right)\right), \\ \#\text{Fix}(w_y) &= 2 \cdot \left(1 - \left(\frac{F(\sqrt{\kappa f_y})}{x}\right)\right). \end{aligned}$$

Next, let  $f \in A$  be a polynomial of degree 3. Then  $E : z^2 = f$  is an elliptic curve over  $\mathbb{F}_q$  (note that  $\infty$  is rational on  $E$ ). Since  $E$  is its own Jacobian,  $h(A[\sqrt{f}])$  is equal to the number of solutions of  $z^2 = f$  over  $\mathbb{F}_q$  plus 1. Now let  $f = f_x f_y$ . Then

$$\begin{aligned} \#\text{Fix}(w_{xy}) &= h(A[\sqrt{f}]) + h(A[\sqrt{\kappa f}]) \\ &= 2 + \#\{(\alpha, \beta) \in \mathbb{F}_q \times \mathbb{F}_q \mid \alpha^2 = f(\beta)\} \\ &\quad + \#\{(\alpha, \beta) \in \mathbb{F}_q \times \mathbb{F}_q \mid \alpha^2 = \kappa f(\beta)\}. \end{aligned}$$

If  $\beta \in \mathbb{F}_q$  is fixed and  $f(\beta) \neq 0$ , then exactly one of  $f(\beta)$  and  $\kappa f(\beta)$  is a square in  $\mathbb{F}_q$ , and we get two solutions of the corresponding equation. On the other hand, if  $f(\beta) = 0$ , then both  $z^2 = f(\beta)$  and  $z^2 = \kappa f(\beta)$  have exactly one solution, namely  $z = 0$ . Overall, we get  $2q$  solutions.  $\square$

**Lemma 4.15.** *Suppose  $R = \{x, y\}$ , and  $\deg(x) = \deg(y) = 2$ . Then*

$$\#\text{Fix}(w_x) \leq 4, \quad \#\text{Fix}(w_y) \leq 4, \quad \#\text{Fix}(w_{xy}) \leq 2(q + 1 + 2\sqrt{q}).$$

*Proof.* The argument is similar to the proof of the previous lemma. Note that  $z^2 = f_x f_y$  corresponds to a curve of genus 1. The Jacobian of such a curve is an elliptic curve, hence has at most  $(q + 1 + 2\sqrt{q})$  rational points over  $\mathbb{F}_q$  by the Hasse-Weil bound.  $\square$

**Lemma 4.16.** *Suppose  $q = 3$  and  $R = \{x, y\}$ , where  $\deg(x) = 1$  and  $\deg(y) = 3$ . Then*

$$\#\text{Fix}(w_x) = 2, \quad \#\text{Fix}(w_y) \geq 2, \quad \#\text{Fix}(w_{xy}) \geq 4.$$

*If  $\#\text{Fix}(w_y) = 2$ , then  $\#\text{Fix}(w_{xy}) = 8$ .*

*Proof.* We have

$$\#\text{Fix}(w_x) = 1 \cdot \left(1 - \left(\frac{F(\sqrt{f_x})}{y}\right)\right) + 1 \cdot \left(1 - \left(\frac{F(\sqrt{\kappa f_x})}{y}\right)\right).$$

Since  $[\mathbb{F}_y : \mathbb{F}_q] = 3$ ,  $\kappa$  is not a square in  $\mathbb{F}_y$ , so exactly one of  $f_x$  or  $\kappa f_x$  is a square modulo  $\mathfrak{p}_y$ . This implies  $\#\text{Fix}(w_x) = 2$ .

Now consider the curves defined by  $z^2 = f_y$ ,  $z^2 = \kappa f_y$  and  $z^2 = \kappa f_x f_y$ . Denote the Jacobians of these curves by  $E_1$ ,  $E_2$  and  $E_3$ , respectively. All three of these abelian varieties are elliptic curves. We have

$$\#\text{Fix}(w_y) = \#E_1(\mathbb{F}_q) \cdot \left(1 - \left(\frac{F(\sqrt{f_y})}{x}\right)\right) + \#E_2(\mathbb{F}_q) \cdot \left(1 - \left(\frac{F(\sqrt{\kappa f_y})}{x}\right)\right).$$

Since  $\deg(x) = 1$ ,  $\kappa$  is not a square in  $\mathbb{F}_x \cong \mathbb{F}_q$ . On the other hand, since  $f_y$  is irreducible,  $f_y \not\equiv 0 \pmod{x}$ . Thus, exactly one of  $f_y$  or  $\kappa f_y$  is a square modulo  $x$ , so  $\#\text{Fix}(w_y) \geq 2$ . Suppose  $\#\text{Fix}(w_y) = 2$ . This is possible only if either  $\#E_1(\mathbb{F}_q) = 1$  or  $\#E_2(\mathbb{F}_q) = 1$ . A case-by-case verification shows that we must have

$$(4.4) \quad f_y = T^3 - T + 1 \quad \text{or} \quad f_y = T^3 - T - 1 :$$

if  $f_y = T^3 - T + 1$  then  $\#E_2(\mathbb{F}_q) = 1$ ; if  $f_y = T^3 - T - 1$  then  $\#E_1(\mathbb{F}_q) = 1$ . In both cases we indeed get  $\#\text{Fix}(w_y) = 2$ .

Next,  $\#\text{Fix}(w_{xy}) = 2 \cdot \#E_3(\mathbb{F}_q)$ . Note that  $z^2 = \kappa f_x f_y$  has a rational solution: if  $f_x = T - c$  with  $c \in \mathbb{F}_q$ , then  $T = c$ ,  $z = 0$  is a solution. Hence,  $E_3$  besides its 0 for the group structure has another rational point, so  $\#E_3(\mathbb{F}_q) \geq 2$ . This implies the desired lower bound on  $\#\text{Fix}(w_{xy})$ . Finally, suppose  $\#\text{Fix}(w_y) = 2$ . From (4.4) we get that the function  $\kappa f_y(s) \in \mathbb{F}_3$ ,  $s \in \mathbb{F}_3$ , is either identically 1 or  $-1$ , and using this, one easily checks that  $\#E_3(\mathbb{F}_q) = 4$ .  $\square$

**Lemma 4.17.** *Suppose  $X$  is a hyperelliptic curve over a field whose characteristic is not 2, and the genus of  $X$  is even. Then any involution of  $X$ , except for the canonical involution, has exactly 2 fixed points.*

*Proof.* If the characteristic is not 2, then the proof of [8, Proposition 1] applies.  $\square$

*Proof of part (2) of Theorem 4.1.* Since the characteristic of  $F$  is odd,  $X^R$  is hyperelliptic if and only if  $\text{Aut}(X^R)$  contains an involution having  $2g(X^R)+2$  fixed points (by the Riemann-Hurwitz formula). If  $R = \{x, y\}$  and  $\{\deg(x), \deg(y)\} = \{1, 2\}$ , then  $g(X^R) = q$  (Theorem 3.1). On the other hand, by Lemma 4.14,  $\#\text{Fix}(w_{xy}) = 2q + 2$ . Thus,  $X^R$  is hyperelliptic in this case, with  $w_{xy}$  being the canonical involution. Thanks to Proposition 4.7, to conclude that this is the only possibility, we need to check that when  $q = 3$ ,  $R = \{x, y\}$  and  $\{\deg(x), \deg(y)\} = \{2, 2\}, \{1, 3\}$ ,  $X^R$  is not hyperelliptic.

Suppose  $q = 3$  and  $\{\deg(x), \deg(y)\} = \{2, 2\}$ . From Theorem 4.6 we know that the canonical involution, if it exists, is an Atkin-Lehner involution. Since  $g(X^R) = 9$ , the canonical involution must have 20 fixed points. On the other hand, from Lemma 4.15 an Atkin-Lehner involution has at most 14 fixed points. Hence  $X^R$  is not hyperelliptic.

Now suppose  $q = 3$  and  $\{\deg(x), \deg(y)\} = \{1, 3\}$ . Then  $g(X^R) = 6$ . Since  $\#\text{Fix}(w_{xy}) \geq 4$ , from Lemma 4.16 and Lemma 4.17, we deduce that if  $X^R$  is hyperelliptic then  $w_{xy}$  is necessarily the canonical involution and  $\#\text{Fix}(w_y) = 2$ .

If this is the case, then  $\#\text{Fix}(w_{xy}) = 8$ . On the other hand, the canonical involution must have 14 fixed points, a contradiction.  $\square$

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