

# On Garland's vanishing theorem for $SL_n$

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**Abstract** This is an expository paper on Garland's vanishing theorem specialized to the case when the linear algebraic group is  $SL_n$ . Garland's theorem can be stated as a vanishing of cohomology groups of certain finite simplicial complexes. The method of the proof is quite interesting on its own. It relates the vanishing of cohomology to the assertion that the minimal positive eigenvalue of a certain combinatorial Laplacian is sufficiently large. Since the 1970s, this idea has found applications in a variety of problems in representation theory, group theory, and combinatorics, so the paper might be of interest to a wide audience. The paper is intended for non-specialists and graduate students.

**Keywords** Garland's method · Combinatorial Laplacians · Buildings

**Mathematics Subject Classification** 20E40 · 05C50 · 22E40

## 1 Introduction

### 1.1 Statement of the theorem

This is an expository paper on Howard Garland's work [8] specialized to the case when the linear algebraic group is  $SL_n$ . Reading [8] requires knowledge of the theory

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of buildings. On the other hand, the ideas in [8], in their essence, are combinatorial. Since the relevant Bruhat–Tits building in the  $SL_n$  case has a simple description in terms of lattices in a finite dimensional vector space, one can give a proof of Garland’s vanishing theorem which requires from the reader only familiarity with linear algebra and some group theory. There is already an excellent Bourbaki Exposé by Borel [3] on Garland’s work. The main difference of our exposition, besides the more detailed proofs, is the absence of any references to the theory of buildings, which makes the article completely self-contained. Since Garland’s result applies to quite general discrete subgroups of  $p$ -adic groups, to give a full account of his work one cannot avoid a discourse on the theory of buildings and representation theory. Other expositions of Garland’s method and its generalizations can be found in [1, 2]; these papers also give nice applications of Garland’s method to group theory and combinatorics, although they do not give a complete proof of Theorem 1.1.

Let  $K$  be a non-archimedean local field with residue field of order  $q$ ; such a field is either isomorphic to a finite extension of the  $p$ -adic numbers  $\mathbb{Q}_p$ , or to the field of formal Laurent series  $\mathbb{F}_q((T))$  over the finite field  $\mathbb{F}_q$  with  $q$  elements. Let  $\Gamma$  be a discrete subgroup of the topological group  $SL_n(K)$ ,  $n \geq 2$ . There is an infinite contractible  $(n - 1)$ -dimensional simplicial complex  $\mathfrak{B}$ , the *Bruhat–Tits building* of  $SL_n(K)$ , on which  $\Gamma$  naturally acts. The complex  $\mathfrak{B}$  can be described in terms of lattices in the vector space  $K^n$ . One way to formulate the main result of [8] for  $SL_n$  is as follows:

**Theorem 1.1** *Assume the quotient  $\mathfrak{B}/\Gamma$  is a finite complex. There is a constant  $q(n)$  depending only on  $n$  such that if  $q \geq q(n)$  then for all  $0 < i < n - 1$  the simplicial cohomology groups  $H^i(\mathfrak{B}/\Gamma, \mathbb{R})$  are zero.*

Such vanishing theorems were originally conjectured by Serre [19]. We will prove Theorem 1.1 in Sect. 4 under a mild assumption on  $\Gamma$ . In the same section we also explain how central division algebras naturally give rise to such groups. We should mention that the restriction on  $q$  being sufficiently large in Theorem 1.1 was removed by Casselman [4], who proved the vanishing of the middle cohomology groups by a completely different method, using representation theory of  $p$ -adic groups.

Garland’s vanishing theorem plays an important role in some problems in representation theory; for example, it puts strict restrictions on the continuous cohomology of topological groups with coefficients in infinite dimensional representations (cf. [4] and Sect. 5.3). An application of Garland’s theorem in arithmetic geometry arises in the calculation of the cohomology groups of certain algebraic varieties possessing rigid-analytic uniformization. More precisely,  $\mathfrak{B}$  can be realized as the skeleton of Drinfeld’s symmetric space  $\Omega^n$ , so the cohomology groups of the algebraic variety uniformized as  $\Omega^n/\Gamma$  are related to the cohomology groups of  $\mathfrak{B}/\Gamma$  (cf. [18]).

## 1.2 Outline of the paper

Now we give an outline of the proof of Theorem 1.1 and the contents of this paper.

Let  $X$  be a finite simplicial complex of dimension  $n$ . Let  $w$  be a “Riemannian metric” on  $X$ , by which we mean, following [7], a function from the non-oriented simplices of  $X$  to the positive real numbers. Let  $C^i(X)$  denote the  $\mathbb{R}$ -vector space of  $i$ -cochains of  $X$  with values in  $\mathbb{R}$ . Define an inner product on  $C^i(X)$ :

$$(f, g) = \sum_{\sigma} w(\sigma) f(\widehat{\sigma}) g(\widehat{\sigma}),$$

where the sum is over all non-oriented  $i$ -simplices of  $X$  and  $\widehat{\sigma}$  is an oriented simplex corresponding to  $\sigma$ . Let  $d: C^i(X) \rightarrow C^{i+1}(X)$  denote the coboundary operator, and  $\delta: C^i(X) \rightarrow C^{i-1}(X)$  denote the adjoint of  $d$  with respect to  $(\cdot, \cdot)$ . Let  $\mathcal{H}^i(X) \subset C^i(X)$  be the subspace of *harmonic cocycles*; by definition, these are the  $i$ -cochains annihilated by both  $d$  and  $\delta$ . It is not hard to show that  $H^i(X, \mathbb{R}) \cong \mathcal{H}^i(X)$ . This isomorphism is a consequence of the “Hodge decomposition” for  $C^i(X)$ . In Sect. 2, after recalling some standard terminology related to simplicial complexes, we prove this well-known fact.

Thus, to prove  $H^i(X, \mathbb{R}) = 0$ , it suffices to prove that there are no non-zero harmonic cocycles. Motivated by the work of Matsushima [13], who reduced the study of harmonic forms on real locally symmetric spaces to the computation of the minimal eigenvalues of certain curvature transformations, Garland reduces the study of  $\mathcal{H}^i(X)$  to estimating the minimal non-zero eigenvalue  $m^i(X)$  of the linear operator  $\Delta = \delta d$  acting on  $C^i(X)$ . Section 3 contains a key part of Garland’s argument: it gives the precise relationship between the vanishing of  $\mathcal{H}^i(X)$  and lower bounds on  $m^i(X)$ , and it also gives a method for estimating  $m^i(X)$  inductively.

In Sect. 4, we describe the Bruhat–Tits building  $\mathfrak{B}$  of  $SL_n(K)$  as a simplicial complex and explain how Theorem 1.1 follows from the results in Sect. 3, assuming a certain lower bound on  $m^i(\text{Lk}(v))$  for vertices of  $\mathfrak{B}$ , where  $\text{Lk}(v)$  denotes the link of the vertex  $v$ . We relegate the proof of this lower bound, which is the most technical part of the paper, to Sect. 5. At the end of Sect. 4 we give a brief discussion of some of the more recent applications of Garland’s method to producing examples of groups having Kazhdan’s property (T).

Based on numerical calculations, in Sect. 5.1 we state a conjecture about the asymptotic behaviour of the eigenvalues of  $\Delta$  acting on  $C^i(\text{Lk}(v))$  for vertices of  $\mathfrak{B}$ , and in Sect. 5.3 we give some evidence for this conjecture. None of the results of this paper are original, except possibly those in Sect. 5.3.

## 2 Simplicial cohomology and harmonic cocycles

In this section we recall the basic definitions from the theory of simplicial cohomology and prove a combinatorial analogue of the Hodge decomposition theorem. This last theorem identifies simplicial cohomology groups with spaces of harmonic cocycles. Its importance for the proof of Theorem 1.1 is that, instead of proving that  $H^i(\mathfrak{B}/\Gamma, \mathbb{R}) = 0$  directly, we will actually show that there are no non-zero harmonic  $i$ -cocycles on  $\mathfrak{B}/\Gamma$ .

### 2.1 Basic concepts

An (abstract) *simplicial complex* is a collection  $X$  of finite non-empty sets, called simplices, such that if  $s$  is an element of  $X$ , so is every non-empty subset of  $s$ . A non-empty subset of a simplex  $s$  is called a *face* of  $s$ . A *simplex of dimension  $i$* , or simply an  *$i$ -simplex*, is a simplex with  $i + 1$  elements. The vertex set  $\text{Ver}(X)$  of  $X$  is the union of its 0-simplices. A subcollection of  $X$  that is itself a complex is called a *subcomplex* of  $X$ . The *dimension* of  $X$  is the largest dimension of one of its simplices, or is infinite if there is no such largest dimension.

Let  $s$  be a simplex of  $X$ . The *star* of  $s$  in  $X$ , denoted  $\text{St}(s)$ , is the subcomplex of  $X$  consisting of the union of all simplices of  $X$  having  $s$  as a face. The *link* of  $s$ , denoted  $\text{Lk}(s)$ , is the subcomplex of  $\text{St}(s)$  consisting of the simplices which are disjoint from  $s$ . If one thinks of  $\text{St}(s)$  as the “unit ball” around  $s$  in  $X$ , then  $\text{Lk}(s)$  is the “unit sphere” around  $s$ .

Let  $X$  and  $Y$  be simplicial complexes. The *join* of  $X$  and  $Y$  is the simplicial complex  $X * Y$  such that  $s \in X * Y$  if either  $s \in X$  or  $s \in Y$ , or  $s = x * y = \{x_0, \dots, x_i, y_0, \dots, y_j\}$ , where  $x = \{x_0, \dots, x_i\} \in X$  and  $y = \{y_0, \dots, y_j\} \in Y$ . It is clear that  $X * Y$  is a simplicial complex and  $\dim X * Y = \dim X + \dim Y + 1$ . Note that, as a special case of this construction,  $\text{St}(s) = s * \text{Lk}(s)$ .

A specific ordering of the vertices of  $s$ , up to an even permutation, is called an *orientation* of  $s$ . Each positive dimensional simplex has two orientations. Denote the set of  $i$ -simplices by  $\widehat{S}_i(X)$ , and the set of oriented  $i$ -simplices by  $S_i(X)$ . Note that  $\widehat{S}_0(X) = S_0(X) = \text{Ver}(X)$ . For  $s \in S_i(X)$ ,  $\bar{s} \in S_i(X)$  denotes the same simplex but with opposite orientation. An  $\mathbb{R}$ -valued  *$i$ -cochain* on  $X$  is a function  $f: S_i(X) \rightarrow \mathbb{R}$  which is alternating if  $i \geq 1$ , i.e.,  $f(s) = -f(\bar{s})$ . (A 0-cochain is just a function on  $\text{Ver}(X)$ .) The  $i$ -cochains naturally form an  $\mathbb{R}$ -vector space which is denoted  $C^i(X)$ . If  $i < 0$  or  $i > \dim X$ , we set  $C^i(X) = 0$ .

The *coboundary operator* is the linear transformation  $d: C^i(X) \rightarrow C^{i+1}(X)$  defined by

$$df([v_0, \dots, v_{i+1}]) = \sum_{j=0}^{i+1} (-1)^j f([v_0, \dots, \widehat{v}_j, \dots, v_{i+1}]),$$

where  $[v_0, \dots, v_{i+1}] \in S_{i+1}(X)$  and the symbol  $\widehat{v}_j$  means that the vertex  $v_j$  is to be deleted from the array. The kernel of  $d: C^i(X) \rightarrow C^{i+1}(X)$  is called the subspace of  *$i$ -cocycles* and denoted  $Z^i(X)$ . As one easily verifies  $d \circ d = 0$  (cf. [14, p. 30]), so  $dC^{i-1}(X) \subset Z^i(X)$ . The  *$i$ -th cohomology group of  $X$*  (with real coefficients) is

$$H^i(X) = Z^i(X) / dC^{i-1}(X).$$

Let  $\mathbf{1} \in C^0(X)$  be the function defined by  $\mathbf{1}(v) = 1$  for all  $v \in \text{Ver}(X)$ . The subspace  $\mathbb{R}\mathbf{1} \subset C^0(X)$  spanned by  $\mathbf{1}$  is the space of constant functions. It is easy to see that  $\mathbb{R}\mathbf{1} \subset Z^0(X)$ . One defines the *reduced  $i$ -th cohomology group*  $\widetilde{H}^i(X)$  of  $X$  by setting  $\widetilde{H}^i(X) = H^i(X)$  if  $i \geq 1$ , and  $\widetilde{H}^0(X) = Z^0(X) / \mathbb{R}\mathbf{1}$ . It is easy to show that the “geometric realization” of  $X$  is connected if and only if  $\widetilde{H}^0(X) = 0$ ; see [14, p. 256].

Now assume  $X$  is finite, i.e., has finitely many vertices. To each simplex  $s$  of  $X$  assign a positive real number  $w(s) = w(\bar{s})$ , which we call the *weight* of  $s$ . Define an inner-product on  $C^i(X)$  by

$$(f, g) = \sum_{s \in \widehat{S}_i(X)} w(s) f(s) g(s), \quad f, g \in C^i(X), \tag{1}$$

where in  $w(s) f(s) g(s)$  we choose some orientation of  $s$ ; this is well-defined since  $f(s) g(s) = f(\bar{s}) g(\bar{s})$ .

Let  $s = [v_0, \dots, v_i] \in S_i(X)$  and  $v \in \text{Ver}(X)$ . If the set  $\{v, v_0, \dots, v_i\}$  is an  $(i + 1)$ -simplex of  $X$ , then we denote by  $[v, s] \in S_{i+1}(X)$  the oriented simplex  $[v, v_0, \dots, v_i]$ . Define a linear transformation  $\delta: C^i(X) \rightarrow C^{i-1}(X)$  by

$$\delta f(s) = \sum_{\substack{v \in \text{Ver}(X) \\ [v, s] \in S_i(X)}} \frac{w([v, s])}{w(s)} f([v, s]).$$

This operator is the adjoint of  $d$  with respect to (1).

**Lemma 2.1** *If  $f \in C^i(X)$  and  $g \in C^{i+1}(X)$ , then  $(df, g) = (f, \delta g)$ .*

*Proof*

$$\begin{aligned} (df, g) &= \sum_{s=[v_0, \dots, v_{i+1}] \in \widehat{S}_{i+1}(X)} w(s) \\ &\quad \cdot \sum_{j=0}^{i+1} f([v_0, \dots, \widehat{v}_j, \dots, v_{i+1}]) g([v_j, v_0, \dots, \widehat{v}_j, \dots, v_{i+1}]) \\ &= \sum_{\sigma \in \widehat{S}_i(X)} w(\sigma) f(\sigma) \sum_{\substack{v \in \text{Ver}(X) \\ [v, \sigma] \in S_{i+1}(X)}} \frac{w([v, \sigma])}{w(\sigma)} g([v, \sigma]) = (f, \delta g). \quad \square \end{aligned}$$

The kernel of  $\delta: C^i(X) \rightarrow C^{i-1}(X)$  will be denoted by  $\mathcal{Z}^i(X)$ .

### 2.2 Combinatorial Hodge decomposition

The intersection

$$\mathcal{H}^i(X) = \mathcal{Z}^i(X) \cap Z^i(X)$$

in  $C^i(X)$  is the subspace of *harmonic  $i$ -cocycles*. (This subspace depends on the choice of the inner-product (1).)

*Remark 2.2* The term *harmonic* comes from the fact that  $f \in C^i(X)$  is in  $\mathcal{H}^i(X)$  if and only if  $f$  is in the kernel of the operator  $d\delta + \delta d$ , which is a combinatorial analogue of the Laplacian.

**Theorem 2.3** *Relative to the inner-product (1), we have the orthogonal direct sum decompositions*

$$C^i(X) = \mathcal{H}^i(X) \oplus dC^{i-1}(X) \oplus \delta C^{i+1}(X), \tag{2}$$

$$Z^i(X) = \mathcal{H}^i(X) \oplus dC^{i-1}(X), \tag{3}$$

$$Z^i(X) = \mathcal{H}^i(X) \oplus \delta C^{i+1}(X). \tag{4}$$

*This implies*

$$H^i(X) = Z^i(X)/dC^{i-1}(X) \cong Z^i(X)/\delta C^{i+1}(X) \cong \mathcal{H}^i(X). \tag{5}$$

*Proof* Let  $f \in dC^{i-1}(X)$  and  $g \in \delta C^{i+1}(X)$ . We can write  $f = df'$  and  $g = \delta g'$  for some  $f' \in C^{i-1}(X)$  and  $g' \in C^{i+1}(X)$ . Since  $d^2 = 0$ , using Lemma 2.1, we get

$$(f, g) = (df', \delta g') = (d^2 f', g') = (0, g') = 0.$$

Hence  $dC^{i-1}(X) \perp \delta C^{i+1}(X)$ . Now suppose  $h \in C^i(X)$  is orthogonal to  $dC^{i-1}(X) \oplus \delta C^{i+1}(X)$ . Then

$$0 = (h, df') = (\delta h, f')$$

for all  $f' \in C^{i-1}(X)$ , which implies  $\delta h = 0$ . Similarly,  $0 = (h, \delta g')$  implies that  $dh = 0$ . Therefore,  $h \in \mathcal{H}^i(X)$ . In other words, the orthogonal complement of  $dC^{i-1}(X) \oplus \delta C^{i+1}(X)$  in  $C^i(X)$  is  $\mathcal{H}^i(X)$ . This proves (2).

A cochain  $f \in C^i(X)$  is in the orthogonal complement of  $\delta C^{i+1}(X)$  if and only if  $(f, \delta g') = (df, g') = 0$  for all  $g' \in C^{i+1}(X)$ , which implies  $df = 0$ . Thus

$$C^i(X) = Z^i(X) \oplus \delta C^{i+1}(X). \tag{6}$$

A similar argument shows that the orthogonal complement of  $dC^{i-1}(X)$  is  $Z^i(X)$ :

$$C^i(X) = Z^i(X) \oplus dC^{i-1}(X). \tag{7}$$

Comparing (6) and (7) with (2), we get (3) and (4). Finally, (3) and (4) imply (5).  $\square$

**Definition 2.4** Following [8], we call the linear transformation  $\Delta = \delta d$  on  $C^i(X)$  the *curvature transformation*. (What we denote by  $\Delta$  in this paper is denoted by  $\Delta^+$  in [3,8].)

By Lemma 2.1, for any  $f, g \in C^i(X)$  we have

$$(\Delta f, g) = (\delta df, g) = (df, dg) = (f, \delta dg) = (f, \Delta g)$$

and

$$(\Delta f, f) = (df, df) \geq 0. \tag{8}$$

Hence  $\Delta$  is a self-adjoint positive operator on  $C^i(X)$ , which implies that  $C^i(X)$  has an orthonormal basis consisting of eigenvectors of  $\Delta$ , and the eigenvalues of  $\Delta$  are non-negative.

**Lemma 2.5** *Let  $i \geq 0$ .*

- (i) *The subspace of  $C^i(X)$  spanned by the eigenfunctions of  $\Delta$  with positive eigenvalues is  $\delta C^{i+1}(X) = Z^i(X)^\perp$ .*
- (ii) *If  $H^i(X) = 0$ , then  $\delta C^{i+1}(X) = Z^i(X)$ .*

*Proof* It is clear from (8) that if  $\Delta f = 0$  then  $df = 0$ . Hence  $\ker \Delta \subseteq Z^i(X)$ . Conversely, if  $f \in Z^i(X)$  then  $\Delta f = \delta df = \delta 0 = 0$ . Therefore the subspace of  $C^i(X)$  spanned by the eigenfunctions of  $\Delta$  with positive eigenvalues is  $Z^i(X)^\perp$ . This latter subspace is  $\delta C^{i+1}(X)$ , as follows from (6). The second claim of the lemma follows from (5). □

### 3 Garland’s method

In this section we discuss what is nowadays called *Garland’s method*. Let  $X$  be a finite simplicial complex. Let

$$\lambda_{\min}^i(X) = \min_{v \in \text{Ver}(X)} m^i(\text{Lk}(v)).$$

Garland’s method shows that a strong enough lower bound on  $\lambda_{\min}^{i-1}(X)$  implies that  $\mathcal{H}^i(X) = 0$  (hence also  $H^i(X, \mathbb{R}) = 0$  by the Hodge decomposition discussed in the previous section). Moreover, the method gives a lower bound on  $m^k(X)$  in terms of  $\lambda_{\min}^{k-1}(X)$ , hence allows to estimate  $\lambda_{\min}^{i-1}(X)$  inductively in certain situations. (We will see an example of such an inductive estimate in Sect. 5.) The observation that Garland’s ideas from [8] apply to any finite simplicial complex satisfying a certain condition is due to Borel; in this section we partly follow [3, Section 1].

The proof of the main results has two parts: It starts with a decomposition of  $(\Delta f, f)$  into a sum  $\sum_v (\Delta f_v, f_v)$  over the vertices of  $X$ , where  $f_v$  is the restriction of  $f$  to the “unit ball” around  $v$ ; this is the content of Sect. 3.1. Then one bounds  $(\Delta f_v, f_v)$  in terms of  $m^{i-1}(\text{Lk}(v))$  by studying the local version of the curvature transformation; this is the content of Sect. 3.2. We combine these two parts in Sect. 3.3 to prove the main results. One of the subtleties is that to make this strategy work one has to choose an appropriate metric (9).

In this section we assume that  $X$  is a finite  $n$ -dimensional complex which has the following property:

$$\text{each simplex of } X \text{ is a face of some } n\text{-simplex.} \tag{\star}$$

For  $s \in S_i(X)$ , let

$$w(s) = \text{the number of (non-oriented) } n\text{-simplices containing } s. \tag{9}$$

Note that, due to  $(\star)$ ,  $w(s) \neq 0$ .

### 3.1 Decomposition of $(\Delta f, f)$ into local factors

We start with a simple lemma.

**Lemma 3.1** *Let  $s \in \widehat{S}_i(X)$  be fixed. Then*

$$\sum_{\substack{\sigma \in \widehat{S}_{i+1}(X) \\ s \subset \sigma}} w(\sigma) = (n - i)w(s).$$

*Proof* Given an  $n$ -simplex  $t$  such that  $s \subset t$ , there are exactly  $n - i$  simplices  $\sigma$  of dimension  $i + 1$  such that  $s \subset \sigma \subset t$ . Hence in the sum of the lemma we count every  $n$ -simplex containing  $s$  exactly  $n - i$  times.  $\square$

For a fixed  $v \in \text{Ver}(X)$  define a linear transformation  $\rho_v : C^i(X) \rightarrow C^i(X)$  by

$$\rho_v f(s) = \begin{cases} f(s) & \text{if } v \in s; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that

$$\rho_v \rho_v = \rho_v,$$

and for  $f, g \in C^i(X)$

$$(\rho_v f, g) = (\rho_v f, \rho_v g) = (f, \rho_v g). \tag{10}$$

Moreover, since any  $i$ -simplex has  $i + 1$  vertices, for  $f \in C^i(X)$  we have the equality

$$\sum_{v \in \text{Ver}(X)} \rho_v f = (i + 1)f. \tag{11}$$

**Lemma 3.2** *For  $f \in C^i(X)$ , we have*

$$i \cdot (\Delta f, f) = \sum_{v \in \text{Ver}(X)} (\Delta \rho_v f, \rho_v f) - (n - i)(f, f).$$

*Proof* First, to simplify the notation in our calculations we introduce new notation. Let  $\sigma \in S_{i+1}(X)$  and  $s \in S_i(X)$  be a face of  $\sigma$ . The orientation on  $\sigma$  induces an orientation on  $s$ ; we define  $[\sigma : s] = \pm 1$  depending on whether this induced orientation is the original orientation of  $s$  or its opposite. With this definition, for  $f \in C^i(X)$  we have

$$df(\sigma) = \sum_{\substack{s \in \widehat{S}_i(X) \\ s \subset \sigma}} [\sigma : s] f(s),$$

where for each face  $s$  of  $\sigma$  we choose some orientation. (Note that  $[\sigma : s]f(s)$  does not depend on the choice of the orientation of  $s$ .)



Let  $v \in \sigma$  be a fixed vertex and  $s_0 \in \widehat{S}_i(X)$  be the unique face of  $\sigma$  not containing  $v$ . Then

$$df(\sigma) = d\rho_v f(\sigma) + [\sigma : s_0] f(s_0).$$

Hence

$$df(\sigma)^2 = d\rho_v f(\sigma)^2 + 2[\sigma : s_0] f(s_0) \sum_{v \in s \subset \sigma} [\sigma : s] f(s) + f(s_0)^2.$$

Summing both sides over all vertices of  $\sigma$ , we get

$$\begin{aligned} (i + 2)df(\sigma)^2 &= \sum_{v \in \sigma} d\rho_v f(\sigma)^2 + 2 \sum_{\substack{s, s' \subset \sigma \\ s \neq s'}} [\sigma : s][\sigma : s'] f(s) f(s') + \sum_{s \subset \sigma} f(s)^2 \\ &= \sum_{v \in \sigma} d\rho_v f(\sigma)^2 + 2 \sum_{s, s' \subset \sigma} [\sigma : s][\sigma : s'] f(s) f(s') - \sum_{s \subset \sigma} f(s)^2 \\ &= \sum_{v \in \sigma} d\rho_v f(\sigma)^2 + 2df(\sigma)^2 - \sum_{s \subset \sigma} f(s)^2. \end{aligned}$$

Hence

$$i \cdot df(\sigma)^2 = \sum_{v \in \sigma} d\rho_v f(\sigma)^2 - \sum_{s \subset \sigma} f(s)^2. \tag{12}$$

Now

$$\begin{aligned} i \cdot (\Delta f, f) &\stackrel{\text{Lem. 2.1}}{=} i \cdot (df, df) = i \sum_{\sigma \in \widehat{S}_{i+1}(X)} w(\sigma) df(\sigma)^2 \\ &\stackrel{(12)}{=} \sum_{\sigma \in \widehat{S}_{i+1}(X)} \sum_{v \in \sigma} w(\sigma) d\rho_v f(\sigma)^2 - \sum_{\sigma \in \widehat{S}_{i+1}(X)} \sum_{s \subset \sigma} w(\sigma) f(s)^2 \\ &= \sum_{v \in \text{Ver}(X)} (d\rho_v f, d\rho_v f) - \sum_{s \in \widehat{S}_i(X)} f(s)^2 \sum_{\substack{\sigma \in \widehat{S}_{i+1}(X) \\ s \subset \sigma}} w(\sigma) \\ &\stackrel{\text{Lem. 3.1}}{=} \sum_{v \in \text{Ver}(X)} (\Delta \rho_v f, \rho_v f) - (n - i) \sum_{s \in \widehat{S}_i(X)} w(s) f(s)^2 \\ &= \sum_{v \in \text{Ver}(X)} (\Delta \rho_v f, \rho_v f) - (n - i)(f, f). \end{aligned}$$

□

### 3.2 Local curvature transformations

For  $f, g \in C^i(\text{Lk}(v))$  define their inner-product by

$$(f, g)_v = \sum_{s \in \widehat{S}_i(\text{Lk}(v))} w_v(s) \cdot f(s) \cdot g(s), \tag{13}$$

where  $w_v(s)$  is the number of  $(n - 1)$ -simplices in  $\text{Lk}(v)$  containing  $s$ . Note that  $\text{Lk}(v)$  is an  $(n - 1)$ -dimensional complex satisfying  $(\star)$ . Another simple observation is that for a simplex  $\sigma$  in  $\text{Lk}(v)$  there is a one-to-one correspondence between the  $n$ -simplices of  $X$  containing  $[v, \sigma]$  and the  $(n - 1)$ -simplices of  $\text{Lk}(v)$  containing  $\sigma$ . Hence

$$w_v(\sigma) = w([v, \sigma]) \quad \text{for any } \sigma \in \text{Lk}(v). \tag{14}$$

Let

$$d_v : C^i(\text{Lk}(v)) \rightarrow C^{i+1}(\text{Lk}(v))$$

be the coboundary operator acting on the cochains of the finite simplicial complex  $\text{Lk}(v)$ . Let  $\delta_v$  be the adjoint of  $d_v$  with respect to (13), and let

$$\Delta_v = \delta_v d_v.$$

Lemma 3.2 essentially decomposes  $\Delta$  into a sum of its restrictions  $\Delta_{\rho_v}$  to  $\text{St}(v)$  over all vertices. We want to relate  $\Delta_{\rho_v}$  to  $\Delta_v$ , and hence to relate the eigenvalues of  $\Delta$  to the eigenvalues of its local version  $\Delta_v$ . For this we need to introduce one more linear operator: For  $i \geq 1$ , define

$$\tau_v : C^i(X) \rightarrow C^{i-1}(X),$$

$$\tau_v f(s) = \begin{cases} f([v, s]) & \text{if } s \in S_{i-1}(\text{Lk}(v)); \\ 0 & \text{otherwise.} \end{cases}$$

Given  $f \in C^i(X)$ , its restriction to  $\text{Lk}(v)$  defines a function in  $C^i(\text{Lk}(v))$ , which by slight abuse of notation we denote by the same letter. With this convention, for  $f \in C^i(X)$  we can consider  $d_v f$  and  $\delta_v f$ , which are now functions on  $\text{Lk}(v)$ . Similarly, we can compute the pairing (13) on  $C^i(X)$ .

**Lemma 3.3** *Assume  $i \geq 1$ . For  $f, g \in C^i(X)$ , we have*

$$(\tau_v f, \tau_v g)_v = (\rho_v f, \rho_v g).$$

*Proof* We have

$$\begin{aligned} (\tau_v f, \tau_v g)_v &= \sum_{\sigma \in \widehat{S}_{i-1}(\text{Lk}(v))} w_v(\sigma) \cdot \tau_v f(\sigma) \cdot \tau_v g(\sigma) \\ &\stackrel{(14)}{=} \sum_{s \in \widehat{S}_i(\text{St}(v))} w(s) \cdot \rho_v f(s) \cdot \rho_v g(s). \end{aligned}$$

Since  $\rho_v f$  is zero away from  $\text{St}(v)$ , the last sum can be extended to the whole  $\widehat{S}_i(X)$ , so the lemma follows.  $\square$

**Lemma 3.4** *Let  $f \in C^i(X)$ . We have*

$$\tau_v d\rho_v f = -d_v \tau_v f, \quad i \geq 1, \tag{15}$$

$$\tau_v \delta f = -\delta_v \tau_v f, \quad i \geq 2, \tag{16}$$

$$\tau_v \Delta \rho_v f = \Delta_v \tau_v f, \quad i \geq 1. \tag{17}$$

*Proof* Let  $s \in S_i(\text{Lk}(v))$ ,  $i \geq 1$ . We have

$$\tau_v d\rho_v f(s) = d\rho_v f([v, s]) = \rho_v(f(s) - f([v, ds])) = -f([v, ds]).$$

In the last term  $ds$  denotes the image of  $s$  under the boundary operator and  $[\cdot]$  is extended linearly to  $\mathbb{Z}[S_i(\text{Lk}(v))]$ . Since  $d$  restricted to  $\text{Lk}(v)$  coincides with  $d_v$ , we have  $f([v, ds]) = d_v \tau_v f(s)$ . This proves (15).

Now assume  $i \geq 2$  and let  $s \in S_{i-2}(\text{Lk}(v))$ . We have

$$\begin{aligned} \tau_v \delta f(s) &= \delta f([v, s]) = \sum_{\substack{x \in \text{Ver}(X) \\ [x, v, s] \in S_i(X)}} \frac{w([x, v, s])}{w([v, s])} f([x, v, s]) \\ &\stackrel{(14)}{=} - \sum_{\substack{x \in \text{Ver}(\text{Lk}(v)) \\ [x, s] \in S_{i-1}(\text{Lk}(v))}} \frac{w_v([x, s])}{w_v(s)} \tau_v f([x, s]) = -\delta_v \tau_v f(s). \end{aligned}$$

This proves (16).

Finally,

$$\tau_v \Delta \rho_v f = \tau_v \delta d\rho_v f \stackrel{(16)}{=} -\delta_v \tau_v d\rho_v f \stackrel{(15)}{=} \delta_v d_v \tau_v f = \Delta_v \tau_v f.$$

This proves (17).  $\square$

**Lemma 3.5** *Assume  $i \geq 1$ . For  $f \in C^i(X)$ , we have*

$$(\Delta \rho_v f, \rho_v f) = (\Delta_v \tau_v f, \tau_v f)_v.$$

*Proof* We have

$$(\Delta \rho_v f, \rho_v f) \stackrel{(10)}{=} (\rho_v \Delta \rho_v f, \rho_v f) \stackrel{\text{Lem. 3.3}}{=} (\tau_v \Delta \rho_v f, \tau_v f)_v \stackrel{(17)}{=} (\Delta_v \tau_v f, \tau_v f)_v. \quad \square$$

**Notation 3.6** Given a simplicial complex  $Y$ , let  $M^i(Y)$  and  $m^i(Y)$  be the maximal and minimal non-zero eigenvalues of  $\Delta$  acting on  $C^i(Y)$ , respectively. Denote

$$\lambda_{\max}^i(Y) = \max_{v \in \text{Ver}(Y)} M^i(\text{Lk}(v)),$$

$$\lambda_{\min}^i(Y) = \min_{v \in \text{Ver}(Y)} m^i(\text{Lk}(v)).$$

**Lemma 3.7** *Assume  $i \geq 1$ . For  $f \in C^i(X)$ , we have*

$$(\Delta_v \tau_v f, \tau_v f)_v \leq M^{i-1}(\text{Lk}(v)) \cdot (\tau_v f, \tau_v f)_v.$$

*If  $\tilde{H}^{i-1}(\text{Lk}(v)) = 0$ , then for  $f \in \mathcal{Z}^i(X)$  we have*

$$(\Delta_v \tau_v f, \tau_v f)_v \geq m^{i-1}(\text{Lk}(v)) \cdot (\tau_v f, \tau_v f)_v.$$

*Proof* We can choose an orthonormal basis  $\{e_1, \dots, e_h\}$  of  $C^{i-1}(\text{Lk}(v))$  with respect to  $(\cdot, \cdot)_v$  which consists of  $\Delta_v$ -eigenvectors. Let  $\{\kappa_1, \dots, \kappa_h\}$  be the corresponding eigenvalues. We have  $\kappa_j \geq 0$  for all  $j$ . Write  $\tau_v f = \sum_{j=1}^h a_j e_j$ . Then

$$\begin{aligned} (\Delta_v \tau_v f, \tau_v f)_v &= \left( \sum_{j=1}^h a_j \kappa_j e_j, \sum_{j=1}^h a_j e_j \right)_v = \sum_{j=1}^h a_j^2 \kappa_j \\ &\leq M^{i-1}(\text{Lk}(v)) \sum_{j=1}^h a_j^2 = M^{i-1}(\text{Lk}(v)) (\tau_v f, \tau_v f)_v. \end{aligned}$$

The second claim will follow from a similar argument if we show that  $\tau_v f$  belongs to the subspace of  $C^{i-1}(\text{Lk}(v))$  spanned by  $\Delta_v$ -eigenfunctions with *positive* eigenvalues. First assume  $i \geq 2$ . In this case  $H^{i-1}(\text{Lk}(v)) = \tilde{H}^{i-1}(\text{Lk}(v)) = 0$ . Hence, thanks to Lemma 2.5, it is enough to show that  $\tau_v f \in \mathcal{H}^{i-1}(\text{Lk}(v))$ . Since by assumption  $\delta f = 0$ , from (16) we get  $\delta_v \tau_v f = -\tau_v \delta f = 0$ . Thus  $\tau_v f \in \mathcal{Z}^{i-1}(\text{Lk}(v))$ .

Now assume  $i = 1$  and  $\tilde{H}^0(\text{Lk}(v)) = 0$ . This last assumption is equivalent to  $\text{Lk}(v)$  being connected. In this case  $Z^0(\text{Lk}(v))$  is spanned by the function  $\mathbf{1} \in C^0(\text{Lk}(v))$  which assumes value 1 on all vertices of  $\text{Lk}(v)$ . By Lemma 2.5, we need to show that  $\mathbf{1}$  is orthogonal to  $\tau_v f$  with respect to the inner-product (13). We compute

$$\begin{aligned} (\mathbf{1}, \tau_v f)_v &= \sum_{x \in \text{Ver}(\text{Lk}(v))} w_v(x) \cdot \tau_v f(x) \\ &\stackrel{(14)}{=} \sum_{x \in \text{Ver}(\text{Lk}(v))} w([v, x]) f([v, x]) = -w(v) \delta f(v) = 0. \end{aligned} \tag{18}$$

and the proof is complete. □

### 3.3 Fundamental inequalities

Now we are ready to prove the main results of this section.

**Theorem 3.8** *Assume  $i \geq 1$ . For  $f \in C^i(X)$  we have*

$$i \cdot (\Delta f, f) \leq ((i + 1) \cdot \lambda_{\max}^{i-1}(X) - (n - i))(f, f).$$

*If  $\tilde{H}^{i-1}(\text{Lk}(v)) = 0$  for every  $v \in \text{Ver}(X)$ , then for  $f \in \mathcal{Z}^i(X)$  we have*

$$i \cdot (\Delta f, f) \geq ((i + 1) \cdot \lambda_{\min}^{i-1}(X) - (n - i))(f, f).$$

*Proof* Combining Lemma 3.2 with Lemma 3.5, we get

$$i \cdot (\Delta f, f) = \sum_{v \in \text{Ver}(X)} (\Delta_v \tau_v f, \tau_v f)_v - (n - i)(f, f). \tag{19}$$

If  $\tilde{H}^{i-1}(\text{Lk}(v)) = 0$  for every  $v \in \text{Ver}(X)$  and  $f \in \mathcal{Z}^i(X)$ , then by Lemma 3.7

$$\begin{aligned} \sum_{v \in \text{Ver}(X)} (\Delta_v \tau_v f, \tau_v f)_v &\geq \sum_{v \in \text{Ver}(X)} m^{i-1}(\text{Lk}(v))(\tau_v f, \tau_v f)_v \\ &\geq \lambda_{\min}^{i-1}(X) \sum_{v \in \text{Ver}(X)} (\tau_v f, \tau_v f)_v. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{v \in \text{Ver}(X)} (\tau_v f, \tau_v f)_v &\stackrel{\text{Lem. 3.3}}{=} \sum_{v \in \text{Ver}(X)} (\rho_v f, \rho_v f) \stackrel{(10)}{=} \sum_{v \in \text{Ver}(X)} (\rho_v f, f) \\ &= \left( \sum_{v \in \text{Ver}(X)} \rho_v f, f \right) \stackrel{(11)}{=} (i + 1)(f, f). \end{aligned}$$

Hence

$$\sum_{v \in \text{Ver}(X)} (\Delta_v \tau_v f, \tau_v f)_v \geq \lambda_{\min}^{i-1}(X)(i + 1)(f, f). \tag{20}$$

Substituting this inequality into (19) we get the second claim of the theorem. The first claim follows from a similar argument.  $\square$

**Corollary 3.9** *Assume  $i \geq 1$ . If  $\tilde{H}^{i-1}(\text{Lk}(v)) = 0$  for every  $v \in \text{Ver}(X)$  and  $\lambda_{\min}^{i-1}(X) > (n - i)/(i + 1)$ , then  $\mathcal{H}^i(X) = 0$ .*

*Proof* Let  $f \in \mathcal{H}^i(X)$ . Then  $df = \delta f = 0$ . We obviously have  $(\Delta f, f) = (df, df) = 0$ . Under our current assumptions, Theorem 3.8 then implies  $(f, f) \leq 0$ , which implies  $f = 0$ .  $\square$

**Theorem 3.10** *Assume  $i \geq 1$ . We have*

$$i \cdot M^i(X) \leq (i + 1) \cdot \lambda_{\max}^{i-1}(X) - (n - i),$$

and

$$i \cdot m^i(X) \geq (i + 1) \cdot \lambda_{\min}^{i-1}(X) - (n - i).$$

*Proof* Let  $f \in C^i(X)$  be an eigenfunction of  $\Delta$  with non-zero eigenvalue  $c \neq 0$ . Then, by Lemma 2.5 (i),  $f = \delta g$  for some  $g \in C^{i+1}(X)$ . By (16)

$$\tau_v f = \tau_v \delta g = -\delta_v \tau_v g.$$

Hence, again by Lemma 2.5 (i),  $\tau_v f$  belongs to the subspace of  $C^{i-1}(\text{Lk}(v))$  spanned by the eigenfunctions of  $\Delta_v$  with positive eigenvalues. As in the proof of Lemma 3.7, this implies

$$(\Delta_v \tau_v f, \tau_v f)_v \geq m^{i-1}(\text{Lk}(v)) (\tau_v f, \tau_v f)_v \geq \lambda_{\min}^{i-1}(X) (\tau_v f, \tau_v f)_v.$$

This inequality, as in the proof of Theorem 3.8, implies (20). Combining (20) with (19) we get

$$ic(f, f) = i(\Delta f, f) \geq ((i + 1) \cdot \lambda_{\min}^{i-1}(X) - (n - i))(f, f),$$

which implies the second inequality of the theorem. The first inequality can be proven by a similar argument. □

*Remark 3.11* Note that in Theorem 3.10 we do not assume  $\tilde{H}^{i-1}(\text{Lk}(v)) = 0$ . Also, since  $\Delta$  is not the zero operator, we must have  $M^i(X) > 0$  for  $i \leq n - 1$ , which implies  $\lambda_{\max}^{i-1}(X) > (n - i)/(i + 1)$ .

**Notation 3.12** For  $m \geq 1$ , let  $I_m$  denote the  $m \times m$  identity matrix and let  $J_m$  denote the  $m \times m$  matrix whose entries are all equal to 1. The minimal polynomial of  $J_m$  is  $x(x - m)$ .

*Example 3.13* Let  $X$  be an  $n$ -simplex. We claim that the eigenvalues of  $\Delta$  acting on  $C^i(X)$  are 0 and  $n + 1$  for any  $0 \leq i \leq n - 1$ . It is easy to see that 0 is an eigenvalue, so we need to show that the only non-zero eigenvalue of  $\Delta$  is  $n + 1$ . It is enough to show that  $m^i(X) = M^i(X) = n + 1$ . First, suppose  $i = 0$ . Since for any simplex of  $X$  there is a unique  $n$ -simplex containing it, one easily checks that  $\Delta$  acts on  $C^0(X)$  as the matrix  $(n + 1)I_{n+1} - J_{n+1}$ . The only eigenvalues of this matrix are 0 and  $n + 1$ . Now let  $i \geq 1$ . The link of any vertex is an  $(n - 1)$ -simplex, so by induction  $\lambda_{\min}^{i-1}(X) = \lambda_{\max}^{i-1}(X) = n$ . Theorem 3.10 implies

$$i \cdot M^i(X) \leq (i + 1)n - (n - i) = i(n + 1)$$

and

$$i \cdot m^i(X) \geq (i + 1)n - (n - i) = i(n + 1).$$

Hence  $n + 1 \leq m^i(X) \leq M^i(X) \leq n + 1$ , which implies the claim. (Of course, for this simple example it is possible, but not completely trivial, to compute the eigenvalues of  $\Delta$  directly.)

**Notation 3.14** Let  $0 \leq j \leq n - 1$ . Given  $s \in \widehat{S}_j(X)$ , its link  $Lk(s)$  in  $X$  has dimension  $n - (j + 1)$  and satisfies  $(\star)$ . For  $0 \leq i \leq n - (j + 1)$ , denote

$$\lambda_{\max}^{i,j}(X) = \max_{s \in \widehat{S}_j(X)} M^i(Lk(s)),$$

$$\lambda_{\min}^{i,j}(X) = \min_{s \in \widehat{S}_j(X)} m^i(Lk(s)).$$

With our earlier notation, we have  $\lambda_{\max}^{i,0}(X) = \lambda_{\max}^i(X)$  and  $\lambda_{\min}^{i,0}(X) = \lambda_{\min}^i(X)$ .

**Corollary 3.15** Let  $0 \leq j < i \leq n - 1$ . We have

$$(i - j) \cdot M^i(X) \leq (i + 1) \cdot \lambda_{\max}^{i-(j+1),j}(X) - (j + 1)(n - i).$$

and

$$(i - j) \cdot m^i(X) \geq (i + 1) \cdot \lambda_{\min}^{i-(j+1),j}(X) - (j + 1)(n - i).$$

*Proof* We will prove the second inequality. The first inequality can be proven by a similar argument.

If  $j = 0$ , then the claim is just Theorem 3.10. Now, given  $j \geq 1$ , assume that we proved the inequality for  $j - 1$ :

$$(i - (j - 1)) \cdot m^i(X) \geq (i + 1) \cdot \lambda_{\min}^{i-j,j-1}(X) - j(n - i). \tag{21}$$

Let  $s \in \widehat{S}_{j-1}(X)$ . The dimension of  $Lk(s)$  is  $n - j$ , so by Theorem 3.10 we have

$$(i - j) m^{i-j}(Lk(s)) \geq (i - j + 1) \lambda_{\min}^{i-j-1,0}(Lk(s)) - ((n - j) - (i - j)).$$

On the other hand, the link of a vertex  $v \in Lk(s)$  in  $Lk(s)$  is the same as the link of the  $j$ -simplex  $[v, s]$  in  $X$ . Thus,

$$\lambda_{\min}^{i-j-1,0}(Lk(s)) \geq \lambda_{\min}^{i-j-1,j}(X),$$

and

$$(i - j) m^{i-j}(Lk(s)) \geq (i - j + 1) \lambda_{\min}^{i-j-1,j}(X) - (n - i).$$

Taking the minimum over all  $s \in \widehat{S}_{j-1}(X)$ , we get

$$(i - j) \lambda_{\min}^{i-j,j-1}(X) \geq (i - j + 1) \lambda_{\min}^{i-j-1,j}(X) - (n - i). \tag{22}$$

Substituting (22) into (21), gives the desired inequality for  $j$ . □

The argument in the previous proof can be easily adapted to prove the following inequality:

$$(i + 1)\lambda_{\min}^{i-1,0}(X) - (n - i) \geq \frac{i}{i - j} \left( (i + 1)\lambda_{\min}^{i-j-1,j}(X) - (j + 1)(n - i) \right).$$

Now substituting this inequality into the second inequality of Theorem 3.8, we get:

**Corollary 3.16** *Assume  $0 \leq j < i \leq n - 1$ . If  $\tilde{H}^{i-1}(\text{Lk}(v)) = 0$  for every  $v \in \text{Ver}(X)$ , then for  $f \in \mathcal{Z}^i(X)$  we have*

$$(i - j) \cdot (\Delta f, f) \geq \left( (i + 1) \cdot \lambda_{\min}^{i-j-1,j}(X) - (j + 1)(n - i) \right) (f, f).$$

*In particular, if  $\lambda_{\min}^{i-j-1,j}(X) > (j + 1)(n - i)/(i + 1)$ , then  $\mathcal{H}^i(X) = 0$ .*

*Remark 3.17* Using (22), it is easy to check that  $\lambda_{\min}^{i-j-1,j}(X) > (j + 1)(n - i)/(i + 1)$  implies  $\lambda_{\min}^{i-k-1,k}(X) > (k + 1)(n - i)/(i + 1)$  for all  $0 \leq k \leq j$ . Hence the strongest assumption for Corollary 3.16 is  $\lambda_{\min}^{0,i-1}(X) > i(n - i)/(i + 1)$ . The advantage in trying to prove this last inequality, besides the fact that it implies all the others, is that the question about the vanishing of  $H^i(X)$  reduces to estimating the minimal non-zero eigenvalues of Laplacians on graphs.

For the purposes of proving Theorem 1.1 we will need a variant of Corollary 3.9. Let  $X$  be an  $n$ -dimensional simplicial complex satisfying  $(\star)$  but which is not necessarily finite. Let  $\Gamma$  be a group acting on  $X$ . This means that  $\Gamma$  acts on the vertices of  $X$  and preserves the simplicial structure of  $X$ , i.e., whenever the vertices  $\{v_0, \dots, v_i\}$  of  $X$  form an  $i$ -simplex,  $0 \leq i \leq \dim X$ , then for any  $\gamma \in \Gamma$  the vertices  $\{\gamma v_0, \dots, \gamma v_i\}$  also form an  $i$ -simplex. Consider the following condition on the action of  $\Gamma$ :

$$\text{St}(v) \cap \text{St}(\gamma v) = \emptyset \quad \text{for any } v \in \text{Ver}(X) \text{ and any } 1 \neq \gamma \in \Gamma. \tag{\dagger}$$

In particular, this implies that the stabilizer of any simplex is trivial.

**Definition 3.18** Let  $X/\Gamma$  be the simplicial complex whose vertices  $\text{Ver}(X/\Gamma)$  are the orbits  $\text{Ver}(X)/\Gamma$  and a subset  $\{\tilde{v}_0, \dots, \tilde{v}_i\}$  of  $\text{Ver}(X/\Gamma)$  forms an  $i$ -simplex,  $0 \leq i \leq \dim X$ , if we can choose a representative  $v_j \in \text{Ver}(X)$  from the orbit  $\tilde{v}_j$  for each  $0 \leq j \leq i$  so that that  $\{v_0, \dots, v_i\}$  form an  $i$ -simplex in  $X$ .

It is obvious that  $X/\Gamma$  is a simplicial complex. Moreover, if  $(\dagger)$  holds then  $S_i(X/\Gamma)$  is in bijection with the orbits  $S_i(X)/\Gamma$  for any  $i$ . Indeed, as is easy to check,  $(\dagger)$  implies that if  $\{v_0, \dots, v_i\}$  and  $\{\gamma_0 v_0, \dots, \gamma_i v_i\}$  are in  $\widehat{S}_i(X)$  for some  $\gamma_0, \dots, \gamma_i \in \Gamma$ , then  $\gamma_0 = \dots = \gamma_i$ .

**Corollary 3.19** *Let  $\Gamma$  be a group acting on  $X$  so that  $(\dagger)$  is satisfied. Assume  $X/\Gamma$  is finite and  $i \geq 1$ . If  $\tilde{H}^{i-1}(\text{Lk}(v)) = 0$  for every  $v \in \text{Ver}(X)$  and  $\lambda_{\min}^{i-1}(X) > (n - i)/(i + 1)$ , then  $H^i(X/\Gamma) = 0$ .*



*Proof*  $X/\Gamma$  is a finite  $n$ -dimensional complex satisfying  $(\star)$ . Let  $v \in \text{Ver}(X)$  and let  $\tilde{v}$  be the image of  $v$  in  $X/\Gamma$ . Due to  $(\dagger)$ ,  $\text{Lk}(v) \cong \text{Lk}(\tilde{v})$ . Hence  $\tilde{H}^{i-1}(\text{Lk}(\tilde{v})) = 0$  for every  $\tilde{v} \in \text{Ver}(X/\Gamma)$  and  $\lambda_{\min}^{i-1}(X/\Gamma) > (n-i)/(i+1)$ . By Corollary 3.9, we have  $\mathcal{H}^i(X/\Gamma) = 0$ . Therefore, by Theorem 2.3,  $H^i(X/\Gamma) = 0$ .  $\square$

### 4 The Bruhat–Tits building of $SL_n(K)$

In this section we describe the Bruhat–Tits building of  $SL_n$ , and the links of its vertices. Then, assuming a certain lower bound on the minimal non-zero eigenvalue of the curvature transformation acting on the links, we prove Theorem 1.1. This is an important application of the ideas developed in the previous section. (The required lower bound on the minimal non-zero eigenvalue will be proven in Sect. 5.) We conclude the section with a brief discussion of some applications of Garland’s method to produce examples of groups having the so-called property (T).

#### 4.1 The building

Let  $n \geq 0$  be a non-negative integer. Let  $K$  be a complete discrete valuation field. Let  $\mathcal{O}$  be the ring of integers in  $K$ ,  $\pi$  be a uniformizer,  $\mathcal{O}/\pi\mathcal{O} \cong \mathbb{F}_q$ , where  $\mathbb{F}_q$  denotes the finite field with  $q$  elements,  $k$  be the residue field, and  $q$  be the order of  $k$ . Let  $\mathcal{V}$  be an  $(n+2)$ -dimensional vector space over  $K$ . A subset  $L \subset \mathcal{V}$  which has a structure of a free  $\mathcal{O}$ -module of rank  $n+2$  such that  $L \otimes_{\mathcal{O}} K = \mathcal{V}$  is called a *lattice*. It is clear that if  $L$  is a lattice then  $xL = \{x \cdot \ell : \ell \in L\}$ ,  $x \in K^\times$ , is also a lattice. We say that  $L$  and  $xL$  are *similar*. Similarity defines an equivalence relation on the set of lattices in  $\mathcal{V}$ . We denote the equivalence class of  $L$  by  $[L]$ .

The *Bruhat–Tits building* of  $SL_{n+2}(K)$  is the simplicial complex  $\mathfrak{B}_n$  with set of vertices  $\{[L] : L \text{ is a lattice in } \mathcal{V}\}$ , where  $\{[L_0], \dots, [L_i]\}$  form an  $i$ -simplex if there is  $L'_j \in [L_j]$  for each  $j$  with

$$\pi L'_i \subsetneq L'_0 \subsetneq L'_1 \subsetneq \dots \subsetneq L'_i.$$

To visualize  $\mathfrak{B}_n$  in some way, fix a basis  $\{e_1, \dots, e_{n+2}\}$  of  $\mathcal{V}$ . It is easy to see that the classes of lattices

$$\mathcal{O}\pi^{a_1}e_1 \oplus \dots \oplus \mathcal{O}\pi^{a_{n+2}}e_{n+2}, \quad a_1, \dots, a_{n+2} \in \mathbb{Z},$$

are in bijection with the elements of  $\mathbb{Z}^{n+2}/\mathbb{Z} \cdot (1, 1, \dots, 1)$ . In particular,  $\mathfrak{B}_n$  is infinite. Next, the vertices corresponding to  $(a_1, \dots, a_{n+2})$  and  $(b_1, \dots, b_{n+2})$  are adjacent in  $\mathfrak{B}_n$  if and only if modulo  $\mathbb{Z} \cdot (1, 1, \dots, 1)$  we have  $a_i \leq b_i \leq a_i + 1$  for all  $i$ . For example, when  $n = 0$ , these vertices form an infinite line as in Fig. 1. When  $n = 1$ , these vertices give a triangulation of  $\mathbb{R}^2$  part of which looks like Fig. 2. It is important to stress that the vertices that we considered above do not give all vertices of  $\mathfrak{B}_n$ , but only of a part of the building, called an *apartment*. For example, it is not hard to see that  $\mathfrak{B}_0$  is an infinite tree in which every vertex is adjacent to exactly  $q + 1$  other vertices. Similarly,  $\mathfrak{B}_n$  is very symmetric in the sense that the simplicial complexes

$$----- (0,2) \text{ --- } (0,1) \text{ --- } (0,0) \text{ --- } (1,0) \text{ --- } (2,0) \text{ -----}$$

Fig. 1

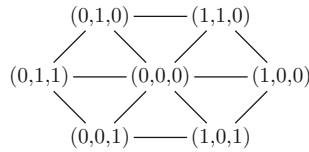


Fig. 2

$St(v)$ ,  $v \in Ver(\mathfrak{B}_n)$ , are all isomorphic to each other. To see what this complex is take a lattice  $\mathcal{L}$  corresponding to  $v$ . Then  $\mathcal{L}/\pi\mathcal{L} = \mathbb{F}_q^{n+2} = V$ , and the vertices of  $St(v)$  are in one-to-one correspondence with the positive dimensional linear subspaces of  $V$  ( $v$  itself corresponds to  $V$ ). Let  $\{V_0, \dots, V_i\}$  be the linear subspaces corresponding to the vertices  $\{v_0, \dots, v_i\}$  of  $St(v)$ . Then  $\{v_0, \dots, v_i\}$  form an  $i$ -simplex if and only if the linear subspaces  $\{V_0, \dots, V_i\}$  fit into an ascending sequence (possibly after reindexing):

$$\mathcal{F}: V_0 \subset V_1 \subset \dots \subset V_i.$$

The  $i$ -simplices of  $Lk(v)$  correspond to those  $\mathcal{F}$  for which  $V_i \neq V$ . Next, we consider more carefully  $Lk(v)$  as a simplicial complex.

### 4.2 Complexes of flags

Fix some  $n \geq 0$ . Let  $V$  be a linear space of dimension  $n + 2$  over the finite field  $\mathbb{F}_q$ . A flag in  $V$  is an ascending sequence

$$\mathcal{F}: F_0 \subset F_1 \subset \dots \subset F_i$$

of distinct linear subspaces  $F_0, \dots, F_i$  of  $V$  such that  $F_0 \neq 0$  and  $F_i \neq V$ . The length of  $\mathcal{F}$  is  $i$ . We will refer to a flag of length  $i$  as  $i$ -flag. In particular, the 0-flags are simply the proper non-zero linear subspaces of  $V$ . Given two flags

$$\mathcal{F}: F_0 \subset F_1 \subset \dots \subset F_i \quad \text{and} \quad \mathcal{G}: G_0 \subset G_1 \subset \dots \subset G_j,$$

we say that  $\mathcal{G}$  refines  $\mathcal{F}$ , and write  $\mathcal{F} \prec \mathcal{G}$ , if for every  $0 \leq k \leq i$  there is  $0 \leq t \leq j$  such that  $F_k = G_t$ . It is convenient also to have the empty flag  $\emptyset$ , which is the empty sequence of linear subspaces; we put  $-1$  for the length of  $\emptyset$ . The refinement defines a partial ordering on the set of flags in  $V$ ; the empty flag is refined by every other flag.

Let  $\mathcal{F}$  be a fixed flag of length  $\ell$ . Consider the following simplicial complex  $X_{\mathcal{F}}$  (when  $\mathcal{F} = \emptyset$  we will also denote this complex by  $X_{\emptyset}^n$ ). The vertices of  $X_{\mathcal{F}}$  are the  $(\ell + 1)$ -flags refining  $\mathcal{F}$ . The vertices  $v_0, \dots, v_h$  form an  $h$ -simplex if the corresponding flags are all refined by a single  $(\ell + h + 1)$ -flag. It is easy to see that  $X_{\mathcal{F}}$  is indeed a finite simplicial complex of dimension  $n - 1 - \ell$ . Since any flag can be refined into

an  $n$ -flag,  $X_{\mathcal{F}}$  satisfies  $(\star)$ . Note that the link of a vertex of the Bruhat–Tits building  $\mathfrak{B}_n$  is isomorphic to  $X_{\emptyset}^n$ .

Now assume  $\mathcal{F} \neq \emptyset$ . Let  $\mathcal{F}: F_0 \subset F_1 \subset \dots \subset F_\ell$ , with  $\ell \geq 0$ . Consider the array of integers  $(t_0, t_1, \dots, t_{\ell+1})$  defined by

$$t_j = \begin{cases} \dim F_0 & \text{if } j = 0; \\ \dim F_j - \dim F_{j-1} & \text{if } 1 \leq j \leq \ell; \\ \dim V - \dim F_\ell & \text{if } j = \ell + 1. \end{cases} \tag{23}$$

It is not hard to see that

$$X_{\mathcal{F}} \cong X_{\emptyset}^{t_0-2} * X_{\emptyset}^{t_1-2} * \dots * X_{\emptyset}^{t_{\ell+1}-2}, \tag{24}$$

where  $X_{\emptyset}^{-1}$  denotes the empty complex.

**Lemma 4.1** *If  $\dim X_{\mathcal{F}} > 0$  then  $X_{\mathcal{F}}$  is connected.*

*Proof* First we show that  $X_{\emptyset}^n$  is connected if  $n \geq 1$ . Let  $x$  and  $y$  be two vertices of  $X_{\emptyset}^n$ , and let  $W_1$  and  $W_2$  be the corresponding subspaces of  $V$ . Choose a 1-dimensional subspace  $L_i$  of  $W_i$ ,  $i = 1, 2$ . Consider the subspace  $P = L_1 + L_2$  of  $V$ . Since  $n \geq 1$ ,  $P \neq V$ , so  $P$  gives a vertex of  $X_{\emptyset}^n$ , which we denote by the same letter. The vertex  $P$  is adjacent to both  $L_1$  and  $L_2$ ,  $L_1$  is adjacent to  $x$ , and  $L_2$  is adjacent to  $y$ , so there is a path from  $x$  to  $y$ .

Now assume  $\mathcal{F} \neq \emptyset$  and  $X_{\mathcal{F}}$  is given by (24). If  $\dim X_{\mathcal{F}} \geq 1$ , then either at least two of  $X_{\emptyset}^{t_j-2}$  are non-empty or at least one  $X_{\emptyset}^{t_j-2}$  has dimension 1. In either case  $X_{\mathcal{F}}$  is clearly connected.  $\square$

**Theorem 4.2** *Assume  $N = \dim X_{\mathcal{F}} \geq 1$ . Then for any  $0 \leq i \leq N - 1$  and  $\varepsilon > 0$  there is a constant  $q(\varepsilon, n)$  depending only on  $\varepsilon$  and  $n$  such that  $m^i(X_{\mathcal{F}}) \geq N - i - \varepsilon$  once  $q > q(\varepsilon, n)$ .*

The proof of this theorem is quite complicated and will be given in Sect. 5.

**Corollary 4.3** *There is a constant  $q(n)$  depending only of  $n$  such that if  $q > q(n)$ , then  $\tilde{H}^i(X_{\mathcal{F}}) = 0$  for all  $0 \leq i \leq N - 1$ .*

*Proof* We use induction on  $N$  and  $i$ . When  $N = 1$  or  $i = 0$ , the claim follows from Lemma 4.1, since  $\tilde{H}^0(X_{\mathcal{F}}) = 0$  is equivalent to  $X_{\mathcal{F}}$  being connected. Now assume  $N > 1$  and  $i \geq 1$ . For any  $v \in \text{Ver}(X_{\mathcal{F}})$ , we have  $\text{Lk}(v) \cong X_{\mathcal{G}}$  for some  $\mathcal{G} \succ \mathcal{F}$ . Since  $\dim X_{\mathcal{G}} = N - 1$ , by the induction assumption  $\tilde{H}^{i-1}(X_{\mathcal{G}}) = 0$  for  $q$  large enough. Next, choosing  $\varepsilon$  small enough in Theorem 4.2, we can make

$$\lambda_{\min}^{i-1}(X_{\mathcal{F}}) > \frac{N - i}{i + 1}.$$

Now the assumptions of Corollary 3.9 are satisfied, so  $\tilde{H}^i(X_{\mathcal{F}}) = 0$ .  $\square$

### 4.3 Main theorem

Since  $\text{Lk}(v)$  is isomorphic to the simplicial complex  $X_{\emptyset}^n$ , the complex  $\mathfrak{B}_n$  is  $(n + 1)$ -dimensional and satisfies  $(\star)$ .

**Theorem 4.4** *Let  $\Gamma$  be a group acting on  $\mathfrak{B}_n$  so that  $(\dagger)$  is satisfied. Assume  $\mathfrak{B}_n/\Gamma$  is finite. There is a constant  $q(n)$  depending only on  $n$  such that if  $q > q(n)$  then  $H^i(\mathfrak{B}_n/\Gamma) = 0$  for  $1 \leq i \leq n$ .*

*Proof* Since  $\text{Lk}(v) \cong X_{\emptyset}^n$  for any  $v \in \text{Ver}(\mathfrak{B}_n)$ , Theorem 4.2 and Corollary 4.3 imply that there is a constant  $q(n)$  depending only on  $n$  such that if  $q > q(n)$  then for any  $1 \leq i \leq n$  we have  $\tilde{H}^{i-1}(\text{Lk}(v)) = 0$  and  $\lambda_{\min}^{i-1}(\mathfrak{B}_n) = m^{i-1}(X_{\emptyset}^n) > (n + 1 - i)/(i + 1)$ . Now the claim follows from Corollary 3.19.  $\square$

*Remark 4.5* It is known that the cohomology groups  $\tilde{H}^i(X_{\mathcal{F}})$  vanish for  $0 \leq i \leq N - 1$ , without any assumptions on  $q$ , by a general result of Solomon and Tits; see [8, Appendix II] for a proof. Assuming this result, to prove Theorem 4.4 we only need the bound  $m^i(X_{\emptyset}^n) \geq n - i - \varepsilon$ . On the other hand, the proof of Theorem 4.2 is inductive, and requires proving this bound for all  $X_{\mathcal{F}}$ . Another observation is that to prove Theorem 4.4 it is enough to prove  $m^0(X_{\mathcal{F}}) \geq N - \varepsilon$  for all  $\mathcal{F}$ . Indeed, the link of any simplex in  $\mathfrak{B}_n$  is isomorphic to some  $X_{\mathcal{F}}$ , so one can appeal to Corollary 3.16 to get the vanishing of the cohomology.

*Remark 4.6* In Sect. 5.1 we will compute that  $m^0(X_{\emptyset}^1) > 1/2$  and  $m^0(X_{\emptyset}^2) > 1$ . Hence for  $n = 1, 2$  Garland’s method proves the vanishing of  $H^1(\mathfrak{B}_n/\Gamma)$  for all  $q$ . On the other hand,  $m^1(X_{\emptyset}^2) = 1/3$  when  $q = 2$ . To apply Corollary 3.19 to show that  $H^2(\mathfrak{B}_2/\Gamma) = 0$  we need  $\lambda_{\min}^1(\mathfrak{B}_2/\Gamma) > 1/3$ , so we need to assume  $q > 2$ .

There is an abundance of groups  $\Gamma$  satisfying  $(\dagger)$ . The most important examples of such groups come from arithmetic. One possible construction proceeds as follows. Let  $F = \mathbb{F}_q(T)$  be the field of rational functions in indeterminate  $T$  with  $\mathbb{F}_q$  coefficients. Fix a place  $\infty = 1/T$  of  $F$ . Let  $A = \mathbb{F}_q[T]$  be the polynomial ring. Let  $K = \mathbb{F}_q((1/T))$  be the completion of  $F$  at  $\infty$ . Let  $D$  be a central division algebra over  $F$  of dimension  $(n + 2)^2$ . Assume  $D$  is split at  $\infty$ , i.e.,  $D \otimes_F K \cong \text{Mat}_{n+2}(K)$ . Let  $\mathcal{D}$  be a maximal  $A$ -order in  $D$ ; see [17] for the definitions. Let  $\mathcal{D}^\times$  be the group of multiplicative units in  $\mathcal{D}$ . The quotient  $\mathcal{D}^\times/\mathbb{F}_q^\times$  can be identified with a discrete, cocompact subgroup of  $\text{PGL}_{n+2}(K)$ . Replacing  $\mathcal{D}^\times/\mathbb{F}_q^\times$  by a subgroup  $\Gamma \subset \mathcal{D}^\times/\mathbb{F}_q^\times$  of finite index if necessary, we get a group which naturally acts on  $\mathfrak{B}_n$  and satisfies  $(\dagger)$ . Moreover, the quotient  $\mathfrak{B}_n/\Gamma$  is finite. For these facts we refer to [9, p. 140], [10, 12, 19]. Theorem 4.4 implies  $H^i(\mathfrak{B}_n/\Gamma) = 0$  for all  $1 \leq i \leq n$ . On the contrary,  $H^{n+1}(\mathfrak{B}_n/\Gamma)$  is usually quite large. Its dimension approximately equals the volume of  $\text{PGL}_{n+2}(K)/\Gamma$  with respect to an appropriately normalized Haar measure on  $\text{PGL}_{n+2}(K)$ ; see [19]. The simplicial complexes  $\mathfrak{B}_n/\Gamma$  are often used in the construction of Ramanujan complexes; see [10, 12].

#### 4.4 Property (T)

Garland's method has been applied to prove that certain groups have Kazhdan's property (T).

Let  $\Gamma$  be a group generated by a finite set  $S$ . Let  $\pi : \Gamma \rightarrow U(H_\pi)$  be a unitary representation. We say that  $\pi$  almost has invariant vectors if for every  $\varepsilon > 0$  there exists a non-zero vector  $u_\varepsilon$  in the Hilbert space  $H_\pi$  such that  $\|\pi(s)u_\varepsilon - u_\varepsilon\| \leq \varepsilon\|u_\varepsilon\|$  for every  $s \in S$ . The group  $\Gamma$  is said to have *property (T)* if every unitary representation of  $\Gamma$  which almost has invariant vectors has a non-zero invariant vector.

Property (T) has important applications to representation theory, ergodic theory, geometric group theory and the theory of networks. For example, Margulis used groups with property (T) to give the first explicit examples of expanding graphs and to solve the Banach–Ruziewicz problem that asks whether the Lebesgue measure is the only normalized rotationally invariant finitely additive measure on the  $n$ -dimensional sphere. We refer to Lubotzky's book [11] for a discussion of property (T) and its applications.

It is known that a group  $\Gamma$  has property (T) if and only if for any unitary representation  $\pi$  of  $\Gamma$ , the first cohomology group  $H^1(\Gamma, \pi)$  is zero. This suggests the following line of attack to prove that  $\Gamma$  has property (T). Suppose that  $\Gamma$  is the fundamental group of a finite simplicial complex  $X$ . By group cohomology,  $H^1(\Gamma, \pi) = H^1(X, E_\pi)$ , where  $E_\pi$  is a local system on  $X$  associated to  $\pi$ . Then one can try to prove the vanishing of  $H^1(X, E_\pi)$  by a generalization of Garland's method. This approach in the case when  $X$  is a 2-dimensional finite simplicial complex was pursued independently by Ballmann and Świątkowski [2], Pansu [15], and Żuk [20]. For example, in [2], the authors prove the following theorem: Assume  $X$  is a 2-dimensional finite simplicial complex,  $\text{Lk}(v)$  is a connected graph for any vertex  $v$  of  $X$ , and  $\lambda_{\min}^0(X) > 1/2$ . Then  $\Gamma = \pi_1(X)$  has property (T). Note that these assumptions are the same as in Corollary 3.9 for  $n = 2$ . They are fulfilled when  $X$  is a finite quotient of a 2-dimensional Bruhat–Tits building. These results gave new explicit examples of groups with property (T) which were significantly different from the earlier known examples. In [5, 6], Dymara and Januszkiewicz applied a generalization of Garland's method to groups acting on buildings of arbitrary type and dimension (e.g. hyperbolic buildings), and produced examples of groups having property (T), not coming from locally symmetric spaces or Euclidean buildings.

### 5 Complexes of flags

The main goal of this section is to prove Theorem 4.2. The notation will be the same as in Sect. 4.2. In particular,  $V$  is a linear space of dimension  $n + 2$  over the finite field  $\mathbb{F}_q$ , and  $\mathcal{F}$  is a (possibly empty) flag in  $V$  of length  $\ell$ . We denote

$$N = \dim X_{\mathcal{F}}.$$

The proof of Theorem 4.2 proceeds by induction on  $N$  and  $i$ . The base case  $N = 1$  follows from a direct calculation. We will carry out this calculation in Sect. 5.1. In

the same subsection we give some explicit examples which provide a sense of the complexity of the eigenvalues of  $\Delta$  acting on  $C^i(X_{\mathcal{F}})$ . These examples suggest a remarkable asymptotic behaviour of the eigenvalues of  $\Delta$  as  $q \rightarrow \infty$ , which we state as a conjecture.

The inductive step, discussed in Sect. 5.2, has two parts. Assuming the claim holds for  $i = 0$  and all  $N$ , the proof of the general case quickly follows from the inequality in Theorem 3.10. On the other hand, the argument which proves the claim for  $i = 0$  and  $N \geq 1$  is fairly intricate. The outline is approximately the following: We start with a  $\Delta$ -eigenfunction  $f \in C^0(X_{\mathcal{F}})$  having eigenvalue  $c > 0$ . The machinery developed in Sect. 3.2 cannot be applied to this function, since we cannot apply the operator  $\tau_v$  directly to  $f$ . Instead, we introduce a parameter  $R \in \mathbb{R}$ , and multiply the values of  $f$  on an appropriate subset of  $\text{Ver}(X_{\mathcal{F}})$  by  $R$ . The resulting function  $f_{\alpha}$  is no longer an eigenfunction of  $\Delta$ , but we get some flexibility because we can vary  $R$ . We apply the machinery of Sect. 3.2 to  $df_{\alpha} \in C^1(X_{\mathcal{F}})$ . Choosing  $R$  appropriately forces some miraculous cancellations, which in the end give the desired bound  $c \geq N - \varepsilon$ .

In Sect. 5.3, we prove some auxiliary results about the eigenvalues of curvature transformations. These results are not used elsewhere in the paper, and are given as some evidence for the conjecture in Sect. 5.1.

### 5.1 The base case and explicit examples

For  $N = 1$  we need to consider only  $\Delta$  acting on  $C^0(X_{\mathcal{F}})$ , since  $0 \leq i \leq N - 1$ .

**Lemma 5.1** *If  $N = 1$ , then  $m^0(X_{\mathcal{F}})$  is equal either to 1 or  $1 - \sqrt{q}/(q + 1)$ .*

*Proof* If  $\dim X_{\mathcal{F}} = 1$  then the length of  $\mathcal{F}$  is  $\ell = n - 2$ . Let  $(t_0, \dots, t_{n-1})$  be defined by (23). Since  $t_i \geq 1$  and  $\sum_{i=0}^{n-1} (t_i - 1) = 2$ , either exactly two  $t_i, t_j, i < j$ , are equal to 2 and all others are 1, or exactly one  $t_i$  is equal to 3 and all others are 1. In the first case  $X_{\mathcal{F}} \cong X_{\emptyset}^0 * X_{\emptyset}^0$ , in the second case  $X_{\mathcal{F}} \cong X_{\emptyset}^1$ .

In the first case  $X_{\mathcal{F}}$  is a  $(q + 1)$ -regular bipartite graph with  $2(q + 1)$  vertices. It is easy to check that  $(q + 1)\Delta$  acts on  $C^0(X_{\mathcal{F}})$  as the matrix

$$(q + 1)I_{2(q+1)} - \begin{pmatrix} 0 & J_{q+1} \\ J_{q+1} & 0 \end{pmatrix}.$$

The minimal polynomial of this matrix is  $x(x - (q + 1))(x - 2(q + 1))$ , so the eigenvalues of  $\Delta$  are 0, 1, and 2.

In the second case,  $X_{\mathcal{F}}$  is isomorphic to the graph whose vertices correspond to 1 and 2-dimensional subspaces of a 3-dimensional vector space  $V$  over  $\mathbb{F}_q$ , two vertices being adjacent if one of the corresponding subspaces is contained in the other. With a slight abuse of terminology, we will call 1 and 2 dimensional subspaces lines and planes, respectively. The number of lines and planes in  $V$  is  $m = q^2 + q + 1$  each. Let  $A = (a_{ij})$  be the  $m \times m$  matrix whose rows are enumerated by the lines in  $V$  and columns by the planes, and  $a_{ij} = -1$  if the  $i$ -th line lies in the  $j$ -th plane, and is 0 otherwise. We can choose a basis of  $C^0(X_{\mathcal{F}})$  so that  $(q + 1)\Delta$  acts as the matrix

$$(q + 1)I_{2m} + \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix},$$

where  $A^t$  denotes the transpose of  $A$ . Let  $M = \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$ . Since any two distinct lines lie in a unique plane and any line lies in  $q + 1$  planes,  $AA^t = qI_m + J_m$ . By a similar argument,  $A^tA = qI_m + J_m$ . Hence

$$M^2 = qI_{2m} + \begin{pmatrix} J_m & 0 \\ 0 & J_m \end{pmatrix}.$$

This implies that  $(M^2 - qI_{2m})(M^2 - (q + 1)^2I_{2m}) = 0$ . Since  $(q + 1)\Delta - (q + 1)I_{2m} = M$ , we conclude that  $(q + 1)\Delta$  satisfies the polynomial equation

$$x(x - (2q + 2))(x^2 - (2q + 2)x + (q^2 + q + 1)) = 0.$$

It is not hard to see that this is in fact the minimal polynomial of  $(q + 1)\Delta$ . Hence the eigenvalues of  $\Delta$  are 0, 2, and  $1 \pm \sqrt{q}/(q + 1)$ . □

Denote by  $\min.\text{pol}_n^i(x)$  the minimal polynomial of  $\Delta$  acting on  $C^i(X_\emptyset^n)$ . The proof of Lemma 5.1 shows that

$$\min.\text{pol}_1^0(x) = x(x - 2)\left(x^2 - 2x + \frac{q^2 + q + 1}{q^2 + 2q + 1}\right).$$

Note that

$$m^0(X_\emptyset^1) = 1 - \frac{\sqrt{q}}{q + 1}$$

is always strictly larger than  $1/2$  and tends to 1 as  $q \rightarrow \infty$ . Moreover, the whole polynomial tends coefficientwise to the polynomial  $x(x - 2)(x - 1)^2$ .

Now assume  $n = 2$ . In this case it is considerably harder to compute the minimal polynomials. With the help of a computer, we deduced that

$$\begin{aligned} \min.\text{pol}_2^0(x) &= x(x - 2)(x - 3)\left(x - \frac{2q^2 + 3q + 2}{q^2 + q + 1}\right) \\ &\quad \cdot \left(x^2 - \frac{4q^2 + 3q + 4}{q^2 + q + 1}x + \frac{4q^2 + 4}{q^2 + q + 1}\right). \end{aligned}$$

This implies

$$m^0(X_\emptyset^2) = \frac{1}{2(q^2 + q + 1)} (4q^2 + 3q + 4 - \sqrt{8q^3 + 9q^2 + 8q})$$

is at least 1.08 and tends to 2 from below as  $q \rightarrow \infty$ . The whole polynomial tends coefficientwise to the polynomial  $x(x - 3)(x - 2)^4$  as  $q \rightarrow \infty$ . Next

$$\begin{aligned} \min.\text{pol}_2^1(x) &= x(x - 1)(x - 2)(x - 3) \left( x^2 - 2x + \frac{q^2 + 1}{q^2 + 2q + 1} \right) \\ &\cdot \left( x^2 - 3x + \frac{2q^2 + 2q + 2}{q^2 + 2q + 1} \right) \left( x^2 - 4x + \frac{4q^2 + 6q + 4}{q^2 + 2q + 1} \right). \end{aligned}$$

In this case

$$m^1(X_{\emptyset}^2) = 1 - \frac{\sqrt{2q}}{q + 1}.$$

It is easy to see that  $1/3 \leq m^1(X_{\emptyset}^2) < 1$ . Moreover,  $m^1(X_{\emptyset}^2)$  is strictly larger than  $1/3$  for  $q > 2$  and tends to 1 as  $q \rightarrow \infty$ ; the whole polynomial tends to  $x(x - 3)(x - 2)^4(x - 1)^4$ .

*Conjecture 5.2* The previous examples, combined with some calculations for  $n = 3$  which we do not list, suggest a remarkable property of the eigenvalues of  $\Delta$  acting on  $C^i(X_{\emptyset}^n)$ ,  $0 \leq i \leq n - 1$ :

- (i) The number of distinct eigenvalues of  $\Delta$  depends only on  $i$ , i.e., does not depend on  $q$ , even though the eigenvalues themselves and the dimension of  $C^i(X_{\emptyset}^n)$  depend on  $q$ .
- (ii) The positive eigenvalues of  $\Delta$ , which in general are neither rational nor integral, tend to the integers

$$n - i, \quad n - i + 1, \quad \dots, \quad n + 1$$

as  $q \rightarrow \infty$ .

### 5.2 Inductive step

Since we proved Theorem 4.2 for  $N = 1$ , we assume  $N \geq 2$ . Let  $1 \leq i \leq N - 1$  be given. Assume for the moment that we proved the bound in Theorem 4.2 for  $\Delta$  acting on  $C^{i-1}(X_{\mathcal{G}})$ , where  $\mathcal{G}$  is any flag with  $\dim \mathcal{G} = N - 1$ . Since for any  $v \in \text{Ver}(X_{\mathcal{F}})$  its link  $\text{Lk}(v)$  is isomorphic to  $X_{\mathcal{G}}$  for some  $\mathcal{G} \succ \mathcal{F}$  with  $\dim X_{\mathcal{G}} = N - 1$ , we get

$$\lambda_{\min}^{i-1}(X_{\mathcal{F}}) \geq (N - 1) - (i - 1) - \varepsilon' = N - i - \varepsilon',$$

where  $\varepsilon' = i \cdot \varepsilon / (i + 1)$ . Then, by Theorem 3.10, we have

$$m^i(X_{\mathcal{F}}) \geq \frac{(i + 1) \cdot \lambda_{\min}^{i-1}(X_{\mathcal{F}}) - (N - i)}{i} \geq N - i - \varepsilon. \tag{25}$$



Therefore, to complete the proof of Theorem 4.2 it remains to show that

$$m^0(X_{\mathcal{F}}) \geq N - \varepsilon. \tag{26}$$

This will occupy the rest of this subsection.

*Remark 5.3* Instead of induction, one can deduce the lower bound (25) directly from (26) using Corollary 3.15. Indeed, the link of any  $(i - 1)$ -simplex in  $X_{\mathcal{F}}$  is isomorphic to some  $X_{\mathcal{G}}$  with  $\dim X_{\mathcal{G}} = N - i$ , so assuming  $m^0(X_{\mathcal{G}}) \geq (N - i) - \varepsilon'$ ,  $\varepsilon' = \varepsilon / (i + 1)$ , Corollary 3.15 gives

$$m^i(X_{\mathcal{F}}) \geq (i + 1)(N - i - \varepsilon') - i(N - i) = N - i - \varepsilon.$$

On the other hand, the proof of Corollary 3.15 uses similar inductive argument as above.

We start by proving some preliminary lemmas. For an integer  $m \geq 1$  we put  $(m)_q = \prod_{k=1}^m (q^k - 1)$ , and we put  $(0)_q = 1$ . The number of  $d$ -dimensional subspaces in an  $m$ -dimensional linear space over  $\mathbb{F}_q$  is equal to

$$\begin{bmatrix} m \\ d \end{bmatrix}_q = \frac{(m)_q}{(d)_q(m - d)_q}.$$

With this notation it is easy to give a formula for the number of  $n$ -flags refining a given flag:

**Lemma 5.4** *Let  $s$  be a simplex in  $X_{\mathcal{F}}$  corresponding to  $\mathcal{G} \succ \mathcal{F}$ . Let  $(r_0, \dots, r_j)$  be the integers defined for  $\mathcal{G}$  by (23). The number of  $N$ -simplices in  $X_{\mathcal{F}}$  containing  $s$  is given by the formula*

$$w(s) = \prod_{k=0}^j \prod_{z=1}^{r_k} \begin{bmatrix} z \\ 1 \end{bmatrix}_q = \prod_{k=0}^j (r_k)_q / (1)_q^{r_k}.$$

Let  $v \in \text{Ver}(X_{\mathcal{F}})$  and let  $\mathcal{G}$  be the corresponding  $(\ell + 1)$ -flag. There is a unique subspace  $G$  in the sequence of  $\mathcal{G}$  which does not occur in  $\mathcal{F}$ . Let

$$\text{Type}(v) = \dim G.$$

Denote the set of types of vertices of  $X_{\mathcal{F}}$  by  $\mathfrak{T}$ . It is easy to see that the vertices of a simplex in  $X_{\mathcal{F}}$  have distinct types. Moreover,  $\#\mathfrak{T} = N + 1$ .

**Lemma 5.5** *Let  $v \in \text{Ver}(X_{\mathcal{F}})$ . Assume  $\alpha \in \mathfrak{T}$  is fixed and  $\alpha \neq \text{Type}(v)$ . Then*

$$\sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ [v,x] \in \widehat{\mathcal{S}}_1(X_{\mathcal{F}}) \\ \text{Type}(x) = \alpha}} w([v, x]) = w(v).$$

*Proof* Let  $\mathcal{G}$  be the flag of length  $i = \ell + 1$  in  $V$  corresponding to  $v$ . Let  $(t_0, \dots, t_{i+1})$  be the array (23) of  $\mathcal{G}$ . Let  $[v, x] \in \widehat{S}_1(X_{\mathcal{F}})$  and  $\mathcal{G}' > \mathcal{G}$  be the corresponding  $(i + 1)$ -flag. There is a unique  $t_a$  such that the array of  $\mathcal{G}'$  is  $(t_0, \dots, t'_a, t''_a, \dots, t_{i+1})$  with  $t'_a + t''_a = t_a$ . Moreover, the type of  $x$  uniquely determines  $a$  and  $t'_a$ . The number of  $[v, x] \in \widehat{S}_1(X_{\mathcal{F}})$  with  $\text{Type}(x) = \alpha$  is equal to  $\left[ \begin{smallmatrix} t_a \\ t'_a \end{smallmatrix} \right]_q$ . Using Lemma 5.4, we compute

$$\sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ [v,x] \in \widehat{S}_1(X_{\mathcal{F}}) \\ \text{Type}(x)=\alpha}} \frac{w([v, x])}{w(v)} = \left[ \begin{smallmatrix} t_a \\ t'_a \end{smallmatrix} \right]_q \frac{(1)_q^{t_a} (t'_a)_q (t''_a)_q}{(t_a)_q (1)_q^{t'_a} (1)_q^{t''_a}} = 1.$$

□

*Remark 5.6* Lemma 5.5 is a refined version of Lemma 3.1. Indeed,

$$\sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ [v,x] \in \widehat{S}_1(X_{\mathcal{F}})}} w([v, x]) = \sum_{\substack{\alpha \in \mathfrak{T} \\ \alpha \neq \text{Type}(v)}} \sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ [v,x] \in \widehat{S}_1(X_{\mathcal{F}}) \\ \text{Type}(x)=\alpha}} w([v, x]) = \sum_{\substack{\alpha \in \mathfrak{T} \\ \alpha \neq \text{Type}(v)}} w(v) = Nw(v).$$

Let  $f \in C^0(X_{\mathcal{F}})$  and let  $R \in \mathbb{R}$  be a fixed constant. For each  $\alpha \in \mathfrak{T}$  define the function  $f_{\alpha} \in C^0(X_{\mathcal{F}})$  by

$$f_{\alpha}(v) = \begin{cases} R \cdot f(v) & \text{if } \text{Type}(v) = \alpha; \\ f(v) & \text{if } \text{Type}(v) \neq \alpha. \end{cases}$$

Also, for  $i \geq 0$  define a linear transformation  $\rho_{\alpha} : C^i(X_{\mathcal{F}}) \rightarrow C^i(X_{\mathcal{F}})$  by

$$\rho_{\alpha} = \sum_{\substack{v \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(v)=\alpha}} \rho_v.$$

**Lemma 5.7** *We have*

$$\sum_{\alpha \in \mathfrak{T}} (1 - \rho_{\alpha})df = (N - 1)df, \tag{27}$$

$$\sum_{\substack{v \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(v)=\alpha}} (\Delta \rho_v df_{\alpha}, \rho_v df_{\alpha}) = (\Delta \rho_{\alpha} df_{\alpha}, \rho_{\alpha} df_{\alpha}), \tag{28}$$

$$(\rho_{\alpha} df_{\alpha}, df_{\alpha}) = (df_{\alpha}, df_{\alpha}) - ((1 - \rho_{\alpha})df, df), \tag{29}$$

$$(\Delta \rho_{\alpha} df_{\alpha}, \rho_{\alpha} df_{\alpha}) = ((1 - \rho_{\alpha})df, df). \tag{30}$$

*Proof* Equation (27) follows from a straightforward calculation:

$$\sum_{\alpha \in \mathfrak{T}} (1 - \rho_{\alpha})df = (N + 1)df - \sum_{v \in \text{Ver}(X_{\mathcal{F}})} \rho_v df = (N + 1)df - 2df = (N - 1)df.$$

To prove (28), expand its right hand-side as

$$(d\rho_\alpha df_\alpha, d\rho_\alpha df_\alpha) = \sum_{\substack{v, v' \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(v) = \text{Type}(v') = \alpha}} (d\rho_v df_\alpha, d\rho_{v'} df_\alpha).$$

Now let  $s = [x, y, z] \in S_2(X_{\mathcal{F}})$ . Since the vertices of the same simplex have distinct types, only one of  $x, y, z$  can be of type  $\alpha$ . Therefore, if  $v \neq v'$  but  $\text{Type}(v) = \text{Type}(v') = \alpha$ , then  $d\rho_v df_\alpha(s) \cdot d\rho_{v'} df_\alpha(s) = 0$ . This implies that in the above sum only the terms with  $v = v'$  are possibly non-zero, so

$$\sum_{\substack{v, v' \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(v) = \text{Type}(v') = \alpha}} (d\rho_v df_\alpha, d\rho_{v'} df_\alpha) = \sum_{\substack{v \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(v) = \alpha}} (d\rho_v df_\alpha, d\rho_v df_\alpha).$$

To prove (29), note that if  $s \in S_1(X_{\mathcal{F}})$  contains a vertex of type  $\alpha$ , then  $(1 - \rho_\alpha)g(s) = 0$  for any  $g \in C^1(X)$ . On the other hand, if  $s$  does not contain a vertex of type  $\alpha$ , then  $(1 - \rho_\alpha)df_\alpha(s) = df_\alpha(s) = df(s) = (1 - \rho_\alpha)df(s)$ . Hence

$$((1 - \rho_\alpha)df_\alpha, df_\alpha) = ((1 - \rho_\alpha)df, df).$$

Now

$$((1 - \rho_\alpha)df, df) = ((1 - \rho_\alpha)df_\alpha, df_\alpha) = (df_\alpha, df_\alpha) - (\rho_\alpha df_\alpha, df_\alpha).$$

Finally, to prove (30), let  $s = [x, y, z] \in S_2(X_{\mathcal{F}})$ . If none of the vertices of  $s$  has type  $\alpha$  then  $d\rho_\alpha df_\alpha(s) = 0$ . If  $s$  has a vertex of type  $\alpha$ , then such a vertex is unique. Without loss of generality, assume  $\text{Type}(x) = \alpha$ . Then

$$d\rho_\alpha df_\alpha([x, y, z]) = f(y) - f(z) = -df([y, z]).$$

Hence

$$\begin{aligned} (d\rho_\alpha df_\alpha, d\rho_\alpha df_\alpha) &= \sum_{\substack{v \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(v) = \alpha}} \sum_{s \in \widehat{S}_1(\text{Lk}(v))} w([v, s]) df(s)^2 \\ &= \sum_{s \in \widehat{S}_1(X_{\mathcal{F}})} (1 - \rho_\alpha)df(s) \cdot df(s) \sum_{\substack{v \in \text{Ver}(\text{Lk}(s)) \\ \text{Type}(v) = \alpha}} w([v, s]) \\ &= ((1 - \rho_\alpha)df, df), \end{aligned}$$

where in the last equality we used Lemma 5.5. □

**Lemma 5.8** *Let  $f \in C^0(X_{\mathcal{F}})$  and suppose  $\Delta f = c \cdot f$ . Then*

$$\sum_{\alpha \in \mathfrak{T}} (\Delta f_\alpha, f_\alpha) = [(N - c)(R - 1)^2 + c(R^2 + N)] \cdot (f, f).$$

*Proof* Fix some type  $\alpha$  and let  $g \in C^0(X_{\mathcal{F}})$  be a function such that  $g(v) = 0$  if  $\text{Type}(v) \neq \alpha$ . Then  $(\Delta g, g) = N \cdot (g, g)$ . Indeed,

$$\begin{aligned} (\Delta g, g) &= (dg, dg) = \sum_{[x,v] \in \widehat{S}_1(X_{\mathcal{F}})} w([x, v])(g(v) - g(x))^2 \\ &= \sum_{\substack{v \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(v) = \alpha}} g(v)^2 \sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ [x,v] \in \widehat{S}_1(X_{\mathcal{F}})}} w([x, v]) \\ &\stackrel{\text{Lem. 3.1}}{=} N \sum_{\substack{v \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(v) = \alpha}} w(v) \cdot g(v)^2 = N \cdot (g, g). \end{aligned}$$

If we apply this to  $g = f_{\alpha} - f$ , then we get

$$(\Delta f_{\alpha}, f_{\alpha}) = N \cdot (f_{\alpha}, f_{\alpha}) - 2(N - c)(f_{\alpha}, f) + (N - c)(f, f). \tag{31}$$

Since the cardinality of  $\mathfrak{T}$  is  $N + 1$ ,

$$\sum_{\alpha \in \mathfrak{T}} f_{\alpha} = (N + R) \cdot f \quad \text{and} \quad \sum_{\alpha \in \mathfrak{T}} (f_{\alpha}, f_{\alpha}) = (N + R^2) \cdot (f, f).$$

Summing (31) over  $\mathfrak{T}$  and using the previous two equalities, we get the claim.  $\square$

**Proposition 5.9** *For any  $\varepsilon > 0$  there is a constant  $q(\varepsilon, n)$  depending only on  $\varepsilon$  and  $n$ , such that if  $q > q(\varepsilon, n)$  then  $m^0(X_{\mathcal{F}}) \geq N - \varepsilon$ .*

*Proof* Since Lemma 5.1 implies this claim for  $N = 1$ , we can assume from now on that  $N \geq 2$ .

Let  $f \in C^0(X_{\mathcal{F}})$  and suppose  $\Delta f = c \cdot f$ . If  $\text{Type}(v) = \alpha$ , then

$$\Delta f_{\alpha}(v) = \sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ [x,v] \in S_1(X_{\mathcal{F}})}} \frac{w([x, v])}{w(v)} (Rf(v) - f(x)) = NRf(v) - C,$$

where  $C$  does not depend on  $R$  since  $\text{Type}(x) \neq \text{Type}(v)$  if  $[x, v] \in S_1(X_{\mathcal{F}})$ . If we take  $R = 1$ , then  $f_{\alpha} = f$ , so  $\Delta f_{\alpha}(v) = c \cdot f(v)$ . We conclude that  $C = (N - c) f(v)$ , and  $\Delta f_{\alpha}(v) = (NR - (N - c)) f(v)$ . From now on we assume that  $R = (N - c)/N$ . With this choice of  $R$  our calculation implies

$$\Delta f_{\alpha}(v) = 0 \quad \text{if} \quad \text{Type}(v) = \alpha. \tag{32}$$

Let  $v \in \text{Ver}(X_{\mathcal{F}})$  be a vertex of type  $\alpha$ . By Lemma 3.5,

$$(\Delta \rho_v df_{\alpha}, \rho_v df_{\alpha}) = (\Delta \tau_v df_{\alpha}, \tau_v df_{\alpha})_v.$$

Since

$$(\mathbf{1}, \tau_v df_\alpha)_v \stackrel{(18)}{=} -w(v)\delta df_\alpha(v) = -w(v)\Delta f_\alpha(v) \stackrel{(32)}{=} 0,$$

we can use the argument in the proof of Lemma 3.7 to conclude

$$(\Delta_v \tau_v df_\alpha, \tau_v df_\alpha)_v \geq m^0(\text{Lk}(v))(\tau_v df_\alpha, \tau_v df_\alpha)_v \geq \lambda_{\min}^0(X_{\mathcal{F}})(\tau_v df_\alpha, \tau_v df_\alpha)_v.$$

(Note that  $\text{Lk}(v)$  is connected since  $\text{Lk}(v) \cong X_{\mathcal{G}}$  for some  $\mathcal{G}$  with  $\dim X_{\mathcal{G}} = N - 1 \geq 1$ .) Hence

$$\begin{aligned} (\Delta \rho_v df_\alpha, \rho_v df_\alpha) &\geq \lambda_{\min}^0(X_{\mathcal{F}})(\tau_v df_\alpha, \tau_v df_\alpha)_v \stackrel{\text{Lem. 3.3}}{=} \lambda_{\min}^0(X_{\mathcal{F}})(\rho_v df_\alpha, \rho_v df_\alpha) \\ &\stackrel{(10)}{=} \lambda_{\min}^0(X_{\mathcal{F}})(\rho_v df_\alpha, df_\alpha). \end{aligned}$$

Summing these inequalities over all vertices of type  $\alpha$  and using (28), we get

$$(\Delta \rho_\alpha df_\alpha, \rho_\alpha df_\alpha) \geq \lambda_{\min}^0(X_{\mathcal{F}}) \cdot (\rho_\alpha df_\alpha, df_\alpha).$$

Using (29) and (30), we can rewrite this inequality as

$$(1 + \lambda_{\min}^0(X_{\mathcal{F}}))((1 - \rho_\alpha)df, df) \geq \lambda_{\min}^0(X_{\mathcal{F}}) \cdot (df_\alpha, df_\alpha).$$

Summing these inequalities over all types and using (27) and Lemma 5.8, we get

$$(1 + \lambda_{\min}^0(X_{\mathcal{F}}))(N - 1)c \geq \lambda_{\min}^0(X_{\mathcal{F}}) \cdot [(N - c)(R - 1)^2 + c(R^2 + N)]. \tag{33}$$

Suppose  $c = m^0(X_{\mathcal{F}})$ . If  $c \geq N$ , then we are done. On the other hand, if  $c < N$ , then  $(N - c)(R - 1)^2$  is positive, so (33) implies

$$(1 + \lambda_{\min}^0(X_{\mathcal{F}}))(N - 1)c \geq \lambda_{\min}^0(X_{\mathcal{F}})c(R^2 + N).$$

Dividing both sides by  $c$  (recall that  $c > 0$ ), we get

$$N - 1 \geq (1 + R^2)\lambda_{\min}^0(X_{\mathcal{F}}).$$

By induction on  $N$ , for any  $\varepsilon > 0$  there is a constant  $q(\varepsilon, n)$  such that  $\lambda_{\min}^0(X_{\mathcal{F}}) \geq N - 1 - \varepsilon$  if  $q \geq q(\varepsilon, n)$ . Thus

$$\varepsilon \geq R^2(N - 1 - \varepsilon).$$

We see that  $R^2 \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Since  $R = (N - c)/N$ , this forces  $c \rightarrow N$ . □

### 5.3 Auxiliary results about eigenvalues

In this subsection we prove that  $M^i(X_{\mathcal{F}}) \leq N + 1$  and  $m^0(X_{\mathcal{F}}) \leq N$ . This implies that if we allow  $q$  to vary, then the lower bound  $m^0(X_{\mathcal{F}}) \geq N - \varepsilon$  in Proposition 5.9 is optimal; in other terms,  $m^0(X_{\mathcal{F}}) \rightarrow N$  as  $q \rightarrow \infty$ , which is consistent with Conjecture 5.2. We also show that  $M^0(X_{\mathcal{F}}) = N + 1$  and its multiplicity is  $N$ , so does not depend on  $q$ .

**Proposition 5.10** *For all  $0 \leq i \leq N - 1$  we have  $M^i(X_{\mathcal{F}}) \leq N + 1$ .*

*Proof* The proof is again by induction on  $N$ . If  $N = 1$ , then the calculations in the proof of Lemma 5.1 show that  $M^0(X_{\mathcal{F}}) = 2$ . Now assume  $N \geq 2$  and  $i \geq 1$ . Assume we proved that  $M^{i-1}(X_{\mathcal{G}}) \leq N$  for any  $\mathcal{G}$  with  $\dim X_{\mathcal{G}} = N - 1$ . Then  $\lambda_{\max}^{i-1}(X_{\mathcal{F}}) \leq N$ , so Theorem 3.10 implies  $M^i(X_{\mathcal{F}}) \leq N + 1$ . It remains to prove that  $M^0(X_{\mathcal{F}}) \leq N + 1$ .

Let  $f \in C^0(X_{\mathcal{F}})$ . By an argument very similar to the proof of Proposition 5.9 we get

$$\begin{aligned} (\Delta \rho_v df_{\alpha}, \rho_v df_{\alpha}) &= (\Delta_v \tau_v df_{\alpha}, \tau_v df_{\alpha})_v \leq \lambda_{\max}^0(X_{\mathcal{F}}) (\tau_v df_{\alpha}, \tau_v df_{\alpha})_v \\ &= \lambda_{\max}^0(X_{\mathcal{F}}) \cdot (\rho_v df_{\alpha}, df_{\alpha}), \end{aligned}$$

which leads to

$$(\Delta \rho_{\alpha} df_{\alpha}, \rho_{\alpha} df_{\alpha}) \leq \lambda_{\max}^0(X_{\mathcal{F}}) \cdot (\rho_{\alpha} df_{\alpha}, df_{\alpha}).$$

By induction,  $\lambda_{\max}^0(X_{\mathcal{F}}) \leq N$ , so using (29) and (30), we can rewrite the previous inequality as

$$(1 + N) \cdot ((1 - \rho_{\alpha})df, df) \leq N \cdot (df_{\alpha}, df_{\alpha}).$$

Assume  $\Delta f = c \cdot f$  is an eigenfunction. Summing the above inequalities over all types and using (27) and Lemma 5.8, we get

$$(N + 1)(N - 1)c \leq N \cdot [(N - c)(R - 1)^2 + c(R^2 + N)].$$

If we put  $R = (N - c)/N$ , then this inequality forces  $c \leq N + 1$ . In particular,  $M^0(X_{\mathcal{F}}) \leq N + 1$ . □

Let  $Y$  be an  $N$ -simplex. Since  $Y$  has a unique simplex of maximal dimension, the weights (9) of the simplices of  $Y$  are all equal to 1. Then, relative to the inner-product (1), we have the orthogonal direct sum decomposition (cf. Lemma 2.5)

$$C^0(Y) = \mathbb{R}\mathbf{1} \oplus \delta C^1(Y),$$

where  $\delta C^1(Y)$  can be explicitly described as the space of functions satisfying

$$\sum_{v \in \text{Ver}(Y)} g(v) = 0.$$

It is easy to check that  $\Delta g = 0$  if and only if  $g \in \mathbb{R}\mathbf{1}$ , and  $\Delta g = (N + 1)g$  if and only if  $g \in \delta C^1(Y)$ . Hence 0 and  $N + 1$  are the only eigenvalues of  $\Delta$  acting on  $C^0(Y)$ , and their multiplicities are 1 and  $N$ , respectively.

**Definition 5.11** We say that  $f \in C^0(X_{\mathcal{F}})$  is *type-constant* if  $f(v) = f(v')$  for all  $v, v' \in \text{Ver}(X_{\mathcal{F}})$  of the same type. We denote the space of type-constant functions by  $\mathcal{C}$ .

Label the vertices of  $Y$  by the elements of  $\mathcal{T}$ . Given a function  $f \in \mathcal{C}$ , define  $\tilde{f} \in C^0(Y)$  by  $\tilde{f}(x) = c_{\alpha}(f)$ ,  $x \in \text{Ver}(Y)$ , where  $\alpha = \text{Type}(x)$  and  $c_{\alpha}(f)$  is the value of  $f$  on vertices of type  $\alpha \in \mathcal{T}$ . It is clear that  $\mathcal{C} \rightarrow C^0(Y)$ ,  $f \mapsto \tilde{f}$ , is an isomorphism of vector spaces which restricts to an isomorphism  $\mathcal{C}_0 \xrightarrow{\sim} \delta C^1(Y)$ , where  $\mathcal{C}_0 \subset \mathcal{C}$  is the subspace of functions  $f \in \mathcal{C}$  satisfying

$$\sum_{\alpha \in \mathcal{T}} c_{\alpha}(f) = 0.$$

**Lemma 5.12** *If  $f \in \mathcal{C}$ , then  $\Delta f \in \mathcal{C}$ . Moreover,  $\widetilde{\Delta f} = \Delta \tilde{f}$ . This implies that for  $f \in \mathcal{C}_0$  we have  $\Delta f = (N + 1)f$ .*

*Proof* For a fixed  $v \in \text{Ver}(X_{\mathcal{F}})$ , we have

$$\begin{aligned} \Delta f(v) &= \sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ [x,v] \in \mathcal{S}_1(X_{\mathcal{F}})}} \frac{w([x,v])}{w(v)} (f(v) - f(x)) \\ &\stackrel{\text{Lem. 3.1}}{=} Nf(v) - \sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ [x,v] \in \mathcal{S}_1(X_{\mathcal{F}})}} \frac{w([x,v])}{w(v)} f(x) \\ &= Nf(v) - \sum_{\substack{\alpha \in \mathcal{T} \\ \alpha \neq \text{Type}(v)}} \sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(x) = \alpha \\ [x,v] \in \mathcal{S}_1(X_{\mathcal{F}})}} \frac{w([x,v])}{w(v)} f(x). \end{aligned}$$

Now

$$\sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(x) = \alpha \\ [x,v] \in \mathcal{S}_1(X_{\mathcal{F}})}} \frac{w([x,v])}{w(v)} f(x) = c_{\alpha}(f) \sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(x) = \alpha \\ [x,v] \in \mathcal{S}_1(X_{\mathcal{F}})}} \frac{w([x,v])}{w(v)} \stackrel{\text{Lem. 5.5}}{=} c_{\alpha}(f).$$

Thus, if we denote  $\beta = \text{Type}(v)$ ,

$$\Delta f(v) = Nc_{\beta}(f) - \sum_{\substack{\alpha \in \mathcal{T} \\ \alpha \neq \beta}} c_{\alpha}(f).$$

It is clear from this that  $\Delta f \in \mathcal{C}$ . Moreover, for any  $z \in \text{Ver}(Y)$  we have

$$\widetilde{\Delta f}(z) = N\widetilde{f}(z) - \sum_{\substack{y \in \text{Ver}(Y_{\mathcal{F}}) \\ y \neq z}} \widetilde{f}(y) = \Delta \widetilde{f}(z).$$

□

**Lemma 5.13** *Let  $f \in C^0(X_{\mathcal{F}})$  be a  $\Delta$ -eigenfunction with eigenvalue  $c$ . If  $c = 0$  or  $N + 1$ , then  $f \in \mathcal{C}$ .*

*Proof* Using the fact that  $X_{\mathcal{F}}$  is connected, it is easy to show that  $\Delta f = 0$  if and only if  $f$  is constant. We prove that if  $\Delta f = (N + 1)f$ , then  $f$  is type-constant.

First, assume  $N = 1$ . Then either  $X_{\mathcal{F}} = X_{\emptyset}^0 * X_{\emptyset}^0$  or  $X_{\mathcal{F}} = X_{\emptyset}^1$ . In either case, the matrix of  $m\Delta$  has the form

$$\begin{pmatrix} mI_m & -A \\ -A^t & mI_m \end{pmatrix},$$

where  $A$  gives the adjacency relations between vertices of type  $\alpha$  and  $\beta$  (there are only two types), and either  $m = q + 1$  or  $m = q^2 + q + 1$ . Let  $\mathbf{x} = (x_1, x_2, \dots, x_{2m})^t$  be an eigenvector with eigenvalue  $2m$ . Let  $\mathbf{x}_{\alpha} = (x_1, x_2, \dots, x_m)^t$  and  $\mathbf{x}_{\beta} = (x_{m+1}, x_2, \dots, x_{2m})^t$ . Then

$$\begin{aligned} mI_m \mathbf{x}_{\alpha} - A \mathbf{x}_{\beta} &= 2m \mathbf{x}_{\alpha} \\ -A^t \mathbf{x}_{\alpha} + mI_m \mathbf{x}_{\beta} &= 2m \mathbf{x}_{\beta}. \end{aligned}$$

Hence  $m\mathbf{x}_{\alpha} = -A\mathbf{x}_{\beta}$  and  $m\mathbf{x}_{\beta} = -A^t\mathbf{x}_{\alpha}$ . This implies  $m^2\mathbf{x}_{\alpha} = AA^t\mathbf{x}_{\alpha}$ . In the first case,  $AA^t = mJ_m$ , so  $m\mathbf{x}_{\alpha} = J_m\mathbf{x}_{\alpha}$ . This implies that  $mx_j = \sum_{i=1}^m x_i$  for all  $1 \leq j \leq m$ . Hence  $x_1 = x_2 = \dots = x_m$ . Similarly, one shows that  $x_{m+1} = \dots = x_{2m}$ . In the second case,  $AA^t = qI_m + J_m$ , so

$$m^2\mathbf{x}_{\alpha} = q\mathbf{x}_{\alpha} + J_m\mathbf{x}_{\alpha}.$$

Hence  $(m^2 - q)x_j = \sum_{i=1}^m x_i$  for all  $1 \leq j \leq m$ , which again implies  $x_1 = x_2 = \dots = x_m$ . Similarly, one shows that  $x_{m+1} = \dots = x_{2m}$ , since  $A^tA = qI_m + J_m$ .

Now assume  $N > 1$  and that we proved the claim for all  $X_{\mathcal{F}}$  of dimension less than  $N$ . Suppose  $f$  is not type-constant. Then there are two vertices  $x, y$  of the same type such that  $f(x) \neq f(y)$ . We claim that we can choose  $x$  and  $y$  so that there is a vertex  $v \in \text{Ver}(X_{\mathcal{F}})$  such that  $x, y \in \text{Lk}(v)$ . We start with  $X_{\mathcal{F}} = X_{\emptyset}^n$ . In that case  $x$  and  $y$  correspond to subspaces  $W_1$  and  $W_2$  of  $\mathbb{F}_q^{n+2}$  of the same dimension. By assumption  $n > 1$ . If  $\dim W_i = 1$ , then  $v$  corresponding to  $W_1 + W_2$  is adjacent to both  $x$  and  $y$ . (Note that  $\dim(W_1 + W_2) = 2 < n + 2$ .) If  $r = \dim W_i > 1$ , choose a line  $\ell_i \in W_i$ . Let  $W_3$  be a subspace of dimension  $r$  which contains  $\ell_1 + \ell_2$ . Let  $z$  be the corresponding vertex. If  $f(x) = f(z)$ , then we replace  $x$  by  $z$  and take  $v$  corresponding to  $\ell_2$ . If  $f(x) \neq f(z)$ , then we replace  $y$  by  $z$  and take  $v$  corresponding to  $\ell_1$ . Now suppose  $X_{\mathcal{F}} \cong X_{\emptyset}^{n_1} * \dots * X_{\emptyset}^{n_s}$ ,  $s \geq 2$ . Our vertices are in the same  $X_{\emptyset}^{n_i}$  since they have the same type, but then any vertex in another  $X_{\emptyset}^{n_j}$  is adjacent to both  $x$  and  $y$ .



Let  $x, y \in \text{Lk}(v)$  be as in the previous paragraph. Let  $\text{Type}(v) = \alpha$ . Consider the function  $\tau_v df_\alpha$ . We have

$$\tau_v df_\alpha(x) = df_\alpha([v, x]) = f(x) - Rf(v) \neq f(y) - Rf(v) = \tau_v df_\alpha(y).$$

Hence  $\tau_v df_\alpha \in C^0(\text{Lk}(v))$  is not type-constant. By induction,  $\tau_v df_\alpha$  does not lie in the subspace of  $C^0(\text{Lk}(v))$  spanned by eigenfunctions with eigenvalue  $N$ . This implies (use the orthonormal decomposition of Lemma 3.7 and Proposition 5.10)

$$(\Delta_v \tau_v df_\alpha, \tau_v df_\alpha)_v < N(\tau_v df_\alpha, \tau_v df_\alpha)_v.$$

This inequality implies, as in the proof of Proposition 5.10, that

$$(\Delta \rho_\alpha df_\alpha, \rho_\alpha df_\alpha) < N(\rho_\alpha df_\alpha, df_\alpha).$$

As in the proof of Proposition 5.10, this leads to

$$(N + 1)(N - 1)c < N[(N - c)(R - 1)^2 + (N + 1)(R^2 + N)],$$

where  $c = N + 1$  and  $R = (N - c)/N$ . But for these  $c$  and  $R$  both sides are equal, so the inequality cannot be strict. □

**Proposition 5.14** *A function  $f \in C^0(X_{\mathcal{F}})$  is a  $\Delta$ -eigenfunction with eigenvalue 0 if and only if  $f$  is constant. A function  $f \in C^0(X_{\mathcal{F}})$  is a  $\Delta$ -eigenfunction with eigenvalue  $N + 1$  if and only if  $f \in \mathcal{C}_0$ . This implies that  $M^0(X_{\mathcal{F}}) = N + 1$  and its multiplicity as an eigenvalue of  $\Delta$  is  $N$ .*

*Proof* It is easy to check that  $\Delta f = 0 \Leftrightarrow df = 0 \Leftrightarrow f$  is constant (since  $X_{\mathcal{F}}$  is connected). By Lemma 5.12, if  $f \in \mathcal{C}_0$ , then  $\Delta f = (N + 1)f$ . Conversely, suppose  $\Delta f = (N + 1)f$ . By Lemma 5.13,  $f$  is type-constant, so by Lemma 5.12,

$$\widetilde{\Delta f} = \widetilde{(N + 1)f} = (N + 1)\widetilde{f} = \Delta \widetilde{f}.$$

This implies  $f \in \mathcal{C}_0$ . □

*Remark 5.15* In [16], we proved a general result about finite buildings which implies that  $M^i(X_{\mathcal{F}}) = N + 1$  for all  $0 \leq i \leq N - 1$ .

**Proposition 5.16**  $m^0(X_{\mathcal{F}}) \leq N$ .

*Proof* Denote  $c = m^0(X_{\mathcal{F}})$  and let  $f$  be a  $\Delta$ -eigenfunction with eigenvalue  $c$ . First we claim that  $c \neq N + 1$ . Indeed,  $\Delta$  is a semi-simple operator and if  $c = N + 1$  then by Proposition 5.10 it has only two distinct eigenvalues, namely 0 and  $N + 1$ . This implies that  $\Delta^2 = (N + 1)\Delta$ . In  $X_{\mathcal{F}}$  we can find two vertices  $x$  and  $y$  which are not adjacent but such that there is another vertex  $v$  which is adjacent to both  $x$  and  $y$ . Let  $g \in C^0(X_{\mathcal{F}})$  be a function such that  $g(x) \neq 0$  but  $g(x') = 0$  if  $x' \neq x$ . Now  $\Delta g(y) = 0$  because this is a sum of the values of  $g$  at  $y$  and the vertices adjacent

to  $y$ , and  $x$  is not one of them. On the the other hand,  $\Delta^2 g(y) \neq 0$  since this is a sum which involves  $g(x)$  with a non-zero coefficient. This contradicts the equality  $\Delta^2 = (N + 1)\Delta$ .

Define a function  $h \in C^0(X_{\mathcal{F}})$  by

$$h(v) = \sum_{\substack{x \in \text{Ver}(X_{\mathcal{F}}) \\ \text{Type}(x) = \text{Type}(v)}} f(x) \quad \text{for any } v \in \text{Ver}(X_{\mathcal{F}}).$$

It is clear that  $h$  is type-constant, and because  $f$  is a  $\Delta$ -eigenfunction, we have  $\Delta h = ch$ . Since  $c \neq 0, N + 1$ , the function  $h$  must be identically 0. Therefore,  $\sum_{\text{Type}(v)=\beta} f(v) = 0$  for any fixed  $\beta \in \mathfrak{T}$ . Obviously the same is also true for  $f_\alpha$ , i.e.,  $\sum_{\text{Type}(v)=\beta} f_\alpha(v) = 0$ . Since  $w(v)$  depends only on the type of  $v$ , we see that  $f_\alpha$  is orthogonal to  $\mathbf{1}$  in  $C^0(X_{\mathcal{F}})$  with respect to the pairing (1). As in the proof of Lemma 3.7, this implies that

$$(\Delta f_\alpha, f_\alpha) \geq c \cdot (f_\alpha, f_\alpha).$$

Summing over all types, we get

$$\sum_{\alpha \in \mathfrak{T}} (\Delta f_\alpha, f_\alpha) \geq c(N + R^2) \cdot (f, f).$$

Comparing this inequality with the expression in Lemma 5.8, we conclude that  $(N - c)(R - 1)^2 \geq 0$ . Since  $R$  is arbitrary, we must have  $c \leq N$ . □

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