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ON GENERATORS OF ARITHMETIC GROUPS OVER FUNCTION FIELDS

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Let $F = \mathbb{F}_q(T)$ be the field of rational functions with $\mathbb{F}_q$-coefficients, and $A = \mathbb{F}_q[T]$ be the subring of polynomials. Let D be a division quaternion algebra over F which is split at $1/T$. For certain A-orders in D we find explicit finite sets generating their groups of units.

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1. Introduction

Notation.

$\mathbb{F}_q$ = the finite field with q elements;
throughout the paper q is assumed to be odd.
$A = \mathbb{F}_q[T]$, T indeterminate.
$F = \mathbb{F}_q(T)$ = the fraction field of A.
$|F|$ = the set of places of F.
For $x \in |F|$, F_x = the completion of F at x.
$\mathcal{O}_x = \{z \in F_x \mid \mathrm{ord}_x(z) \geq 0\}$ = the ring of integers of F_x.
$\mathbb{F}_x$ = the residue field of $\mathcal{O}_x$; $\deg(x) = [\mathbb{F}_x : \mathbb{F}_q]$.
For $0 \neq f \in A$, $\deg(f)$ = the degree of f as a polynomial in T, and $\deg(0) = +\infty$.
For $f/g \in F$, $\deg(f/g) = \deg(f) - \deg(g)$.
$\mathrm{ord} = -\deg$ defines a valuation on F; the corresponding place is denoted by ∞.
$K = F_\infty$.
$\mathcal{O} = \mathcal{O}_\infty$.
$\pi = T^{-1}$ = uniformizer at infinity.

Let D be a quaternion division algebra over F such that $D \otimes_F K \cong \mathbb{M}_2(K)$. Let Λ be an A-order in D, i.e. Λ, as a subring of D containing A, is also an A-module containing four linearly independent generators over F. The subgroup $\Gamma := \Lambda^\times$ of units of Λ consists of elements $\lambda \in \Lambda$ whose reduced norm $\mathrm{Nr}(\lambda)$ is in $\mathbb{F}_q^\times$. It is known that Γ is infinite finitely generated group. The problem of finding explicit sets of generators for Γ arises naturally in the study of these arithmetic groups. In this paper we develop a method for finding such explicit sets.

Now we explain what we mean by "explicit" and state the main result of the paper. Let $\mathfrak{r} \in A$ be the discriminant of D — this is the product of the monic generators of the primes in A where D ramifies (see Sec. 2). There exists $\mathfrak{a} \in A$ such that D is isomorphic to the F-algebra with basis $1, i, j, ij$ satisfying the relations

$$i^2 = \mathfrak{a}, \quad j^2 = \mathfrak{r}, \quad ij = -ji \tag{1.1}$$

(see Lemma 2.1). Let Λ be the A-order

$$\Lambda = A \oplus Ai \oplus Aj \oplus Aij \tag{1.2}$$

which is the order having the simplest presentation in terms of the basis $\{1, i, j, ij\}$. Intrinsically, this is the level-$\mathfrak{a}$ Eichler order in D. In this case,

$$\Gamma = \left\{ a + bi + cj + dij \;\middle|\; \begin{array}{c} a, b, c, d \in A \\ a^2 - \mathfrak{a}b^2 - \mathfrak{r}c^2 + \mathfrak{a}\mathfrak{r}d^2 \in \mathbb{F}_q^\times \end{array} \right\}.$$

Now the problem is to find finitely many quadruples $(a_n, b_n, c_n, d_n) \in A^4$, $1 \le n \le N$ for some N, such that $\gamma_n = a_n + b_n i + c_n j + d_n ij \in \Lambda$ are in Γ and generate Γ. Finding *minimal* sets of generators is a difficult computational problem — usually one has to carry out lengthy computer calculations for each specific case to find a minimal explicit set of generators, cf. [2]. For a discussion of the analogous problem over number fields we refer to [1, 14], where the reader will find a few examples where explicit minimal sets of generators are computed. (See [9] for some examples in the function field case.)

Definition 1.1. Put $\deg(0) = 0$ and for $\lambda = a + bi + cj + dij \in \Lambda$ define

$$\|\lambda\| = \max(\deg(a), \deg(b), \deg(c), \deg(d)).$$

Instead of trying to find minimal sets of generators for Γ, we look for possibly larger sets which are easy to describe. The main result of the paper is the following.

Theorem 1.2. *The finite set*

$$S = \{\gamma \in \Gamma \mid \|\gamma\| \le 4\deg(\mathfrak{a}\mathfrak{r}) + 6\}$$

generates Γ.

To prove Theorem 1.2 we use the action of Γ on the Bruhat–Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(K)$. The key is to quantify the discontinuity of the action of Γ on $\mathcal{T}$. More precisely, for a ball $\mathcal{T}_B$ of radius B around a fixed vertex and $\gamma \in \Gamma$, we show that

$\gamma\mathcal{T}_B \cap \mathcal{T}_B = \emptyset$ once $\|\gamma\| > 2B$. To deduce Theorem 1.2 from this result, one needs strong bounds on the diameter of the quotient graph $\Gamma\backslash\mathcal{T}$. Such a bound follows from the property of $\Gamma\backslash\mathcal{T}$ being covered by a Ramanujan graph.

A problem of similar nature over $\mathbb{Q}$ was considered by Chalk and Kelly in [3] (see also [1, 5, 14]). The approach in [3] is analytic in nature and relies on the study of isometric circles of Γ.

The strategy of the proof of Theorem 1.2 can be adapted to any A-order in D given as the free A-module generated by four explicit elements of D, although the calculations in Sec. 4 will be more complicated. The final answer will be similar to Theorem 1.2 and will depend on the discriminant of the order. We also note that the assumption on q being odd can be removed at the expense of redoing the calculations in Sec. 4 specifically for the case when q is even. This is due to the fact that in characteristic two quaternion algebras have different presentation from (1.1), cf. [13, p. 5].

Remark 1.3. After this paper was written, Böckle and Butenuth [2] obtained a result similar to Theorem 1.2 by a different method, which gives a sharper bound.

Remark 1.4. Theorem 1.2 raises the following natural questions:

(1) How large is the set S?
(2) What is the minimal σ independent of Γ such that there exists a constant δ (also independent of Γ) with the property that the set

$$\{\gamma \in \Gamma \mid \|\gamma\| \leq \sigma \deg(\mathfrak{ar}) + \delta\}$$

generates Γ?

Proposition 2.6 gives lower bounds on the quantities in question, since it says that a minimal generating set for Γ has approximately $q^{\deg \mathfrak{ar}}$ elements.

2. Arithmetic of Quaternion Algebras

In this section we recall some facts about quaternion algebras. The standard reference for this material is [13].

Let D be a *quaternion algebra* over F, i.e. a four-dimensional F-algebra with center F which does not possess non-trivial two-sided ideals. A quaternion algebra is either a division algebra or is isomorphic to the algebra of 2×2 matrices. If L is a field containing F, then $D \otimes_F L$ is a quaternion algebra over L. Let $x \in |F|$ and denote $D_x := D \otimes_F F_x$. We say that D *ramifies* (respectively, *splits*) at x if D_x is a division algebra (respectively, is isomorphic to $\mathbb{M}_2(F_x)$). Let $R \subset |F|$ be the set of places where D ramifies. It is known that R is a finite set of even cardinality, and conversely, for any choice of a finite set $R \subset |F|$ of even cardinality there is a unique, up to an isomorphism, quaternion algebra ramified exactly at the places in R. In particular, $D \cong \mathbb{M}_2(F)$ if and only if $R = \emptyset$.

Explicitly quaternion algebras can be given as follows. For $a, b \in F^\times$, let $H(a,b)$ be the F-algebra with basis $1, i, j, ij$ (as an F-vector space), where i, j satisfy

$$i^2 = a, \quad j^2 = b, \quad ij = -ji.$$

$H(a,b)$ is a quaternion algebra, and any quaternion algebra D is isomorphic to $H(a,b)$ for some $a, b \in F^\times$ (although a and b are not uniquely determined by D, e.g. $H(a,b) \cong H(b,a)$).

From now on we assume that D is a division algebra (equivalently $R \neq \emptyset$). Let $L \neq F$ be a non-trivial field extension of F. Then L embeds into D, i.e. there is an F-isomorphism of L onto an F-subalgebra of D, if and only if $[L : F] = 2$ and places in R do not split in L.

There is a canonical involution $\alpha \mapsto \alpha'$ on D which is the identity on F and satisfies $(\alpha\beta)' = \beta'\alpha'$. The *reduced trace* of α is $\mathrm{Tr}(\alpha) = \alpha + \alpha'$; the *reduced norm* of α is $\mathrm{Nr}(\alpha) = \alpha\alpha'$; the *reduced characteristic polynomial* of α is

$$f(x) = (x - \alpha)(x - \alpha') = x^2 - \mathrm{Tr}(\alpha)x + \mathrm{Nr}(\alpha).$$

If $\alpha \notin F$, then the reduced trace and norm of α are the images of α under the trace and norm of the quadratic field extension $F(\alpha)/F$.

From now on we also assume that $\infty \notin R$. For $x \in |F| - \infty$, denote by (x) the prime ideal of A corresponding to x. Let $f_x \in A$ be the monic generator of (x), and let $\mathfrak{r} = \prod_{x \in R} f_x$. Given $a, b \in A$, with b irreducible and coprime to a, let

$$\left(\frac{a}{b}\right) = \begin{cases} 1 & \text{if } a \text{ is a square mod } (b), \\ -1 & \text{otherwise} \end{cases}$$

be the Legendre symbol. Let

$$\mathrm{Odd}(R) = \begin{cases} 1 & \text{if } \deg(x) \text{ is odd for all } x \in R, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1. (1) *Suppose* $\mathrm{Odd}(R) = 0$. *There is a monic irreducible polynomial* $\mathfrak{a} \in A$ *of even degree which is coprime to* $\mathfrak{r}$ *and satisfies*

$$\left(\frac{\mathfrak{a}}{f_x}\right) = -1 \quad \textit{for all } x \in R.$$

For such $\mathfrak{a}$, $D \cong H(\mathfrak{a}, \mathfrak{r})$.

(2) *Suppose* $\mathrm{Odd}(R) = 1$. *Let* $\xi \in \mathbb{F}_q$ *be a non-square. Then* $D \cong H(\xi, \mathfrak{r})$.

Proof. The proof of this lemma is quite standard; we give the details for completeness.

(1) By the Chinese Remainder Theorem [10, Proposition 1.4], there exists $a \in A$ such that $(a/f_x) = -1$ for all $x \in R$. Consider the set of polynomials $\{a + \mathfrak{r}b \mid b \in A\}$. By the strong form of the function field analog of Dirichlet's theorem [10, Theorem 4.8], this set contains irreducible monic polynomials $\mathfrak{a}$ of even degrees.

Fix such $\mathfrak{a}$. It is clear that $\mathfrak{a}$ satisfies $(\mathfrak{a}/f_x) = -1$ for all $x \in R$. Next, by the analog of Quadratic Reciprocity [10, Theorem 3.3]

$$\left(\frac{f_x}{\mathfrak{a}}\right) = (-1)^{\frac{q-1}{2}\deg(\mathfrak{a})\deg(f_x)}\left(\frac{\mathfrak{a}}{f_x}\right).$$

Since $\deg(\mathfrak{a})$ is even, the right-hand side is equal to -1. Hence

$$\left(\frac{\mathfrak{r}}{\mathfrak{a}}\right) = \prod_{x\in R}\left(\frac{f_x}{\mathfrak{a}}\right) = (-1)^{\#R} = 1. \tag{2.1}$$

Now we show that $D \cong H(\mathfrak{a},\mathfrak{r})$. It is enough to check that $H(\mathfrak{a},\mathfrak{r})$ is ramified exactly at the places in R. For this we need to show that the Hilbert symbol $(\mathfrak{a},\mathfrak{r})_x$ is -1 if and only if $x \in R$, cf. [13, p. 32]. By [11, p. 210], for $x \in |F| - (\mathfrak{a}) - \infty$, $(\mathfrak{a},\mathfrak{r})_x = 1$ if and only if $\mathfrak{a}^{\mathrm{ord}_x(\mathfrak{r})}$ is a square modulo (x). Now $\mathrm{ord}_x(\mathfrak{r}) = 0$ if $x \in |F| - R - \infty$, and $\mathrm{ord}_x(\mathfrak{r}) = 1$ if $x \in R$. Observe that the image of $\mathfrak{a}$ is not a square in $\mathbb{F}_x$ for $x \in R$ by the choice of $\mathfrak{a}$. Therefore, $H(\mathfrak{a},\mathfrak{r})$ is ramified at the places in R and is unramified at $|F| - R - \infty - (\mathfrak{a})$. By the same argument, $H(\mathfrak{a},\mathfrak{r})$ is unramified at $(\mathfrak{a})$, since $\mathfrak{r}^{\mathrm{ord}_{(\mathfrak{a})}(\mathfrak{a})} = \mathfrak{r}$ is a square modulo $(\mathfrak{a})$ by (2.1). Finally, $H(\mathfrak{a},\mathfrak{r})$ is unramified at ∞ since the number of places where a quaternion algebra ramifies is even.

(2) Note that ξ is not a square in $\mathbb{F}_x$, $x \in R$, since $\deg(x)$ is odd. Now apply the argument in the previous paragraph. □

Definition 2.2. Let $\mathcal{R}$ be a Dedekind domain with quotient field L and let B be a quaternion algebra over L. For any finite-dimensional L-vector space V, an $\mathcal{R}$-*lattice* in V is a finitely generated $\mathcal{R}$-submodule M in V such that $L \otimes_{\mathcal{R}} M \cong V$. An $\mathcal{R}$-*order* in B is a subring Λ of B, having the same unity element as B, and such that Λ is an $\mathcal{R}$-lattice in B. A *maximal* $\mathcal{R}$-*order* in B is an $\mathcal{R}$-order which is not contained in any other $\mathcal{R}$-order in B.

Let Λ be an A-order in D. It is known that Λ is maximal if and only if $\Lambda_x := \Lambda \otimes_A \mathcal{O}_x$ is a maximal $\mathcal{O}_x$-order in D_x for all $x \in |F| - \infty$. A maximal $\mathcal{O}_x$-order in D_x is unique if $x \in R$ — it is the integral closure of $\mathcal{O}_x$ in D_x. On the other hand, for $x \notin R$, Λ_x is maximal if and only if there is an invertible element $u \in \mathbb{M}_2(F_x)$ such that $u\Lambda_x u^{-1} = \mathbb{M}_2(\mathcal{O}_x)$.

Definition 2.3. Suppose $\mathfrak{n} \in A$ is squarefree and coprime to $\mathfrak{r}$. Λ is an *Eichler order of level*-$\mathfrak{n}$ if Λ_x is maximal for all $x \in R$, and for $x \in |F| - R - \infty$ it is isomorphic to the subring of $\mathbb{M}_2(\mathcal{O}_x)$ given by the matrices

$$\left\{\begin{pmatrix} a & b \\ \mathfrak{n}c & d \end{pmatrix} \middle|\ a,b,c,d \in \mathcal{O}_x\right\}.$$

Suppose $\Lambda = Ae_1 \oplus Ae_2 \oplus Ae_3 \oplus Ae_4$, where $e_1, \ldots, e_4$ is a basis of D as an F-vector space. The *discriminant* of Λ is the ideal of A generated by $\det(\mathrm{Tr}(e_ie_j))_{i,j}$. It is known that the discriminant of any order is divisible by $(\mathfrak{r})^2$. Moreover, the maximal orders are uniquely characterized by the fact that their discriminants are

$(\mathfrak{r})^2$, and the level-$\mathfrak{n}$ Eichler orders are uniquely characterized by the fact that their discriminants are equal to $(\mathfrak{n}\mathfrak{r})^2$. By a theorem of Eichler, since D splits at ∞, all maximal A-orders are conjugate in D; the same is true also for the level-$\mathfrak{n}$ Eichler orders (see [13, p. 89]).

Definition 2.4. The order $\Lambda = A \oplus Ai \oplus Aj \oplus Aij$ in $H(\mathfrak{a}, \mathfrak{r})$ will be called the *standard order*. By computing its discriminant, we see that Λ is a level-$\mathfrak{a}$ Eichler order. In particular, Λ is maximal if and only if $\mathfrak{a} \in \mathbb{F}_q^\times$.

Given an A-order Λ, the group $\Lambda^\times$ of its invertible elements consists of

$$\{\lambda \in \Lambda \mid \mathrm{Nr}(\lambda) \in \mathbb{F}_q^\times\}.$$

If $\lambda \in \Lambda^\times$ is a torsion element, then it is algebraic over $\mathbb{F}_q$. This easily implies that λ is torsion if and only if $\mathrm{Tr}(\lambda) \in \mathbb{F}_q$; such elements will be called *elliptic*. An element $\lambda \in \Lambda^\times$ which is not elliptic will be called *hyperbolic*.

Lemma 2.5. *If λ is hyperbolic, then its image in $\mathrm{GL}_2(K)$ under an embedding $D \hookrightarrow \mathbb{M}_2(K)$ has two distinct K-rational eigenvalues.*

Proof. The reduced characteristic polynomial of λ is $h_\lambda := x^2 + \mathrm{Tr}(\lambda)x + \kappa$, where $\kappa \in \mathbb{F}_q^\times$. Since $\lambda \notin F^\times$, $F(\lambda)$ is quadratic, and therefore h_λ is irreducible. Next, since $s := \mathrm{Tr}(\lambda) \in A$ has degree ≥ 1, $s^2 - 4\kappa$ is a non-zero polynomial of even degree whose leading coefficient is a square. This implies that h_λ splits over K and has distinct roots. □

Denote

$$g(R) = 1 + \frac{1}{q^2-1}\prod_{x\in R}(q^{\deg(x)} - 1) - \frac{q}{q+1} \cdot 2^{\#R-1} \cdot \mathrm{Odd}(R).$$

Let Λ be the standard order in $H(\mathfrak{a}, \mathfrak{r})$, where $\mathfrak{a}$ is a monic irreducible polynomial of even degree if $\mathrm{Odd}(R) = 0$, and $\mathfrak{a} = \xi$ is a non-square in $\mathbb{F}_q$ if $\mathrm{Odd}(R) = 1$, cf. Lemma 2.1. Denote $\Gamma := \Lambda^\times$. Note that $\mathbb{F}_q^\times$ is in the center of Γ.

Proposition 2.6. (1) *$\Gamma/\mathbb{F}_q^\times$ has non-trivial torsion if and only if* $\mathrm{Odd}(R) = 1$.

(2) *If $\mathrm{Odd}(R) = 1$, then Γ can be generated by $g(R) + 2^{\#R-1}$ elements, and a set of generators of Γ has at least $g(R)$ elements.*

(3) *If $\mathrm{Odd}(R) = 0$, then $\Gamma/\mathbb{F}_q^\times$ is a free group of rank*

$$1 + (q^{\deg(\mathfrak{a})} + 1)(g(R) - 1).$$

Proof. Parts (1) and (2) follow from Theorem 5.7 and its proof in [9]. (3) Let Υ be a maximal order containing Λ. Define $\Gamma' = \Upsilon^\times$. By [9, Theorem 5.7], $\Gamma'/\mathbb{F}_q^\times$ is a free group of rank $g(R)$. It is not hard to show that $[\Gamma'/\mathbb{F}_q^\times : \Gamma/\mathbb{F}_q^\times] = q^{\deg(\mathfrak{a})} + 1$, cf. [8, p. 212]. Now the claim that $\Gamma/\mathbb{F}_q^\times$ is a free group of rank

$$1 + (q^{\deg(\mathfrak{a})} + 1)(g(R) - 1)$$

follows from Schreier's theorem [12, p. 29]. □

3. Geometry on the Bruhat–Tits Tree

We start by recalling some of the terminology from [12]. Let $\mathcal{G}$ be an (oriented) connected graph; see [12, Definition 1, p. 13]. We denote by $\mathrm{Ver}(\mathcal{G})$ and $\mathrm{Ed}(\mathcal{G})$ the sets of vertices and oriented edges of $\mathcal{G}$, respectively. By $e \in \mathrm{Ed}(\mathcal{G})$, $o(e), t(e) \in \mathrm{Ver}(\mathcal{G})$ and $\bar{e} \in \mathrm{Ed}(\mathcal{G})$ denote its origin, terminus and inversely oriented edge, respectively. We will assume that for any $v \in \mathrm{Ver}(\mathcal{G})$ the number of edges e with $o(e) = v$ is finite; this number is the *degree* of v. $\mathcal{G}$ is *m-regular* if every vertex in $\mathcal{G}$ has degree m. The *distance* $d(v, w)$ between $v, w \in \mathrm{Ver}(\mathcal{G})$ in $\mathcal{G}$ is the obvious combinatorial distance, i.e. the number of edges in a shortest path without backtracking connecting v and w. The *diameter* $D(\mathcal{G})$ of a finite graph $\mathcal{G}$ is the maximum of the distances between its vertices. A graph in which a path without backtracking connecting any two vertices v and w is unique is called a *tree*; the unique path between v and w is called *geodesic*.

Let Γ be a group acting on a graph $\mathcal{G}$, i.e. Γ acts via automorphisms. We say that $v, w \in \mathrm{Ver}(\mathcal{G})$ are *Γ-equivalent* if there is $\gamma \in \Gamma$ such that $\gamma v = w$. Γ acts with *inversion* if there is $\gamma \in \Gamma$ and $e \in \mathrm{Ed}(\mathcal{G})$ such that $\gamma e = \bar{e}$. If Γ acts without inversion, then we have a natural quotient graph $\Gamma\backslash\mathcal{G}$ such that $\mathrm{Ver}(\Gamma\backslash\mathcal{G}) = \Gamma\backslash\mathrm{Ver}(\mathcal{G})$ and $\mathrm{Ed}(\Gamma\backslash\mathcal{G}) = \Gamma\backslash\mathrm{Ed}(\mathcal{G})$, cf. [12, p. 25].

Recall the notation $K := F_\infty$ and $\mathcal{O} := \mathcal{O}_\infty$. Let V be a two-dimensional K-vector space. Let Λ be an $\mathcal{O}$-lattice in V. For any $x \in K^\times$, $x\Lambda$ is also a lattice. We call Λ and $x\Lambda$ equivalent lattices. The equivalence class of Λ is denoted by $[\Lambda]$.

Let $\mathcal{T}$ be the graph whose vertices $\mathrm{Ver}(\mathcal{T}) = \{[\Lambda]\}$ are the equivalence classes of lattices in V, and two vertices $[\Lambda]$ and $[\Lambda']$ are adjacent if we can choose representatives $L \in [\Lambda]$ and $L' \in [\Lambda']$ such that $L' \subset L$ and $L/L' \cong \mathbb{F}_q$. One shows that $\mathcal{T}$ is an infinite tree which is $(q+1)$-regular, cf. [12, Chap. II]. This is the *Bruhat–Tits tree* of $\mathrm{PGL}_2(K)$.

Fix a vertex $[\Lambda]$ of $\mathcal{T}$. The set of vertices of $\mathcal{T}$ at distance n from $[\Lambda]$ is in natural bijection with $\mathbb{P}^1(L/\pi^n L)$. An *end* of $\mathcal{T}$ is an equivalence class of half-lines, two half-lines being equivalent if they differ in a finite graph. Taking the projective limit over n, we get a bijection

$$\partial\mathcal{T} := \text{set of ends of } \mathcal{T} \cong \mathbb{P}^1(\mathcal{O}) = \mathbb{P}^1(K),$$

which is independent of the choice of $[\Lambda]$. Given two vectors u_1, u_2 spanning V, we denote $[u_1, u_2] = [\mathcal{O}u_1 \oplus \mathcal{O}u_2]$. $\mathrm{GL}_2(K)$, as the group of K-automorphisms of V, acts on the Bruhat–Tits tree $\mathcal{T}$ via $g[u_1, u_2] = [gu_1, gu_2]$. It is easy to check that this action preserves the distance between any two vertices, and the induced action on $\partial\mathcal{T}$ agrees with the usual action of $\mathrm{GL}_2(K)$ on $\mathbb{P}^1(K)$ through fractional linear transformations.

Fix the standard basis $e_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ of V, and let $O := [e_1, e_2]$. Since $\mathrm{GL}_2(K)$ acts transitively on the vertices of $\mathcal{T}$ and the stabilizer of O is $\mathrm{GL}_2(\mathcal{O})K^\times$, we have a bijection

$$\begin{aligned} \mathrm{GL}_2(K)/\mathrm{GL}_2(\mathcal{O})K^\times &\xrightarrow{\sim} \mathrm{Ver}(\mathcal{T}), \\ g &\mapsto g \cdot O. \end{aligned} \tag{3.1}$$

Using the Iwasawa decomposition, one easily shows that the set of matrices

$$\left\{ \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \middle| \begin{array}{c} k \in \mathbb{Z} \\ u \in K,\ u \pmod{\pi^k \mathcal{O}} \end{array} \right\} \tag{3.2}$$

is a system of representatives for $\mathrm{Ver}(\mathcal{T})$; see [4, (1.5)]. The map (3.1) then becomes

$$\begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \mapsto [\pi^k e_1, u e_1 + e_2].$$

Note that under this bijection the identity matrix corresponds to O.

Definition 3.1. We say that a matrix $M \in \mathrm{GL}_2(K)$ is in *reduced form* if it belongs to the set of matrices in (3.2). For two matrices $M, M' \in \mathrm{GL}_2(K)$, we write $M \sim M'$ if they represent the same vertex in $\mathcal{T}$.

Lemma 3.2. *The distance between* $\left(\begin{smallmatrix} \pi^k & u \\ 0 & 1 \end{smallmatrix} \right)$ *(in reduced form) and* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)$ *is*

$$\begin{cases} |k| & \text{if } u = 0 \quad \text{or} \quad \mathrm{ord}(u) \geq 0, \\ k - 2 \cdot \mathrm{ord}(u) & \text{if } u \neq 0 \quad \text{and} \quad \mathrm{ord}(u) < 0. \end{cases}$$

(*Note that for a matrix in reduced form* $k > \mathrm{ord}(u)$.)

Proof. This is an easy calculation. □

Given two distinct points $P, Q \in \mathbb{P}^1(K)$, there is a unique path in $\mathcal{T}$, without backtracking and infinite in both directions, whose ends are P and Q; this is the *geodesic* $\mathcal{A}(P, Q)$ connecting the two boundary points of $\mathcal{T}$. For example, the geodesic connecting $0 = (0 : 1)$ and $\infty = (1 : 0)$ is the subgraph of $\mathcal{T}$ with vertices $\{ \left(\begin{smallmatrix} \pi^k & 0 \\ 0 & 1 \end{smallmatrix} \right) \mid k \in \mathbb{Z} \}$.

Assume $\gamma \in \mathrm{GL}_2(K)$ has two distinct K-rational eigenvalues a and b. The eigenvectors corresponding to a and b can be regarded as two well-defined points on $\mathbb{P}^1(K)$ — if $\left(\begin{smallmatrix} x \\ y \end{smallmatrix} \right)$ is an eigenvector, then the corresponding point is $(x : y)$. Let $\mathcal{A}(\gamma)$ be the geodesic connecting these points. Suppose $\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix} \right)$ are eigenvectors corresponding to a and b, respectively. Since $\left(\begin{smallmatrix} x_1 & x_2 \\ y_1 & y_2 \end{smallmatrix} \right)$ maps ∞ to $(x_1 : y_1)$ and 0 to $(x_2 : y_2)$,

$$\mathrm{Ver}(\mathcal{A}(\gamma)) = \left\{ \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \pi^k & 0 \\ 0 & 1 \end{pmatrix} \middle| \, k \in \mathbb{Z} \right\}. \tag{3.3}$$

Definition 3.3. The *distance* from $\mathcal{A}(\gamma)$ to O is the minimum of the distances from the vertices on $\mathcal{A}(\gamma)$ to O. Let $\mathcal{P}$ be a straight path in $\mathcal{T}$ [12, p. 62], i.e. a doubly infinite chain $\{\ldots, v_{i-1}, v_i, v_{i+1}, \ldots\}$ of adjacent vertices, where v_i $(i \in \mathbb{Z})$ is adjacent to v_{i+1} and $v_{i-1} \neq v_{i+1}$. We say that γ induces a *translation of* $\mathcal{P}$ *of amplitude* m if either $\gamma(v_i) = v_{i+m}$ for all i, or $\gamma(v_i) = v_{i-m}$ for all i.

Lemma 3.4. *The action of* γ *on* $\mathcal{T}$ *induces a translation of* $\mathcal{A}(\gamma)$ *of amplitude* $|\mathrm{ord}(a) - \mathrm{ord}(b)|$.

Proof. This is easy to see after choosing the eigenvectors of γ as a basis of V. □

Lemma 3.5. *Suppose* $x_1y_1 \neq 0$ *and* $x_2y_2 \neq 0$. *Define* $x := x_1/y_1$ *and* $y := x_2/y_2$. *Suppose* $\operatorname{ord}(x) \geq B$ *and* $\operatorname{ord}(y) \geq B$, *or* $\operatorname{ord}(x) \leq -B$ *and* $\operatorname{ord}(y) \leq -B$ *for some* $B \geq 0$. *Then the distance from* $\mathcal{A}(\gamma)$ *to* O *is at least* B.

Proof. Note that $x \neq y$, as $\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} y \\ 1 \end{smallmatrix}\right)$ are eigenvectors for a and b. Without loss of generality we can assume $\operatorname{ord}(x) \geq \operatorname{ord}(y)$. The geodesic $\mathcal{A}(\gamma)$ has vertices

$$\begin{pmatrix} x\pi^k & y \\ \pi^k & 1 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

First, consider the case when $k > 0$. Then $\left(\begin{smallmatrix} 1 & 0 \\ -\pi^k & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O})$, so by multiplying $\left(\begin{smallmatrix} x\pi^k & y \\ \pi^k & 1 \end{smallmatrix}\right)$ by this matrix from the right we see that

$$\begin{pmatrix} x\pi^k & y \\ \pi^k & 1 \end{pmatrix} \sim \begin{pmatrix} (x-y)\pi^k & y \\ 0 & 1 \end{pmatrix}.$$

On the other hand, $\operatorname{ord}((x-y)\pi^k) \geq k + \operatorname{ord}(y) > \operatorname{ord}(y)$, so the resulting matrix is in reduced form. The distance from this matrix to O is

$$\operatorname{ord}((x-y)\pi^k) \geq k + \operatorname{ord}(y) \quad \text{if } \operatorname{ord}(y) \geq 0,$$

and

$$\operatorname{ord}((x-y)\pi^k) - 2\operatorname{ord}(y) \geq k - \operatorname{ord}(y) \quad \text{if } \operatorname{ord}(y) < 0.$$

In either case, we conclude that the distance is at least $1 + |\operatorname{ord}(y)| \geq 1 + B$.

The matrix with $k = 0$ is adjacent to the matrix with $k = 1$, so from the previous paragraph we conclude that the corresponding matrix and O are at a distance at least $(1 + B) - 1 = B$.

Now suppose $k < 0$. Then

$$\begin{pmatrix} x\pi^k & y \\ \pi^k & 1 \end{pmatrix} \sim \begin{pmatrix} x & \pi^{-k}y \\ 1 & \pi^{-k} \end{pmatrix} \sim \begin{pmatrix} \pi^{-k}(y-x) & x \\ 0 & 1 \end{pmatrix}.$$

If $\operatorname{ord}(x) = \operatorname{ord}(y)$, then $\operatorname{ord}(x-y) \geq \operatorname{ord}(y) = \operatorname{ord}(x)$. Hence $-k + \operatorname{ord}(x-y) > \operatorname{ord}(x)$, and the above matrix is in reduced form. The distance from O is

$$-k + \operatorname{ord}(x-y) \geq -k + \operatorname{ord}(y) > B \quad \text{if } \operatorname{ord}(y) \geq 0,$$

or

$$-k + \operatorname{ord}(x-y) - 2\operatorname{ord}(y) \geq -k - \operatorname{ord}(y) > B \quad \text{if } \operatorname{ord}(y) < 0.$$

If $\operatorname{ord}(x) > \operatorname{ord}(y)$, then $\operatorname{ord}(x-y) = \operatorname{ord}(y)$. If $\operatorname{ord}(\pi^{-k}(x-y)) = -k + \operatorname{ord}(y) \leq \operatorname{ord}(x)$, then

$$\begin{pmatrix} \pi^{-k}(y-x) & x \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} \pi^{-k}(y-x) & 0 \\ 0 & 1 \end{pmatrix},$$

so the distance is $|-k+\mathrm{ord}(y)|$. If $\mathrm{ord}(y) \geq B$, then this last quantity is obviously $\geq 1+B$. On the other hand, if $\mathrm{ord}(y) \leq -B$, then, due to the assumption of the lemma, $-k+\mathrm{ord}(y) \leq \mathrm{ord}(x) \leq -B$. Thus, $|-k+\mathrm{ord}(y)| \geq B$ Finally, if $-k+\mathrm{ord}(y) > \mathrm{ord}(x)$, then the distance is

$$-k+\mathrm{ord}(x-y) = -k+\mathrm{ord}(y) > B \quad \text{if } \mathrm{ord}(x) \geq 0,$$

or

$$\begin{aligned} &-k+\mathrm{ord}(x-y)-2\mathrm{ord}(x) \\ &\quad = -k+\mathrm{ord}(y)-2\mathrm{ord}(x) > -\mathrm{ord}(x) \geq B \quad \text{if } \mathrm{ord}(x) < 0. \end{aligned}$$ □

Remark 3.6. The inequalities $\mathrm{ord}(x) \gg 0$, $\mathrm{ord}(y) \gg 0$ (respectively, $\mathrm{ord}(x) \ll 0$, $\mathrm{ord}(y) \ll 0$) essentially mean that both x and y are in a small neighborhood of 0 (respectively, ∞). With this in mind, one can visualize the previous lemma as follows: for a sufficiently small interval on $\mathbb{R}$ the geodesic in the Poincaré upper half-plane $\mathcal{H}$ connecting any two distinct points in that interval is far from a fixed point in $\mathcal{H}$.

Notation 3.7. For $B \geq 0$, let $\mathcal{T}_B$ be the finite subtree of $\mathcal{T}$ with set of vertices

$$\mathrm{Ver}(\mathcal{T}_B) = \{v \in \mathrm{Ver}(\mathcal{T}) \mid d(v, O) \leq B\}.$$

Lemma 3.8. *Let n be the amplitude of translation with which γ acts on $\mathcal{A}(\gamma)$. Let m be the distance from $\mathcal{A}(\gamma)$ to O. If $2m+n > 2B$, then $\gamma\mathcal{T}_B \cap \mathcal{T}_B = \emptyset$.*

Proof. Let $P \in \mathcal{A}(\gamma)$ be the vertex closest to O. Then $m = d(P, O)$ and

$$d(O, \gamma O) = d(O, P) + d(P, \gamma P) + d(\gamma P, \gamma O) = 2m+n,$$

cf. [12, Proposition 24(iv), p. 63]. On the other hand, if $\gamma\mathcal{T}_B \cap \mathcal{T}_B \neq \emptyset$, then $d(O, \gamma O) \leq 2B$. Thus $2m+n \leq 2B$. □

Proposition 3.9. *Assume Γ acts without inversion on $\mathcal{T}$ and $\Gamma\backslash\mathcal{T}$ is finite. Let $B = D(\Gamma\backslash\mathcal{T})$. Let S denote the set of $\gamma \in \Gamma$ such that $\gamma\mathcal{T}_B \cap \mathcal{T}_B \neq \emptyset$. Then S generates Γ.*

Proof. Let $\bar{O} \in \Gamma\backslash\mathcal{T}$ be the image of O. Let $\bar{v} \in \mathrm{Ver}(\Gamma\backslash\mathcal{T})$. Consider a path P of shortest length connecting $\bar{O}$ and $\bar{v}$. Obviously P is a subtree of $\Gamma\backslash\mathcal{T}$, hence by [12, Proposition 14, p. 25] it lifts to $\mathcal{T}$. This implies that there is a vertex v in $\mathcal{T}$ which maps to $\bar{v}$ and $d(O, v) = d(\bar{O}, \bar{v})$. In particular, $v \in \mathcal{T}_B$, so $\mathcal{T}_B$ surjects onto $\Gamma\backslash\mathcal{T}$ under the quotient map $\mathcal{T} \to \Gamma\backslash\mathcal{T}$.

Let $\mathrm{real}(\mathcal{T})$ be the *realization* of $\mathcal{T}$; see [12, p. 14]. Recall that $\mathrm{real}(\mathcal{T})$ is a CW-complex where each edge of $\mathcal{T}$ is homeomorphic to the interval $[0, 1] \subset \mathbb{R}$. Let U be the open subset of $\mathrm{real}(\mathcal{T})$ consisting of points at distance $< 1/3$ from $\mathrm{real}(\mathcal{T}_B)$.

Then $\gamma U \cap U \neq \emptyset$ if and only if $\gamma \mathcal{T}_B \cap \mathcal{T}_B \neq \emptyset$, and $U \to \mathrm{real}(\Gamma\backslash\mathcal{T})$ is surjective. The claim of the proposition now follows from [12, (1), p. 30]. □

Let Γ be as in Proposition 2.6. Γ acts naturally on $\mathcal{T}$ (see Sec. 4). The action is without inversion and the quotient graph $\Gamma\backslash\mathcal{T}$ is finite; see [9, Lemma 5.1].

Lemma 3.10. *Let* $V := \#\mathrm{Ver}(\Gamma\backslash\mathcal{T})$.

(1) *If* $\mathrm{Odd}(R) = 1$, *then*

$$V = \frac{2}{(q-1)(q^2-1)} \prod_{x \in R} (q^{\deg(x)} - 1) + \frac{q}{q+1} 2^{\#R-1}.$$

(2) *If* $\mathrm{Odd}(R) = 0$, *then*

$$V = \frac{2(q^{\deg(\mathfrak{a})} + 1)}{(q-1)(q^2-1)} \prod_{x \in R} (q^{\deg(x)} - 1).$$

Proof. (1) This follows from [9, Theorem 5.5]. (2) Since $\mathbb{F}_q^\times \lhd \Gamma$ acts trivially on $\mathcal{T}$ and $\Gamma/\mathbb{F}_q^\times$ is a free group, $\Gamma\backslash\mathcal{T}$ is $(q+1)$-regular. Thus, if we denote by E the number of (non-oriented) edges of $\Gamma\backslash\mathcal{T}$, then $E = (q+1)V/2$. On the other hand, by [12, Theorem 4′, p. 27], the rank of $\Gamma/\mathbb{F}_q^\times$ is equal to $E+1-V$. Now the expression for V follows from Proposition 2.6. □

Lemma 3.11. *Let* $\mathcal{G}$ *be an* m*-regular Ramanujan graph on* n *vertices. Then*

$$D(\mathcal{G}) \leq 2\log_{m-1}(n) + \log_{m-1}(4).$$

Proof. For the definition of Ramanujan graphs, see [6]. The claim of the lemma is part of [6, Proposition 7.3.11]. □

Proposition 3.12. $D(\Gamma\backslash\mathcal{T}) \leq 2\deg(\mathfrak{a}\mathfrak{r}) + 3$.

Proof. Let $I = (T) \lhd A$ be the ideal generated by T. Let Υ be a maximal order containing Λ. Denote by $\Gamma(I)$ the principal level-I congruence subgroup of $\Upsilon^\times$, cf. [7]. Let $\Gamma' := \Gamma \cap \Gamma(I)$. $\Gamma\backslash\mathcal{T}$ is naturally a quotient of $\Gamma'\backslash\mathcal{T}$, so

$$D(\Gamma\backslash\mathcal{T}) \leq D(\Gamma'\backslash\mathcal{T}).$$

Moreover, since $[\Gamma : \Gamma'] < q^4$, we have $V' < q^4 V$, where $V' := \#\mathrm{Ver}(\Gamma'\backslash\mathcal{T})$.

By [7, Theorem 1.2], $\Gamma'\backslash\mathcal{T}$ is a $(q+1)$-regular Ramanujan graph, so the previous paragraph and Lemma 3.11 give the bound

$$D(\Gamma\backslash\mathcal{T}) \leq 2\log_q(q^4 V) + 1.$$

Now the proposition follows from the formulae in Lemma 3.10. (Note that $\deg(\mathfrak{r}) = \sum_{x\in R} \deg(x)$, and $\deg(\mathfrak{a}) = 0$ when $\mathrm{Odd}(R) = 1$.) □

4. Proof of Theorem 1.2

Let $(\mathfrak{a},\mathfrak{b}) = (\mathfrak{a}',\mathfrak{r})$ (respectively, $(\mathfrak{a},\mathfrak{b}) = (\mathfrak{r},\xi)$) if $\mathrm{Odd}(R) = 0$ (respectively, $\mathrm{Odd}(R) = 1$), with $\mathfrak{a}',\xi$ chosen as in Lemma 2.1. Let Λ be the standard order in $H(\mathfrak{a},\mathfrak{b})$ and $\Gamma := \Lambda^\times$. By Proposition 2.6, this is a finitely generated group, and we would like to find an explicit set of generators. We start by embedding $H(\mathfrak{a},\mathfrak{b})$ into $\mathbb{M}_2(K)$. The map

$$i \mapsto \begin{pmatrix} \sqrt{\mathfrak{a}} & 0 \\ 0 & -\sqrt{\mathfrak{a}} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ \mathfrak{b} & 0 \end{pmatrix}$$

gives an embedding $H(\mathfrak{a},\mathfrak{b}) \hookrightarrow \mathbb{M}_2(K)$. Indeed, since $\mathfrak{a} \in A$ is monic and has even degree, the equation $x^2 = \mathfrak{a}$ has a solution in K. Thus, the above matrices are indeed in $\mathbb{M}_2(K)$. It remains to observe that the given matrices satisfy the same relations as i and j. Under this embedding Γ is the subgroup of $\mathrm{GL}_2(K)$ consisting of matrices

$$\Gamma = \left\{ \begin{pmatrix} a+b\sqrt{\mathfrak{a}} & c+d\sqrt{\mathfrak{a}} \\ \mathfrak{b}(c-d\sqrt{\mathfrak{a}}) & a-b\sqrt{\mathfrak{a}} \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d, \in A \\ \det = a^2 - b^2\mathfrak{a} - c^2\mathfrak{b} + d^2\mathfrak{a}\mathfrak{b} \in \mathbb{F}_q^\times \end{matrix} \right\}.$$

Proposition 4.1. *Fix some $B \geq 0$. If $\gamma \in \Gamma$ is hyperbolic and satisfies $\|\gamma\| > 2B$, then $\gamma\mathcal{T}_B \cap \mathcal{T}_B = \emptyset$.*

Proof. To simplify the notation, we put $\deg(0) = 0$. Let the image of γ in $\mathrm{GL}_2(K)$ be the matrix

$$\begin{pmatrix} a+b\sqrt{\mathfrak{a}} & c+d\sqrt{\mathfrak{a}} \\ \mathfrak{b}(c-d\sqrt{\mathfrak{a}}) & a-b\sqrt{\mathfrak{a}} \end{pmatrix}.$$

Let $\alpha,\beta \in K$ be the eigenvalues of γ. We know that $\mathrm{Nr}(\gamma) = \alpha\beta =: \kappa \in \mathbb{F}_q^\times$ and $\mathrm{Tr}(\gamma) = \alpha+\beta = 2a \notin \mathbb{F}_q$. Thus, $\beta = \kappa\alpha^{-1}$ and $\mathrm{ord}(\beta) = -\mathrm{ord}(\alpha)$. Without loss of generality, we assume $\mathrm{ord}(\alpha) \geq 0$, so $\mathrm{ord}(\beta) \leq 0$. Since $\deg(a) \geq 1$, $\mathrm{ord}(\beta) = \mathrm{ord}(a) \leq -1$. Using Lemma 3.4, we conclude that γ acts on $\mathcal{A}(\gamma)$ by translations with amplitude $2\deg(a)$. Let m be the distance from $\mathcal{A}(\gamma)$ to O. Lemma 3.8 implies that

$$\text{if } m + \deg(a) > B, \text{ then } \gamma\mathcal{T}_B \cap \mathcal{T}_B = \emptyset. \tag{4.1}$$

In particular, if $\deg(a) > B$, then $\gamma\mathcal{T}_B \cap \mathcal{T}_B = \emptyset$.

Suppose $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is an eigenvector. Then $\mathfrak{b}(c-d\sqrt{\mathfrak{a}}) = 0$, which forces $c = d = 0$. Thus, we must have $a^2 - b^2\mathfrak{a} \in \mathbb{F}_q^\times$. If $\deg(b) > \deg(a) - \deg(\mathfrak{a})/2$, then this is not possible. It is easy to see that we reach the same conclusion in the case when $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ is an eigenvector.

Assume $\deg(b) > \deg(a) - \deg(\mathfrak{a})/2$. Define $s := \deg(b) + \deg(\mathfrak{a})/2 - \deg(a) > 0$. From the previous paragraph we know that there are eigenvectors for α and β of the form $\left(\begin{smallmatrix}x\\1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}y\\1\end{smallmatrix}\right)$, with $x \neq 0$, $y \neq 0$. Now

$$x(a+b\sqrt{\mathfrak{a}}) + (c+d\sqrt{\mathfrak{a}}) = \alpha x$$

$$x\mathfrak{b}(c-d\sqrt{\mathfrak{a}}) + (a-b\sqrt{\mathfrak{a}}) = \alpha.$$

From this we get

$$x = -\frac{c + d\sqrt{\mathfrak{a}}}{a + b\sqrt{\mathfrak{a}} - \alpha} = \frac{\alpha - a + b\sqrt{\mathfrak{a}}}{\mathfrak{b}(c - d\sqrt{\mathfrak{a}})}. \tag{4.2}$$

Similarly,

$$y = -\frac{c + d\sqrt{\mathfrak{a}}}{a + b\sqrt{\mathfrak{a}} - \kappa\alpha^{-1}} = \frac{\kappa\alpha^{-1} - a + b\sqrt{\mathfrak{a}}}{\mathfrak{b}(c - d\sqrt{\mathfrak{a}})}. \tag{4.3}$$

Hence

$$\frac{x}{y} = \frac{a + b\sqrt{\mathfrak{a}} - \kappa\alpha^{-1}}{a + b\sqrt{\mathfrak{a}} - \alpha} \tag{4.4}$$

and

$$x - y = \frac{(c + d\sqrt{\mathfrak{a}})(\kappa\alpha^{-1} - \alpha)}{(a + b\sqrt{\mathfrak{a}} - \alpha)(a + b\sqrt{\mathfrak{a}} - \kappa\alpha^{-1})} = x\frac{\alpha - \kappa\alpha^{-1}}{(a + b\sqrt{\mathfrak{a}} - \kappa\alpha^{-1})}. \tag{4.5}$$

From our assumption $\deg(b) > \deg(a) - \deg(\mathfrak{a})/2$ and (4.4), we get

$$\operatorname{ord}(x) = \operatorname{ord}(y) =: n.$$

Similarly from (4.5), we get

$$\operatorname{ord}(x - y) = n + s.$$

By (3.3), the geodesic $\mathcal{A}(\gamma)$ has vertices

$$\operatorname{Ver}(\mathcal{A}(\gamma)) = \left\{ \begin{pmatrix} x\pi^k & y \\ \pi^k & 1 \end{pmatrix} \middle| \, k \in \mathbb{Z} \right\}.$$

If $k \geq 0$, then

$$\begin{pmatrix} x\pi^k & y \\ \pi^k & 1 \end{pmatrix} \sim \begin{pmatrix} (x - y)\pi^k & y \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} \pi^{k+n+s} & y \\ 0 & 1 \end{pmatrix}.$$

Since $y \neq 0$ and $k + n + s > n = \operatorname{ord}(y)$, the last matrix is in reduced form. By Lemma 3.2, the distance of this matrix from O is $k + |n| + s \geq s$. If $k < 0$, then

$$\begin{pmatrix} x\pi^k & y \\ \pi^k & 1 \end{pmatrix} \sim \begin{pmatrix} x & y\pi^{-k} \\ 1 & \pi^{-k} \end{pmatrix} \sim \begin{pmatrix} y\pi^{-k} & x \\ \pi^{-k} & 1 \end{pmatrix}.$$

A similar calculation shows that the distance from this matrix to O is again greater or equal to s. Now

$$s + \deg(a) = \deg(b) + \deg(\mathfrak{a})/2.$$

If $\deg(b) > B - \deg(\mathfrak{a})/2$, then this last quantity is greater than B. Therefore, by (4.1), $\gamma\mathcal{T}_B \cap \mathcal{T}_B = \emptyset$. Note that if $\deg(a) \leq B$, then $\deg(b) > B - \deg(\mathfrak{a})/2$ implies $\deg(b) > \deg(a) - \deg(\mathfrak{a})/2$. Overall, we have shown that if $\deg(a) > B$ or $\deg(b) > B - \deg(\mathfrak{a})/2$, then $\gamma\mathcal{T}_B \cap \mathcal{T}_B = \emptyset$.

Now assume $\deg(a) \leq B$ and $\deg(b) \leq B - \deg(\mathfrak{a})/2$, but $\deg(c) > 2B$ or $\deg(d) > 2B$. From our earlier discussion we know that $\binom{1}{0}$ or $\binom{0}{1}$ are not eigenvectors (since otherwise $c = d = 0$). Now, either $\operatorname{ord}(c + d\sqrt{\mathfrak{a}}) < -2B$ or

$\mathrm{ord}(c - d\sqrt{\mathfrak{a}}) < -2B$ (and only one of the inequalities holds due to $\mathrm{Nr}(\gamma) \in \mathbb{F}_q^\times$). First, assume $\mathrm{ord}(c + d\sqrt{\mathfrak{a}}) < -2B$. Since $\mathrm{ord}(a + b\sqrt{\mathfrak{a}} - \alpha) \geq -B$ and $\mathrm{ord}(a + b\sqrt{\mathfrak{a}} - \kappa\alpha^{-1}) \geq -B$, from the first equality in (4.2) and (4.3) we get $\mathrm{ord}(x) < -B$ and $\mathrm{ord}(y) < -B$. Next, assume $\mathrm{ord}(c - d\sqrt{\mathfrak{a}}) < -2B$. Since $\mathrm{ord}(\alpha - a + b\sqrt{\mathfrak{a}}) \geq -B$ and $\mathrm{ord}(\kappa\alpha^{-1} - a + b\sqrt{\mathfrak{a}}) \geq -B$, from the second equality in (4.2) and (4.3) we get $\mathrm{ord}(x) > B$ and $\mathrm{ord}(y) > B$. In either case, Lemma 3.5 implies that the distance from $\mathcal{A}(\gamma)$ to O is greater than B. As before, from (4.1) we conclude $\gamma\mathcal{T}_B \cap \mathcal{T}_B = \emptyset$. □

Proposition 4.2. *If $\gamma \in \Gamma$ is elliptic and satisfies $\|\gamma\| > B$, then $\gamma\mathcal{T}_B \cap \mathcal{T}_B = \emptyset$.*

Proof. To simplify the notation, we again put $\deg(0) = 0$. If γ is elliptic and satisfies the inequality of the proposition, then obviously $\gamma \notin \mathbb{F}_q^\times$. By Proposition 2.6, the existence of such γ is possible if and only if $\mathrm{Odd}(R) = 1$. Hence, we can assume that

$$\gamma = \begin{pmatrix} a + b\sqrt{\mathfrak{r}} & c + d\sqrt{\mathfrak{r}} \\ \xi(c - d\sqrt{\mathfrak{r}}) & a - b\sqrt{\mathfrak{r}} \end{pmatrix}.$$

Since γ is elliptic, $a \in \mathbb{F}_q$. Assume $\deg(c) > B$. Consider $\tau := \gamma \cdot j \in \Gamma$:

$$\tau = \begin{pmatrix} \xi(c + d\sqrt{\mathfrak{r}}) & a + b\sqrt{\mathfrak{r}} \\ \xi(a - b\sqrt{\mathfrak{r}}) & \xi(c - d\sqrt{\mathfrak{r}}) \end{pmatrix}.$$

Note that τ is hyperbolic since $\mathrm{Tr}(\tau) = 2\xi c \notin \mathbb{F}_q$. Suppose $\gamma\mathcal{T}_B \cap \mathcal{T}_B \neq \emptyset$. Since j fixes O, we get

$$d(O, \tau O) = d(O, \gamma O) \leq 2B.$$

On the other hand, τ acts on its geodesic by translation with amplitude $2\deg(c) > 2B$, so $d(O, \tau O) > 2B$ — a contradiction. Thus, from now on we can assume $\deg(c) \leq B$. If $\deg(b) > B$, then $\deg(a^2 - b^2\mathfrak{r}) > 2B$. Since

$$(a^2 - b^2\mathfrak{r}) - \xi(c^2 - d^2\mathfrak{r}) \in \mathbb{F}_q^\times,$$

we must have $\deg(b) = \deg(d) =: s$. In this case we have

$$\gamma = \begin{pmatrix} z_1\pi^{-s} & z_2\pi^{-s} \\ z_3\pi^{-s} & z_4\pi^{-s} \end{pmatrix} \sim \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix},$$

where $z_1, \ldots, z_4 \in \mathcal{O}^\times$. Moreover, since $\det(\gamma) \in \mathbb{F}_q^\times$,

$$z_1 z_4 - z_2 z_3 \in \pi^{2s}\mathcal{O}^\times.$$

This implies

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \sim \begin{pmatrix} \pi^{2s} & 0 \\ 0 & 1 \end{pmatrix}.$$

The distance from this matrix to O is obviously $2s$. On the other hand, the matrix corresponding to the vertex γO under (3.2) is γ, so $d(O, \gamma O) = 2s > 2B$. The previous argument works also under the assumption $\deg(d) > B$, and leads to the same conclusion. □

Proof of Theorem 1.2. Let $B = 2\deg(\mathfrak{ab})+3$. By Proposition 3.12, $D(\Gamma\backslash\mathcal{T}) \leq B$. By Propositions 4.1 and 4.2, the set of elements $\gamma \in \Gamma$ such that $\gamma\mathcal{T}_B \cap \mathcal{T}_B \neq \emptyset$ is contained in S. Now the theorem follows from Proposition 3.9. □

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