

The Number of Rational Points on Drinfeld Modular Varieties over Finite Fields

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Drinfeld and Vladut proved that Drinfeld modular curves have many \mathbb{F}_{q^2} -rational points compared to their genera. We propose a conjectural generalization of this result to higher dimensional Drinfeld modular varieties, and prove a theorem giving evidence for the conjecture.

1 Introduction

Let q be a power of a prime p and let \mathbb{F}_q denote the finite field with q elements. Let X be a smooth, geometrically connected, d -dimensional variety defined over \mathbb{F}_q . Fix an algebraic closure $\overline{\mathbb{F}}_q$ of \mathbb{F}_q . Also, fix a prime number $\ell \neq p$ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field \mathbb{Q}_ℓ of ℓ -adic numbers. Grothendieck's theory of étale cohomology produces the ℓ -adic cohomology groups with compact supports

$$H^*(X) := H_c^*(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_\ell). \quad (1.1)$$

These groups are finite-dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces endowed with an action of the Galois group $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. It is known that $H^i(X) = 0$ for $i > 2d$ (cf. [28, Chapter VI]). Denote by $h^i(X) := \dim_{\overline{\mathbb{Q}}_\ell} H^i(X)$ the (compact) ℓ -adic Betti numbers of X .

Let Frob_q be the inverse of the standard topological generator $x \mapsto x^q$ of $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, that is, the so-called *geometric Frobenius element*. Assume $H^i(X) \neq 0$. Denote the eigenvalues of Frob_q acting on $H^i(X)$ by $\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,s}$ (here $s = h^i(X)$). Deligne proved that $\{\alpha_{i,j}\}$ are algebraic numbers. Moreover, for any isomorphism $\iota: \overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$ the absolute value $|\iota(\alpha_{i,j})|$ is independent of ι and is equal to $q^{m/2}$ for some $0 \leq m \leq i$; see [6, Theorem 3.3.1].

For an integer $n \geq 1$ denote by \mathbb{F}_{q^n} the degree n extension of \mathbb{F}_q , and let $X(\mathbb{F}_{q^n})$ be the set of \mathbb{F}_{q^n} -rational points on X . By the Grothendieck-Lefschetz trace formula [28, Theorem 13.1]

$$\#X(\mathbb{F}_{q^n}) = \sum_{i \geq 0} (-1)^i \text{Tr}(\text{Frob}_q^n | H^i(X)) = \sum_{i \geq 0} (-1)^i \sum_{j=1}^{h^i(X)} \alpha_{i,j}^n. \quad (1.2)$$

Denote $\text{WD}(X, \mathbb{F}_{q^n}) := \sum_{i,j} |\iota(\alpha_{i,j}^n)|$. If one combines (1.2) with Deligne's bounds, then there results the estimate

$$\#X(\mathbb{F}_{q^n}) \leq \text{WD}(X, \mathbb{F}_{q^n}) \leq \sum_{i \geq 0} q^{in/2} h^i(X). \quad (1.3)$$

When X is a curve, this estimate is equivalent to Weil's famous bound. When X is projective, the second inequality in (1.3) is always an equality thanks to Deligne's Theorem 1.6 in [5].

Since the early 1980's, partly due to Goppa's construction of algebra-geometric codes, the question of the "optimality" of the bound (1.3) received a considerable amount of attention. More precisely, it became important to know whether there exist varieties over \mathbb{F}_q which have many rational points compared to their Betti numbers. One way to formulate this problem is as follows: assume d , q , and n are fixed. For a smooth, geometrically connected, d -dimensional variety X over \mathbb{F}_q put $h(X) := \sum_i h^i(X)$. How large can the ratio $\#X(\mathbb{F}_{q^n})/h(X)$ be when $h(X) \gg q$? Not much is known about this question beyond dimension 1.

We recall the principal results for the case of curves, that is, for $d = 1$. Refining an idea of Ihara, Drinfeld and Vladut [37] proved

$$\limsup_{h(X) \rightarrow \infty} \left(\frac{\#X(\mathbb{F}_{q^n})}{h(X)} \right) \leq \frac{q^{n/2} - 1}{2}. \quad (1.4)$$

Remark 1.1. Note that Weil's bound only gives $\limsup(\#X(\mathbb{F}_{q^n})/h(X)) \leq q^{n/2}$.

Now the modular curves (classical, Shimura, Drinfeld) enter the picture in a key manner, since they provide examples of curves which attain the previous bound for $n = 2$

(and in fact the modular curves are the only known such examples). We recall the result for Drinfeld modular curves, which is due to Vladut [27]. First we need to introduce some notation.

Let T be a transcendental parameter over \mathbb{F}_q , and let $A = \mathbb{F}_q[T]$ be the ring of polynomials in T with \mathbb{F}_q coefficients. Let $\mathfrak{n} \triangleleft A$ be an ideal, and let $M_{\mathfrak{n}}^{d+1}$ be the Drinfeld modular scheme parametrizing Drinfeld A -modules of rank $(d+1)$ with full level \mathfrak{n} structure (we refer to Section 4 for the definitions). Drinfeld proved that $M_{\mathfrak{n}}^{d+1} \rightarrow \text{Spec}(A[\mathfrak{n}^{-1}])$ is a smooth affine scheme of pure relative dimension d . Its fibre over a prime $\mathfrak{q} \triangleleft A[\mathfrak{n}^{-1}]$ will be denoted by $M_{\mathfrak{n},\mathfrak{q}}^{d+1}$. The group $\text{GL}_{d+1}(A/\mathfrak{n})$ acts on $M_{\mathfrak{n},\mathfrak{q}}^{d+1}$. Denote by $X_{\mathfrak{n},\mathfrak{q}}^{d+1}$ the quotient of $M_{\mathfrak{n},\mathfrak{q}}^{d+1}$ under the action of $(A/\mathfrak{n})^\times$ embedded into $\text{GL}_{d+1}(A/\mathfrak{n})$ as the subgroup of scalar matrices.

Assume $\mathfrak{n} = \mathfrak{p} \neq (T)$ is a prime of *odd* degree. In [27, Chapter II] Vladut shows that $X_{\mathfrak{p},T}^2$ is a smooth, affine, geometrically connected curve defined over \mathbb{F}_q , $h(X_{\mathfrak{p},T}^2) \rightarrow \infty$ when $\deg(\mathfrak{p}) \rightarrow \infty$ and

$$\liminf_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{\#X_{\mathfrak{p},T}^2(\mathbb{F}_{q^2})}{h(X_{\mathfrak{p},T}^2)} \right) \geq \frac{q-1}{2}. \quad (1.5)$$

Therefore, by comparing with (1.4), we have

$$\lim_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{\#X_{\mathfrak{p},T}^2(\mathbb{F}_{q^2})}{h(X_{\mathfrak{p},T}^2)} \right) = \frac{q-1}{2}. \quad (1.6)$$

This result can be extended to other Drinfeld modular curves having different types of level structures, and also to their canonical compactifications; see [27] or [14].

Remark 1.2. That Vladut's result in [27] can be rewritten as (1.5) follows from the following two facts. First, as is easy to check, $h(X_{\mathfrak{p},T}^2) = 2g + c$, where g is the genus of the compactification of $X_{\mathfrak{p},T}^2$ and c is the number of the cusps. Second, $c/g \rightarrow 0$ when $\deg(\mathfrak{p}) \rightarrow \infty$, as follows from the formulae for g and c in terms of the level (cf. [14]).

In this paper we would like to propose a conjectural generalization of the Vladut-Drinfeld result (1.6) to an arbitrary $d \geq 1$. Fix q and d , and let $\mathfrak{n} := d+1$.

Definition 1.3. A prime ideal $\mathfrak{p} \triangleleft A$ is *admissible* if $x \mapsto x^d$ is an automorphism of $(A/\mathfrak{p})^\times / \mathbb{F}_q^\times$.

It is easy to show that there are infinitely many admissible primes; see Lemma 4.7. (Note that the primes of odd degree are admissible when $d = 1$.) In Section 4 we will prove that for an admissible prime $\mathfrak{p} \neq \mathfrak{q}$, $X_{\mathfrak{p},\mathfrak{q}}^n$ is a smooth, affine, geometrically connected variety of dimension d defined over \mathbb{F}_q . Moreover, $h(X_{\mathfrak{p},\mathfrak{q}}^n) \rightarrow \infty$ when $\deg(\mathfrak{p}) \rightarrow \infty$.

Conjecture 1.4.

$$\lim_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{\#X_{\mathfrak{p},T}^n(\mathbb{F}_{q^n})}{h(X_{\mathfrak{p},T}^n)} \right) = \frac{1}{n} \prod_{i=1}^{n-1} (q^i - 1), \quad (1.7)$$

where the limit is over the admissible primes not equal to (T) . \square

When $n = 2$, this is exactly (1.6). Next, we formulate another conjecture which at first might seem unrelated to the problem we are interested in. Let F be the fraction field of A . Fix a separable closure \bar{F} of F . Denote by $\eta : A[n^{-1}] \hookrightarrow F$ the generic point of $\text{Spec}(A[n^{-1}])$ and by $M_{n,\eta}^n := M_n^n \otimes_{A[n^{-1}]} F$ the generic fibre of M_n^n . Consider the cohomology groups $H^i(M_{n,\eta}^n) := H_c^i(M_{n,\eta}^n \otimes_F \bar{F}, \bar{\mathbb{Q}}_\ell)$, which are finite-dimensional $\bar{\mathbb{Q}}_\ell$ -vector spaces endowed with an action of $\text{Gal}(\bar{F}/F)$. Let $h^i(M_{n,\eta}^n) := \dim_{\bar{\mathbb{Q}}_\ell} H^i(M_{n,\eta}^n)$.

Conjecture 1.5. For any $i \neq d$,

$$\lim_{\deg(n) \rightarrow \infty} \left(\frac{h^i(M_{n,\eta}^n)}{h^d(M_{n,\eta}^n)} \right) = 0. \quad (1.8)$$

\square

This conjecture is motivated by a (vague) general principle that the largest and the most interesting cohomology group of a variety related to automorphic forms is the middle cohomology group; it is trivial to check that the conjecture is true when $d = 1$. A statement analogous to Conjecture 1.5 for Shimura varieties was proved by Clozel [3]. The same statement for the moduli varieties of \mathcal{D} -elliptic sheaves in the division algebra case follows from one of the main results in [26] (\mathcal{D} -elliptic sheaves are generalizations of Drinfeld modules).

Now consider the virtual $\text{Gal}(\bar{F}/F)$ -module $\mathcal{H} = \sum_{i \geq 0} (-1)^i H^i(M_{n,\eta}^n)$. Write \mathcal{H} as a sum of irreducible modules with integral coefficients $\mathcal{H} = \sum_{j \geq 0} a_j \mathcal{H}_j$. It is easy to see that Conjecture 1.5 implies

$$\sum_{i \geq 0} h^i(M_{n,\eta}^n) \sim \sum_{j \geq 0} |a_j| \dim_{\bar{\mathbb{Q}}_\ell} \mathcal{H}_j, \quad (1.9)$$

where \sim means that the left-hand side divided by the right-hand side tends to 1 as $\deg(n) \rightarrow \infty$.

The main result of this paper is the following evidence for Conjecture 1.4 (see Theorem 4.19).

Theorem 1.6. Fix a proper prime ideal $\mathfrak{q} \triangleleft A$. Let $\mathbb{F}_{\mathfrak{q}} := A/\mathfrak{q}$ and $\kappa := [\mathbb{F}_{\mathfrak{q}} : \mathbb{F}_{\mathfrak{q}}]$, so that $\#\mathbb{F}_{\mathfrak{q}} = q^\kappa$. Denote the degree n extension of $\mathbb{F}_{\mathfrak{q}}$ by $\mathbb{F}_{\mathfrak{q}}^{(n)}$. Let \mathfrak{p} be an admissible prime not

equal to q . Then,

$$\limsup_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{\#X_{\mathfrak{p},q}^n(\mathbb{F}_q^{(n)})}{h(X_{\mathfrak{p},q}^n)} \right) \leq \frac{1}{n} \prod_{i=1}^{n-1} (q^{\kappa^i} - 1). \quad (1.10)$$

If (1.9) is true, then

$$\lim_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{\#X_{\mathfrak{p},q}^n(\mathbb{F}_q^{(n)})}{h(X_{\mathfrak{p},q}^n)} \right) = \frac{1}{n} \prod_{i=1}^{n-1} (q^{\kappa^i} - 1). \quad (1.11)$$

□

In particular, by taking $q = (T)$, we see that Conjecture 1.5 implies Conjecture 1.4. We will also prove the following theorem (see Theorem 4.20).

Theorem 1.7. With notation of Theorem 1.6, suppose (1.9) is true. Then,

$$\lim_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{\text{WD}(X_{\mathfrak{p},q}^n, \mathbb{F}_q^{(n)})}{h(X_{\mathfrak{p},q}^n)} \right) = q^{\kappa n(n-1)/2}. \quad (1.12)$$

□

One can compare the limits in Theorems 1.6 and 1.7 from two opposite viewpoints. On the one hand, since $(1/n) \prod_{i=1}^{n-1} (q^{\kappa^i} - 1) < q^{\kappa n(n-1)/2}$, Drinfeld modular varieties $X_{\mathfrak{p},q}^n$ never have as many $\mathbb{F}_q^{(n)}$ -rational points as the Weil-Deligne bound allows when $\deg(\mathfrak{p})$ is large. On the other hand, the degree of $(1/n) \prod_{i=1}^{n-1} (q^{\kappa^i} - 1)$ as a polynomial in q is the same as the degree of $q^{\kappa n(n-1)/2}$, so $\#X_{\mathfrak{p},q}^n(\mathbb{F}_q^{(n)})$ asymptotically comes close to $\text{WD}(X_{\mathfrak{p},q}^n, \mathbb{F}_q^{(n)})$, and one can say that the varieties $X_{\mathfrak{p},q}^n$ have many $\mathbb{F}_q^{(n)}$ -rational points compared to their Betti numbers.

The organization of the paper and the outline of the proof of Theorem 1.6 are as follows: the definition and the main properties of Drinfeld modular varieties M_n^n are recalled in Section 4. In the same section we show that $X_{\mathfrak{p},q}^n$ are defined over \mathbb{F}_q and are geometrically irreducible when \mathfrak{p} is admissible. The proof relies on the analogue of the Weil pairing for Drinfeld modules constructed by van der Heiden. Next, we show that for $\deg(\mathfrak{p}) \gg 0$ the $\mathbb{F}_q^{(n)}$ -rational points on $X_{\mathfrak{p},q}^n$ are exactly the supersingular points. It is possible to compute the number of supersingular points using a mass-formula. After this we turn to $h(X_{\mathfrak{p},q}^n)$. Under the assumption (1.9), we use Laumon's proof of a special case of Langlands conjecture over function fields to reduce the calculation of the asymptotic size of $h(X_{\mathfrak{p},q}^n)$ to the calculation of the dimension of a certain space of cusp forms on GL_n . A theorem of Harder relates the dimension of this space of cusp forms to the Euler-Poincaré characteristic of the quotient $\mathcal{B}(n)$ of the Bruhat-Tits building of PGL_n under the action of level- n principal congruence subgroup of $\text{GL}_n(A)$; see Section 3.3. The calculation of the Euler-Poincaré characteristic is carried out in Section 3.2 (we have left

the most tedious part of the calculation for the appendix). Our methods are combinatorial. The final result expresses the Euler-Poincaré characteristic as a sum of the special values of a partial zeta-function of F . This can be interpreted as a Gauss-Bonnet type formula in the non-Archimedean setting, and is of independent interest. Once we know the asymptotic sizes of $h(X_{p,q}^n)$ and $\#X_{p,q}^n(\mathbb{F}_q^{(n)})$, Theorem 1.6 easily follows. For Theorem 1.7, we use the Ramanujan-Petersson conjecture proven in this setting by Laumon.

Remark 1.8. A very general problem about the asymptotics of the number of rational points on varieties over \mathbb{F}_q can be formulated as follows. Let $\mathfrak{V}(q, d)$ be the set of all smooth, geometrically connected, d -dimensional varieties over \mathbb{F}_q . Consider the set of rational numbers

$$S(q, d, n) := \left\{ \frac{\#X(\mathbb{F}_{q^n})}{h(X)} \mid X \in \mathfrak{V}(q, d) \right\}. \quad (1.13)$$

By (1.3) the elements of $S(q, d, n)$ all lie in the interval $[0, q^{dn}]$. Now one might ask what are the accumulation points of $S(q, d, n)$. This is apparently a very hard question, and it is unreasonable to expect that all accumulation points can be explicitly described. Even in the case of curves there are still some fundamental open problems. For example, the largest accumulation point of $S(p, 1, n)$ is not known for any p unless n is even, in which case the answer is $(p^{n/2} - 1)/2$. Using infinite class field towers, Serre proved [30] that the largest accumulation point of any $S(p, 1, n)$ is nonzero. Since then, many other authors have improved Serre's result; see [9], for example.

For the higher dimensional varieties there are only a few partial results. Using explicit formulae, Lachaud and Tsfasman proved certain generalizations of the Drinfeld-Vladut bound (1.4); see [22, 35]. In [35] one also finds some examples of infinite sets of smooth proper varieties whose $\#X(\mathbb{F}_{q^n})/h(X)$ can be shown to converge and the limit can be explicitly determined. These examples are obtained either by taking appropriate products of curves or by considering hypersurfaces given by Fermat-type equations. We should mention that the general results in [22] fall quite short of giving upper bounds on $\limsup(\#X_{p,q}^n(\mathbb{F}_q^{(n)})/h(X_{p,q}^n))$ of the same magnitude as in Theorem 1.6.

From this general perspective, our Conjecture 1.4 specifies an accumulation point of $S(q, d, d + 1)$.

2 Conventions

The purpose of this section is to introduce the terminology and notation which will be used in later sections of the paper.

2.1 Simplicial complexes

Recall that an n -dimensional simplex s (or an n -*simplex*, for short) is the smallest convex set in a real vector space containing $n + 1$ points v_0, v_1, \dots, v_n in general position. The points v_i are the *vertices* of the simplex s . Any simplex spanned by a subset of $\{v_0, v_1, \dots, v_n\}$ is called a *face* of s . A *simplicial complex* \mathcal{D} is a collection of simplices such that a face of a simplex of \mathcal{D} is in \mathcal{D} , and the intersection of two simplices of \mathcal{D} is a face of each of them. The *dimension* of \mathcal{D} is the supremum of the dimensions of its simplices. A subcollection \mathcal{D}' of \mathcal{D} that contains all the faces of its elements is called a *subcomplex* of \mathcal{D} . A *pointed n -simplex* is an n -simplex with a distinguished vertex.

A Δ -*complex* is a quotient space of a collection of disjoint simplices obtained by identifying certain of their faces via canonical linear homeomorphisms that preserve the ordering of vertices (cf. [20]). From the point of view of homology theory, Δ -complexes are equivalent to simplicial complexes. In fact, it is easy to see that a simplicial complex is a Δ -complex, and a Δ -complex is homeomorphic to a simplicial complex. (Note also that simplicial complexes are the Δ -complexes whose simplices are uniquely determined by their vertices.) Denote the set of i -simplices (resp., pointed i -simplices) of a Δ -complex \mathcal{D} by $S_i(\mathcal{D})$ (resp., $\widehat{S}_i(\mathcal{D})$). We also denote the set of vertices of \mathcal{D} by $\text{Ver}(\mathcal{D})$, so $\text{Ver}(\mathcal{D}) = S_0(\mathcal{D})$.

The homology groups $H_*(\mathcal{D}, R)$ (and the cohomology groups $H^*(\mathcal{D}, R)$) of a Δ -complex \mathcal{D} with coefficients in a ring R are defined in a usual manner; see [20, Chapter 2]. We simply write $H_*(\mathcal{D})$ for $H_*(\mathcal{D}, \mathbb{Q})$. Assume \mathcal{D} is n -dimensional, and $H_i(\mathcal{D})$ are finite dimensional. The *Euler-Poincaré characteristic* of \mathcal{D} is

$$\chi(\mathcal{D}) := \sum_{i=0}^n (-1)^i \dim_{\mathbb{Q}} H_i(\mathcal{D}). \quad (2.1)$$

If \mathcal{D} is finite, then, as is easy to check, $\chi(\mathcal{D}) = \sum_{i=0}^n (-1)^i \#S_i(\mathcal{D})$.

Let G be a group acting on the vertices of \mathcal{D} . We say that G *preserves the simplicial structure of \mathcal{D}* , or simply, G *acts on \mathcal{D}* , if for any n -simplex $\{v_0, \dots, v_n\}$ of \mathcal{D} and any $g \in G$ the set $\{gv_0, \dots, gv_n\}$ is also an n -simplex of \mathcal{D} . If G acts on \mathcal{D} , then one easily constructs a Δ -complex \mathcal{D}/G , which is naturally the quotient space of this action. For a simplex $w \in \mathcal{D}$, we denote by $\text{Stab}_G(w)$ or G_w the stabilizer of w in G .

2.2 Parabolic subgroups

Let n be a positive integer. An *ordered partition* of n is an expression of n as an *ordered* sum of positive integers. We will write ordered partitions as row vectors. Let $\mathbf{P}(n)$ be the set of all ordered partitions of n , so $\mathbf{p} = (p_1, \dots, p_h) \in \mathbf{P}(n)$ if $n = p_1 + \dots + p_h$,

and all $p_i \in \mathbb{Z}_{>0}$. It is easy to check that $\mathbf{P}(n)$ has 2^{n-1} elements. Define the *length* of $\mathbf{p} = (p_1, \dots, p_h)$ to be $\ell(\mathbf{p}) = h$.

To each $\mathbf{p} = (p_1, \dots, p_h) \in \mathbf{P}(n)$ we associate the subgroup $P_{\mathbf{p}}$ of GL_n consisting of matrices of the form

$$\begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1h} \\ 0 & G_{22} & \cdots & G_{2h} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & G_{hh} \end{pmatrix}, \quad (2.2)$$

where G_{ij} is a $p_i \times p_j$ block. The group $P_{\mathbf{p}}$ is a semidirect product $P_{\mathbf{p}} = M_{\mathbf{p}} \rtimes U_{\mathbf{p}}$, where $M_{\mathbf{p}} \cong GL_{p_1} \times \cdots \times GL_{p_h}$ is characterized by the condition that $G_{ij} = 0$ unless $i = j$, and the normal subgroup $U_{\mathbf{p}}$ is characterized by the condition that each G_{ii} is the identity matrix in GL_{p_i} . The groups $P_{\mathbf{p}}$ are called the *standard parabolic subgroups* of GL_n . The decomposition $P_{\mathbf{p}} = M_{\mathbf{p}} \rtimes U_{\mathbf{p}}$ is called the *Levi decomposition* of $P_{\mathbf{p}}$.

Let Λ be the set $\{2, 3, \dots, n\}$. To each subset $I \subseteq \Lambda$ we associate an ordered partition $\mathbf{p}(I) \in \mathbf{P}(n)$ as follows. First, put $\mathbf{p}(\Lambda) = (n)$. If $I \subsetneq \Lambda$, let

$$\Lambda - I = \{i_1 < i_2 < \cdots < i_k\}. \quad (2.3)$$

Now let $\mathbf{p}(I) = (i_1 - 1, i_2 - i_1, \dots, i_k - i_{k-1}, n + 1 - i_k)$. Note that $\mathbf{p}(\emptyset) = (1, 1, \dots, 1)$. It is easy to see that $I \rightarrow \mathbf{p}(I)$ is a one-to-one correspondence between the subsets of Λ and the elements of $\mathbf{P}(n)$. Denote by P_I, M_I, U_I the groups $P_{\mathbf{p}(I)}, M_{\mathbf{p}(I)}, U_{\mathbf{p}(I)}$, respectively.

2.3 Notation

From now on, unless specified otherwise, the following notation is fixed:

- (1) $n \geq 2$ is a fixed integer;
- (2) $G = GL_n$;
- (3) B is the Borel subgroup of upper-triangular matrices of G ;
- (4) Z is the center of G ;
- (5) \mathbb{F}_q is the finite field of q elements, where q is a power of the prime p ;
- (6) $A = \mathbb{F}_q[T]$ is the ring of polynomials in T with coefficients in \mathbb{F}_q ;
- (7) $F = \mathbb{F}_q(T)$ is the fraction field of A (equivalently, the field of rational functions on $\mathbb{P}_{\mathbb{F}_q}^1$);
- (8) F_v is the completion of F at the place v ;
- (9) π_v is a uniformizer of F_v ;

- (10) ord_v is the canonical valuation on F_v normalized by $\text{ord}_v(\pi_v) = 1$;
- (11) $\mathcal{O}_v = \{x \in F_v \mid \text{ord}_v(x) \geq 0\}$ is the ring of integers in F_v ;
- (12) $\mathfrak{p}_v = \pi_v \mathcal{O}_v$ is the maximal ideal of \mathcal{O}_v ;
- (13) \mathbb{F}_v is the residue field $\mathcal{O}_v/\mathfrak{p}_v$;
- (14) q_v is the order of the finite field \mathbb{F}_v ;
- (15) \mathbb{A} is the ring of adèles of F ;
- (16) \mathbb{A}^\times is the group of ideles of F ;
- (17) $\mathcal{O} = \prod_v \mathcal{O}_v$;
- (18) $\mathcal{K}(\mathfrak{n}_v) = \{M \in G(\mathcal{O}_v) \mid M \equiv 1 \pmod{\mathfrak{n}_v}\}$, where \mathfrak{n}_v is an ideal of \mathcal{O}_v , is the *principal congruence subgroup* of $G(\mathcal{O}_v)$ of level \mathfrak{n}_v .

Let n be a monic polynomial in A . We denote by the same letter the ideal generated by n in A . If $\mathfrak{p} \triangleleft A$ is a prime ideal, we denote the residue field A/\mathfrak{p} by $\mathbb{F}_\mathfrak{p}$. Consider the map $\text{deg} : A \rightarrow \mathbb{Z}$ which to each polynomial $f(T) \in A$ associates its degree in T (by convention, $\text{deg}(0) = \infty$). This induces a valuation w on F by $w(a/b) = \text{deg}(b) - \text{deg}(a)$, where $a, b \in A$. The place corresponding to this valuation is denoted by ∞ . This place will play a special role in what follows. A natural uniformizer at ∞ is $1/T$. Finally, denote

- (a) $\Gamma = G(A)$;
- (b) $\Gamma(n) = \ker(G(A) \rightarrow G(A/n))$;
- (c) $\mathbb{A}_f = \prod'_{v \neq \infty} F_v$;
- (d) $\mathbb{A}_f^\times = \prod'_{v \neq \infty} F_v^\times$;
- (e) $\mathcal{O}_f = \prod_{v \neq \infty} \mathcal{O}_v$.

Note that $\mathbb{A} = \mathbb{A}_f \times F_\infty$, $\mathbb{A}^\times = \mathbb{A}_f^\times \times F_\infty^\times$, $\mathcal{O} = \mathcal{O}_f \times \mathcal{O}_\infty$, and \mathcal{O}_f is the completion of A with respect to the ideal topology.

3 Quotients of the Bruhat-Tits building

In this section we compute the Euler-Poincaré characteristic of the quotient of the Bruhat-Tits building of $\text{PGL}_n(F_\infty)$ under the action of $\Gamma(n)$, and relate this number to the dimension of a space of cusp forms on $G(\mathbb{A})$.

3.1 Definition of the building and its basic properties

Let \mathcal{V} be an n -dimensional vector space over F_∞ . An \mathcal{O}_∞ -*lattice* in \mathcal{V} is a free finitely generated \mathcal{O}_∞ -module of maximal rank ($= n$) in \mathcal{V} . Two lattices L_1 and L_2 are *similar* if there exists $x \in F_\infty^\times$ with $x \cdot L_1 = L_2$. This defines an equivalence relation on the set of lattices in \mathcal{V} . We denote the equivalence class of L by $[L] := \{xL \mid x \in F_\infty^\times\}$. Since F_∞ is a local field, $[L]$ can be identified with $\{\pi_\infty^i L \mid i \in \mathbb{Z}\}$.

The *Bruhat-Tits building* of $\mathrm{PGL}_n(\mathbb{F}_\infty)$ is the simplicial complex \mathcal{B} with the set of vertices $\{[L] \mid L \text{ is a lattice in } \mathcal{V}\}$ and the set of i -simplices consisting of $\{[L_0], \dots, [L_i]\}$, such that there is $L'_j \in [L_j]$ for each j with

$$L'_0 \supseteq L'_1 \supseteq \dots \supseteq L'_i \supseteq \pi_\infty L'_0. \quad (3.1)$$

\mathcal{B} is $(n-1)$ -dimensional. Indeed, any i -simplex as above produces the flag

$$L'_0/\pi_\infty L'_0 \supset L'_1/\pi_\infty L'_0 \supset \dots \supset L'_i/\pi_\infty L'_0 \supset 0 \quad (3.2)$$

in the n -dimensional \mathbb{F}_q -vector space $L'_0/\pi_\infty L'_0$.

Fix a basis $E = \{e_1, \dots, e_n\}$ of \mathcal{V} . For any n -tuple $i_1, \dots, i_n \in \mathbb{Z}$ denote by $[i_1, i_2, \dots, i_n]$ the equivalence class of the lattice $\pi_\infty^{i_1} e_1 \mathcal{O}_\infty \oplus \dots \oplus \pi_\infty^{i_n} e_n \mathcal{O}_\infty$. The maximal subcomplex \mathcal{A} of \mathcal{B} having set of vertices

$$\mathrm{Ver}(\mathcal{A}) = \{[i_1, i_2, \dots, i_n] \mid i_1, \dots, i_n \in \mathbb{Z}\} \quad (3.3)$$

is called the *standard apartment* of \mathcal{B} . The maximal subcomplex \mathcal{W} of \mathcal{B} having set of vertices

$$\mathrm{Ver}(\mathcal{W}) = \{[i_1, i_2, \dots, i_n] \mid i_1 \leq i_2 \leq \dots \leq i_n\} \quad (3.4)$$

is called the *standard Weyl chamber* of \mathcal{B} . Note that every vertex of \mathcal{A} has a unique representative of the form $[0, i_2, \dots, i_n]$. We will denote $0 := [0, \dots, 0] \in \mathcal{A}$.

The group $G(\mathbb{F}_\infty)$ operates transitively on the vertices of \mathcal{B} by $g[L] := [gL]$. Since $G(\mathbb{F}_\infty)$ preserves the inclusions of lattices, it acts on \mathcal{B} . Let $G^0(\mathbb{F}_\infty)$ be the kernel of the homomorphism $G(\mathbb{F}_\infty) \rightarrow \mathbb{Z}$ given by $g \mapsto \mathrm{ord}_\infty(\det(g))$. As is easy to check, the quotient $\mathcal{X} := \mathcal{B}/G^0(\mathbb{F}_\infty)$ is naturally isomorphic to a $(n-1)$ -simplex. By enumerating the vertices of \mathcal{X} from 0 to $n-1$, we get a map

$$\mathrm{Type} : \mathrm{Ver}(\mathcal{B}) \longrightarrow \{0, \dots, n-1\}, \quad (3.5)$$

which to each $v \in \mathrm{Ver}(\mathcal{B})$ assigns the number of its image in $\mathrm{Ver}(\mathcal{X})$.

Lemma 3.1. Let Γ' be a subgroup of $G^0(\mathbb{F}_\infty)$, and let $s = \{v_0, \dots, v_m\}$ be a m -simplex of \mathcal{B} . Then, $\mathrm{Stab}_{\Gamma'}(s) = \bigcap_{i=0}^m \mathrm{Stab}_{\Gamma'}(v_i)$. \square

Proof. Let $g \in G^0(\mathbb{F}_\infty)$. Since s is a face of some $(n-1)$ -simplex, $\mathrm{Type}(v_i) = \mathrm{Type}(v_j)$ if and only if $v_i = v_j$. On the other hand, $G^0(\mathbb{F}_\infty)$ preserves the type of each vertex. Therefore, $gs = s$ if and only if $gv_i = v_i$ for $0 \leq i \leq m$. \blacksquare

Lemma 3.2. Let $v = [i_1, \dots, i_n] \in \mathcal{A}$. Then, $\text{Stab}_\Gamma(v)$ is the group of all matrices $(a_{jk}) \in \Gamma$, with $\deg(a_{jk}) \leq i_k - i_j$. \square

Proof. First, consider the stabilizer of v in $G(\mathbb{F}_\infty)$. Let $D = \text{diag}(\pi_\infty^{i_1}, \dots, \pi_\infty^{i_n})$, so that $D \cdot 0 = v$. If $gv = v$, then

$$D^{-1}gD \in \text{Stab}_{G(\mathbb{F}_\infty)}(0) = G(\mathcal{O}_\infty) \cdot Z(\mathbb{F}_\infty). \quad (3.6)$$

Hence, $\text{Stab}_{G(\mathbb{F}_\infty)}(v)$ is the group $\{(\pi_\infty^{i_j - i_k} \alpha_{jk}) \mid (\alpha_{jk}) \in G(\mathcal{O}_\infty)\} \cdot Z(\mathbb{F}_\infty)$. Now note that $\text{Stab}_\Gamma(v) = \text{Stab}_{G(\mathbb{F}_\infty)}(v) \cap \Gamma$. As $A \cap \pi_\infty^m \mathcal{O}_\infty$ is the set of polynomials of degree $\leq -m$, the claim follows. \blacksquare

Theorem 3.3. \mathcal{W} is a fundamental domain for the action of Γ on \mathcal{B} . \square

Proof. See [32]. \blacksquare

Remark 3.4. The isomorphism $\mathcal{B}/\Gamma \cong \mathcal{W}$ can be proven using some algebraic geometry (the proof in [32] is different). The idea is the following: let $I_\infty \cong \mathcal{O}_{\mathbb{P}_{\mathbb{F}_q}^1}(-1)$ be the sheaf of ideals of the point $\infty = 1/T$ on $\mathbb{P}_{\mathbb{F}_q}^1$. Two vector bundles V and V' on $\mathbb{P}_{\mathbb{F}_q}^1$ are said to be I_∞ -equivalent if there is $m \in \mathbb{Z}$ such that $V' \cong I_\infty^{\otimes m} \otimes V$. As in [31, Section II.2.1], one shows that there is a bijection between $\text{Ver}(\mathcal{B}/\Gamma)$ and the set of I_∞ -equivalence classes of rank- n vector bundles on $\mathbb{P}_{\mathbb{F}_q}^1$. On the other hand, by a theorem of Grothendieck every vector bundle V over the projective line is a direct sum of line bundles, so can be written as

$$I_\infty^{\otimes i_1} \oplus I_\infty^{\otimes i_2} \oplus \dots \oplus I_\infty^{\otimes i_n}, \quad (3.7)$$

where $i_1 \leq i_2 \leq i_3 \leq \dots \leq i_n$ (cf. [19, Corollary V.2.14]). The map

$$I_\infty^{\otimes i_1} \oplus I_\infty^{\otimes i_2} \oplus \dots \oplus I_\infty^{\otimes i_n} \longmapsto [i_1, i_2, \dots, i_n] \quad (3.8)$$

establishes a bijection between the I_∞ -equivalence classes of rank- n vector bundles and the vertices of \mathcal{W} .

3.2 Euler-Poincaré characteristic of $\mathcal{B}/\Gamma(n)$

Denote by $\mathcal{B}(n)$ the Δ -complex $\mathcal{B}/\Gamma(n)$. Clearly $\mathcal{B}(n)$ is connected since a path between two vertices in \mathcal{B} descends to a path between the images of these vertices in $\mathcal{B}(n)$.

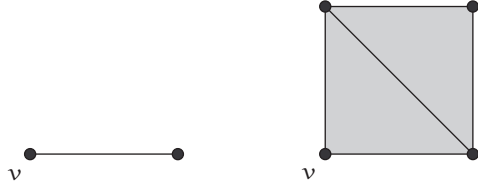


Figure 3.1 $K(v)$ for $n = 2, 3$.

Remark 3.5. One can show that $\mathcal{B}(n)$ is in fact a simplicial complex; see [11, Theorem 4.13]. (Although the running hypothesis in [11] is $n = 3$, the proof readily generalizes to an arbitrary $n \geq 2$.) A key intermediate fact which goes into the proof is the following.

Let $\{v_0, \dots, v_i\}$ and $\{u_0, \dots, u_i\}$ be i -simplices of \mathcal{B} . Suppose there are $\gamma_0, \dots, \gamma_i \in \Gamma(n)$ with $\gamma_0 v_0 = u_0, \dots, \gamma_i v_i = u_i$. Then, there is some $\gamma \in \Gamma(n)$ with $\gamma v_0 = u_0, \dots, \gamma v_i = u_i$.

This property is very specific to $\Gamma(n)$ and is false for general congruence subgroups. For example, take $n = 2$ and consider the quotient $\mathcal{B}'(n)$ of \mathcal{B} (a tree in this case) by the Hecke congruence subgroup $\Gamma_0(n)$. Then, $\mathcal{B}'(n)$ quite often has two distinct edges joining the same two vertices, that is, $\mathcal{B}'(n)$ is not a simplicial complex.

Nevertheless, treating $\mathcal{B}(n)$ as a Δ -complex will be sufficient for our purposes as we are primarily interested in its homology groups.

Define the operators d_2, d_3, \dots, d_n on $\text{Ver}(\mathcal{A})$ by

$$d_j([i_1, \dots, i_n]) = [i_1, i_2, \dots, i_{j-1}, i_j + 1, i_{j+1} + 1, \dots, i_n + 1]. \quad (3.9)$$

Note that we could have defined d_1 by the same formula, but then d_1 is simply the identity map since $[i_1 + m, \dots, i_n + m] = [i_1, \dots, i_n]$ for any $m \in \mathbb{Z}$. It is clear that d_j 's commute with each other and any vertex of \mathcal{W} can be obtained from $\mathbf{0}$ by a unique (up to permutations) sequence of d_j 's. Let $v \in \text{Ver}(\mathcal{W})$, and $v = d_2^{s_2} \cdots d_n^{s_n}(\mathbf{0})$, where $s_j \geq 0$ and if $s_j = 0$, then d_j^0 means the identity map. Define $\deg_j(v) = s_j$. The map $v \mapsto d_2^{s_2} \cdots d_n^{s_n}$ gives a one-to-one correspondence between the vertices of \mathcal{W} and the monomials in d_j 's.

For any $k \geq 0$, let \mathcal{W}_k be the maximal subcomplex of \mathcal{W} having set of vertices

$$\text{Ver}(\mathcal{W}_k) = \{v \mid \deg_2(v) \leq k, \dots, \deg_n(v) \leq k\}. \quad (3.10)$$

Let v be a vertex of \mathcal{W} . Denote by $K(v)$ the maximal subcomplex of \mathcal{W} having set of vertices

$$\text{Ver}(K(v)) = \{d_2^{s_2} \cdots d_n^{s_n}(v) \mid 0 \leq s_j \leq 1 \text{ for } 2 \leq j \leq n\}; \quad (3.11)$$

see Figure 3.1.

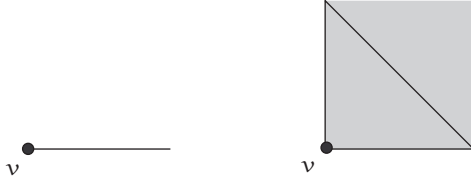


Figure 3.2 $K^0(v)$ for $n = 2, 3$.

For a fixed $2 \leq m \leq n$, define $K_m^x(v)$, for $x = 0$ or 1 , to be the subcomplex with

$$\text{Ver}(K_m^x(v)) = \{d_2^{s_2} \cdots d_n^{s_n}(v) \mid 0 \leq s_j \leq 1 \text{ for } 2 \leq j \leq n, s_m = x\}. \quad (3.12)$$

Let $K^0(v) = K(v) - \bigcup_{m=2}^n K_m^1(v)$ be the set of simplices in $K(v)$ which do not completely lie in one of the $K_m^1(v)$'s; see Figure 3.2. Note that $K^0(v)$ is not a simplicial complex since not every face of a simplex in $K^0(v)$ lies in $K^0(v)$.

Since $\Gamma(n)$ is a normal subgroup of Γ , $\Gamma(n) \backslash \Gamma$ is a group, which we denote by $\Upsilon(n)$. It is well known that $\Upsilon(n) \cong \mathbb{F}_q^\times \ltimes \text{SL}_n(A/n)$. For a simplex $w \in \mathcal{W}$, denote the image of $\Gamma_w = \text{Stab}_\Gamma(w)$ in $\Upsilon(n)$ by $\overline{\Gamma}_w$.

Definition 3.6. Let \mathcal{S} be a set of simplices in \mathcal{W} (e.g., a simplicial subcomplex). Define

$$\tilde{\chi}(\mathcal{S}) = \sum_{w \in \mathcal{S}} (-1)^{\dim(w)} (\#\overline{\Gamma}_w)^{-1}. \quad (3.13)$$

Notation 3.7. For $m \geq 1$, let

$$\phi(m) = (-1)^{m+1} \left(\prod_{i=1}^m (q^i - 1) \right)^{-1}. \quad (3.14)$$

Proposition 3.8.

$$\tilde{\chi}(K^0(v)) = \begin{cases} \phi(n), & \text{if } v = \mathbf{0}; \\ 0, & \text{otherwise.} \end{cases} \quad (3.15)$$

□

Proof. See the appendix. ■

Definition 3.9. A vertex $v \in \text{Ver}(\mathcal{W}_k)$ is a *corner* of \mathcal{W}_k if $\deg_j(v)$ is equal either to 0 or k for all $2 \leq j \leq n$. Clearly, \mathcal{W}_k has 2^{n-1} corners.

Let v be a corner of \mathcal{W}_k . Let $I_0(v) \subset \{2, \dots, n\}$ be the set of indices i such that $\deg_i(v) = 0$, and let $I_k(v)$ be the complement of $I_0(v)$ in $\{2, \dots, n\}$. Let $\mathbb{K}(v)$ be the set of simplices of the form $\{v, s\}$, where $\text{Ver}(s)$ is a subset (possibly empty) of the set $\{d_i(v) \mid i \in I_0(v)\}$. Note that $\mathbb{K}(v) = v$ if and only if $v = d_2^k d_3^k \cdots d_n^k(\mathbf{0})$.

Proposition 3.10. With notation as above,

$$\tilde{\chi}(\mathcal{W}_k) = \sum_v \tilde{\chi}(\mathbb{K}(v)), \quad (3.16)$$

where the sum is over all corners of \mathcal{W}_k . \square

Proof. Let v be a corner. Let $\mathcal{W}_k(v)$ be the maximal subcomplex of \mathcal{W}_k having set of vertices

$$\text{Ver}(\mathcal{W}_k(v)) = \{v' \mid v' \in \text{Ver}(\mathcal{W}_k), \deg_j(v') = k \text{ if } j \in I_k(v)\}. \quad (3.17)$$

In particular, $\mathcal{W}_k(\mathbf{0}) = \mathcal{W}_k$. Let $\Xi(v)$ be the subset of corners of \mathcal{W}_k contained in $\mathcal{W}_k(v)$; we denote the set of all corners of \mathcal{W}_k by Ξ (so $\Xi = \Xi(\mathbf{0})$). Denote by

$$\mathcal{W}_k^0(v) = \mathcal{W}_k(v) - \bigcup_{\substack{y \in \Xi(v) \\ y \neq v}} \mathcal{W}_k(y), \quad (3.18)$$

the set of simplices of $\mathcal{W}_k(v)$ which are not completely contained in one of $\mathcal{W}_k(y)$, $y \in \Xi(v)$, $y \neq v$. Clearly \mathcal{W}_k is the disjoint union $\coprod_{v \in \Xi} \mathcal{W}_k^0(v)$. Hence,

$$\tilde{\chi}(\mathcal{W}_k) = \sum_{v \in \Xi} \tilde{\chi}(\mathcal{W}_k^0(v)). \quad (3.19)$$

Next, since $\mathcal{W}_k^0(\mathbf{0})$ is the disjoint union $\coprod_{v \in \mathcal{W}_{k-1}} K^0(v)$, the proof of Proposition 3.8 implies $\tilde{\chi}(\mathcal{W}_k^0(\mathbf{0})) = \tilde{\chi}(\mathbb{K}(\mathbf{0}))$. Note that each $\mathcal{W}_k^0(v)$, $v \neq \mathbf{0}$, is isomorphic to $\mathcal{W}_k^0(\mathbf{0})$ in the building of $\text{PGL}_m(\mathbb{F}_\infty)$ for some $m < n$, so one can adapt the argument for $v = \mathbf{0}$ to an arbitrary corner (essentially by induction) to show that $\tilde{\chi}(\mathcal{W}_k^0(v)) = \tilde{\chi}(\mathbb{K}(v))$ for any $v \in \Xi$. Combined with (3.19), this proves the proposition. \blacksquare

For $v \in \Xi$, let $\mathbf{p}_v := \mathbf{p}(I_0(v)) \in \mathbf{P}(n)$. This gives a one-to-one correspondence between the corners of \mathcal{W}_k and the elements of $\mathbf{P}(n)$. For $\mathbf{p} = (p_1, \dots, p_h) \in \mathbf{P}(n)$, let

$$\theta(\mathbf{p}) = n^2 - \sum_{i=1}^h \sum_{j=1}^h p_i p_j = \sum_{i < j} p_i p_j, \quad \Phi(\mathbf{p}) = \phi(p_1) \cdots \phi(p_h). \quad (3.20)$$

Proposition 3.11. Let v be a corner of \mathcal{W}_k . Let \mathbf{p}_v be the corresponding partition of n . Let $d = \deg(n)$. Assume $k \geq d - 1$. Then,

$$\tilde{\chi}(\mathbb{K}(v)) = \Phi(\mathbf{p}_v) \cdot q^{-d \cdot \theta(\mathbf{p}_v)}. \quad (3.21)$$

□

Proof. Given a partition \mathbf{p} , let us denote by $G_{\mathbf{p}}$ the group $M_{\mathbf{p}}(\mathbb{F}_q)U_{\mathbf{p}}(A/n)$.

Let σ be an i -simplex in $\mathbb{K}(v)$. Write $\sigma = \{v, d_{j_1}(v), \dots, d_{j_i}(v)\}$, where $j_1, \dots, j_i \in I_0(v)$. Let $\mathbf{p}_{\sigma} := \mathbf{p}(I_0(v) - \{j_1, \dots, j_i\})$. Using Lemmas 3.1 and 3.2, it is easy to check that if $k \geq d - 1$, then $\overline{\Gamma}_{\sigma} = G_{\mathbf{p}_v} \cap G_{\mathbf{p}_{\sigma}}$. Now the claim of the proposition follows from a simple calculation, similar to the calculations in the proof of Proposition 3.8. ■

Theorem 3.12. Let $d = \deg(n)$. Then,

$$\chi(\mathcal{B}(n)) = [\Gamma : \Gamma(n)] \sum_{\mathbf{p} \in \mathcal{P}(n)} \Phi(\mathbf{p}) \cdot q^{-d\theta(\mathbf{p})}. \quad (3.22)$$

□

Proof. Denote by $\mathcal{B}_k(n)$ the subcomplex of $\mathcal{B}(n)$ which maps onto \mathcal{W}_k under the quotient map $\mathcal{B}(n)/\Upsilon(n) \xrightarrow{\sim} \mathcal{W}$. Let $w \in \mathcal{W}_k$ be an i -simplex. The number of i -simplices in $\mathcal{B}(n)$ which map to w is equal to $[\Upsilon(n) : \overline{\Gamma}_w]$. Hence, using Euler's formula,

$$\chi(\mathcal{B}_k(n)) = \sum_{w \in \mathcal{W}_k} (-1)^{\dim(w)} [\Upsilon(n) : \overline{\Gamma}_w] = [\Gamma : \Gamma(n)] \cdot \tilde{\chi}(\mathcal{W}_k). \quad (3.23)$$

From Propositions 3.10 and 3.11, we conclude that there is an equality $\chi(\mathcal{B}_k(n)) = \chi(\mathcal{B}_{d-1}(n))$ for any $k \geq d - 1$. On the other hand, since $\mathcal{B}_k(n) \subset \mathcal{B}_{k+1}(n)$ and $\bigcup_{k=1}^{\infty} \mathcal{B}_k(n) = \mathcal{B}(n)$, Harder's results in [16, 18] imply that there are isomorphism $H_i(\mathcal{B}(n)) \cong H_i(\mathcal{B}_k(n))$, $0 \leq i \leq n - 1$, when k is large enough. Thus, $\chi(\mathcal{B}(n)) = \chi(\mathcal{B}_{d-1}(n))$ (see also [20, Proposition 3.33]). Now the formula of the theorem follows from Propositions 3.10 and 3.11. ■

Remark 3.13. We should mention that by a different method Stuhler proved a formula for the Euler characteristic of the quotient of \mathcal{B} under the action of a small enough arithmetic subgroup of SL_n ; see [33, Section 4, Theorem 3]. His formula is valid over arbitrary function fields but is less explicit than the formula in Theorem 3.12. The proof we presented above is more elementary than Stuhler's argument, being tailored to the specific situation we are interested in. Indeed, we use the fact that \mathcal{B}/Γ has a very simple structure (this is not true for higher genus function fields), and deduce the formula for $\chi(\mathcal{B}(n))$ from an identity for q -multinomial coefficients (Lemma A.2).

Example 3.14. Let $n = 2$. We compute $\chi(\mathcal{B}(n))$ by applying the formula in Theorem 3.12. The ordered partitions of 2 are (2) and (1, 1). Now $\theta((2)) = 0$, $\theta((1, 1)) = 1$, and

$$\phi(1) = \frac{1}{q-1}, \quad \phi(2) = \frac{-1}{(q^2-1)(q-1)}. \quad (3.24)$$

Hence,

$$\frac{\chi(\mathcal{B}(n))}{[\Gamma : \Gamma(n)]} = \frac{-1}{(q^2-1)(q-1)} + \frac{1}{(q-1)^2 q^d} = \frac{1}{(q-1)^2} \left(\frac{1}{q^d} - \frac{1}{q+1} \right). \quad (3.25)$$

This recovers the formula in [15, Corollary 5.8]; see also [31, Chapter II].

Example 3.15. Let $n = 3$. There are four ordered partitions of 3, namely, (3), (2, 1), (1, 2), (1, 1, 1). We have

$$\theta((3)) = 0, \quad \theta((2, 1)) = \theta((1, 2)) = 2, \quad \theta((1, 1, 1)) = 3. \quad (3.26)$$

Next, $\phi(3) = [(q^3-1)(q^2-1)(q-1)]^{-1}$. Thus,

$$\begin{aligned} \frac{\chi(\mathcal{B}(n))}{[\Gamma : \Gamma(n)]} &= \phi(3) + 2\phi(1)\phi(2)q^{-2d} + \phi(1)^3 q^{-3d} \\ &= \frac{1}{(q-1)^3} \left(\frac{1}{(q^2+q+1)(q+1)} - \frac{2}{(q+1)q^{2d}} + \frac{1}{q^{3d}} \right). \end{aligned} \quad (3.27)$$

This recovers the formula in [11, Corollary 6.11].

Remark 3.16. As we mentioned earlier, $[\Gamma : \Gamma(n)] = (q-1)\#\mathrm{SL}_n(A/n)$. Therefore, this number can be expressed in terms of q and the degrees of primary components of n . For example, assume $n = p_1 p_2 \cdots p_s$ is square-free (p_i are prime). Let $d_i := \deg(p_i)$. Then,

$$[\Gamma : \Gamma(n)] = (q-1) \frac{\prod_{i=1}^s \prod_{j=0}^{n-1} (q^{n d_i} - q^{j d_i})}{\prod_{i=1}^s (q^{d_i} - 1)}. \quad (3.28)$$

Corollary 3.17.

$$\lim_{\deg(n) \rightarrow \infty} \frac{\chi(\mathcal{B}(n))}{[\Gamma : \Gamma(n)]} = \phi(n). \quad (3.29)$$

□

Proof. This is a trivial consequence of Theorem 3.12. ■

Remark 3.18. Let $\zeta_F(s) = \prod_v (1 - q_v^{-s})^{-1}$ be the zeta-function of F , where the product is over all valuations of F . It is well known (and is easy to show) that

$$\zeta_F(s) = \frac{1}{(1 - q^{-s})(1 - q^{1-s})}. \quad (3.30)$$

Let

$$\zeta_{F,\infty}(s) := \prod_{v \neq \infty} (1 - q_v^{-s})^{-1} = \frac{1}{1 - q^{1-s}}. \quad (3.31)$$

For any $m \geq 1$,

$$\prod_{i=0}^{m-1} \zeta_{F,\infty}(-i) = \frac{1}{(1 - q)(1 - q^2) \cdots (1 - q^m)} = -\phi(m). \quad (3.32)$$

Hence, Theorem 3.12 relates the Euler-Poincaré characteristic of $\mathcal{B}(n)$ (equivalently, the Euler-Poincaré characteristic of $\Gamma(n)$) to the values at negative integers of the partial zeta-function $\zeta_{F,\infty}$. This can be interpreted as a Gauss-Bonnet type formula; see [29]. The contributions of the “cusps” correspond to the contributions of the corners different from 0. Corollary 3.17 says that these contributions are minuscule when $\deg(n)$ is large.

3.3 Harder’s theorem

To state the main result of this section, we need to recall some notions from the theory of automorphic forms.

For a subset $I \subseteq \{2, \dots, n\}$ let V_I be the vector space of \mathbb{C} -valued locally constant functions on $G(F_\infty)$ which are left $P_I(F_\infty)$ -invariant. Let (V_I, ρ_I) be the representation $\rho_I : G(F_\infty) \rightarrow \text{End}(V_I)$ of $G(F_\infty)$ induced by right translations on $P_I(F_\infty) \backslash G(F_\infty)$. Each (V_I, ρ_I) is a subrepresentation of $(V_\emptyset, \rho_\emptyset)$ since any $P_I(F_\infty)$ -invariant function is automatically $B(F_\infty)$ -invariant ($B = P_\emptyset$). The *special* (or *Steinberg*) representation of $G(F_\infty)$ is the representation on the space $V_\emptyset / \sum_{I \neq \emptyset} V_I$. We will denote this representation by Sp .

Let \mathcal{K} be an open subgroup of $G(\mathcal{O})$. An *automorphic cusp form* for \mathcal{K} is a \mathbb{C} -valued function φ on $G(\mathbb{A})$ which is left $G(F)$ -invariant, right $\mathcal{K} \cdot Z(F_\infty)$ -invariant and which satisfies the condition

$$\int_{U_I(F) \backslash U_I(\mathbb{A})} \varphi(ug) du = 0 \quad (3.33)$$

for each I and $g \in G(\mathbb{A})$ (here du is a normalized Haar measure on the compact group $U_I(F) \backslash U_I(\mathbb{A})$). Denote the \mathbb{C} -vector space of automorphic cusp forms for \mathcal{K} by $W(\mathcal{K})$.

We say that the cusp form φ is *special at ∞* if the right $G(\mathbb{F}_\infty)$ -translates of φ generate a $G(\mathbb{F}_\infty)$ -module isomorphic with a direct sum of a finite number of copies of Sp . Denote the subspace of $W(\mathcal{K})$ spanned by the cusp forms which are special at ∞ by $W_{\text{sp}}(\mathcal{K})$.

Let \mathcal{J} be the *Iwahori subgroup* of $G(\mathbb{F}_\infty)$. By definition, \mathcal{J} is the inverse image of $B(\mathbb{F}_\infty)$ under the reduction modulo \mathfrak{p}_∞ homomorphism $G(\mathcal{O}) \rightarrow G(\mathbb{F}_\infty)$. Let \mathfrak{n} be an ideal in A . For a place $v \neq \infty$ of F let \mathfrak{n}_v be the ideal generated by \mathfrak{n} in \mathcal{O}_v under the injection $A \rightarrow \mathcal{O}_v$. Let $\mathcal{K}(\mathfrak{n})_f = \prod_{v \neq \infty} \mathcal{K}(\mathfrak{n}_v)$. Note that $\mathcal{K}(\mathfrak{n})_f$ is the adelic version of $\Gamma(\mathfrak{n})$, and in fact, $\Gamma(\mathfrak{n}) = G(F) \cap \mathcal{K}(\mathfrak{n})_f$. Let $\mathcal{K}(\mathfrak{n}) = \mathcal{K}(\mathfrak{n})_f \times \mathcal{J}$. Denote $W_{\text{sp}}(\mathcal{K}(\mathfrak{n}))$ simply by $W_{\text{sp}}(\mathfrak{n})$.

Theorem 3.19 (Harder). The space $W_{\text{sp}}(\mathfrak{n})$ is finite dimensional, and

$$\dim_{\mathbb{C}} W_{\text{sp}}(\mathfrak{n}) = \left[(A/\mathfrak{n})^\times : \mathbb{F}_q^\times \right] \cdot \dim_{\mathbb{Q}} H_{n-1}(\mathcal{B}(\mathfrak{n})). \quad (3.34)$$

For all $0 < i < n - 1$,

$$H_i(\mathcal{B}(\mathfrak{n})) = 0. \quad (3.35)$$

□

We briefly indicate the ideas which go into the proof of this deep result. For the proof itself see [18] (and also [16, 17]).

First, one shows that $H^i(\mathcal{B}(\mathfrak{n}))$ is canonically isomorphic to $H^i(\Gamma(\mathfrak{n}), \mathbb{Q})$ for any $0 \leq i \leq n - 1$; the argument is outlined in [29, Section 1.6, Remark 1]. For a discrete cocompact subgroup Γ' of $G(\mathbb{F}_\infty)$, Serre conjectured that $H^i(\Gamma', \mathbb{Q})$ vanish for $0 < i < n - 1$. This conjecture was initially proven by Garland [10], under the assumption that q is large enough, and by Casselman [2] in general. Garland's argument relates the vanishing of cohomology groups to the estimates of the eigenvalues of a certain combinatorial Laplace operator; Casselman's argument uses the theory of admissible representations of $G(\mathbb{F}_\infty)$. For a non-cocompact congruence subgroup of $G(\mathcal{O}_\infty)$, such as $\Gamma(\mathfrak{n})$, the vanishing of the middle cohomology groups was proven by Harder, using representation-theoretic methods similar to Casselman's.

Now we discuss the first part of Theorem 3.19. Let \mathcal{K} be an open subgroup of $G(\mathcal{O})$. Since $G(\mathcal{O})$ is compact, \mathcal{K} has finite index in $G(\mathcal{O})$. Write $\mathcal{K} = \mathcal{K}_f \times \mathcal{K}_\infty$, where \mathcal{K}_f is an open subgroup of $G(\mathcal{O}_f)$ and \mathcal{K}_∞ is an open subgroup of $G(\mathcal{O}_\infty)$. As results from the *strong approximation theorem* for SL_n , the determinant induces a bijection

$$G(F) \backslash G(\mathbb{A}_f) / \mathcal{K}_f \xrightarrow{\cong} F^\times \backslash \mathbb{A}_f^\times / \det \mathcal{K}_f, \quad (3.36)$$

where $G(F)$ is embedded diagonally into $G(\mathbb{A})$. It is well known that

$$F^\times \backslash \mathbb{A}_f^\times / \mathcal{O}_f^\times \cong \text{Pic}(A) = 1. \quad (3.37)$$

We conclude that the double coset space on the left-hand side of (3.36) is finite, as $\det \mathcal{K}_f$ has finite index in \mathcal{O}_f^\times . Let S denote a set of representatives of this finite coset space, and for $x \in S$ let $\Gamma_x := G(F) \cap x\mathcal{K}_f x^{-1}$, where the intersection takes place in the group $G(\mathbb{A}_f)$. Each Γ_x is an arithmetic subgroup of $G(F)$. We get the bijection

$$G(F) \backslash G(\mathbb{A}) / \mathcal{K}Z(F_\infty) \xrightarrow{\cong} \bigsqcup_{x \in S} \Gamma_x \backslash G(F_\infty) / \mathcal{K}_\infty Z(F_\infty). \quad (3.38)$$

Since the stabilizer of a maximal flag in \mathbb{F}_∞^n is isomorphic to $B(\mathbb{F}_\infty)$, the stabilizer in $G(F_\infty)$ of a pointed $(n-1)$ -simplex of \mathcal{B} is isomorphic to J . Therefore,

$$\begin{aligned} \text{PGL}_n(F_\infty) / J &\cong \widehat{S}_{n-1}(\mathcal{B}), \\ G(F) \backslash G(\mathbb{A}) / \mathcal{K}(n)Z(F_\infty) &\xrightarrow{\cong} \bigsqcup_{x \in S} \widehat{S}_{n-1}(\mathcal{B}(n)_x), \end{aligned} \quad (3.39)$$

where $\mathcal{B}(n)_x$ denotes the quotient of \mathcal{B} by $\Gamma(n)_x := G(F) \cap x\mathcal{K}(n)_f x^{-1}$. Note that all $\mathcal{B}(n)_x$ are isomorphic to $\mathcal{B}(n)_1 = \mathcal{B}(n)$. Next, it is not hard to check that

$$F^\times \backslash \mathbb{A}_f^\times / \det \mathcal{K}(n)_f \cong (A/n)^\times / \mathbb{F}_q^\times, \quad (3.40)$$

compare [21, equation (6.6)]. Hence $\#S = [(A/n)^\times : \mathbb{F}_q^\times]$. The upshot is that

$$W_{\text{sp}}(\mathfrak{n}) \xrightarrow{\cong} \bigoplus_{x \in (A/n)^\times / \mathbb{F}_q^\times} H_1^{n-1}(\mathcal{B}, \mathbb{C})^{\Gamma(n)_x}, \quad (3.41)$$

where $H_1^{n-1}(\mathcal{B}, \mathbb{C})^{\Gamma(n)_x}$ is a space of \mathbb{C} -valued functions on the pointed $(n-1)$ -simplices of \mathcal{B} which satisfy some conditions. These conditions turn out to be the following (cf. [4, 10]): f is in $H_1^{n-1}(\mathcal{B}, \mathbb{C})^{\Gamma(n)_x}$ if and only if (i) f is a cochain; (ii) f is harmonic, which means f is in the kernel of a certain operator δ acting on the cochains; (iii) f is $\Gamma(n)_x$ -invariant and has finite support modulo $\Gamma(n)_x$. The final step consists of showing

$$H_1^{n-1}(\mathcal{B}, \mathbb{C})^{\Gamma(n)_x} \cong H^{n-1}(\mathcal{B}(n)_x, \mathbb{C}), \quad (3.42)$$

which is a non-Archimedean version of Hodge decomposition.

Corollary 3.20.

$$\lim_{\deg(\mathfrak{n}) \rightarrow \infty} \frac{[(A/\mathfrak{n})^\times : \mathbb{F}_q^\times] \cdot [\Gamma : \Gamma(\mathfrak{n})]}{\dim_{\mathbb{C}} W_{\text{sp}}(\mathfrak{n})} = \prod_{i=1}^n (q^i - 1). \quad (3.43)$$

□

Proof. First of all, Theorem 3.19 implies

$$\chi(\mathcal{B}(\mathfrak{n})) = 1 + (-1)^{n-1} \dim_{\mathbb{Q}} H_{n-1}(\mathcal{B}(\mathfrak{n})). \quad (3.44)$$

(Recall that $\mathcal{B}(\mathfrak{n})$ is connected, so $H_0(\mathcal{B}(\mathfrak{n})) \cong \mathbb{Q}$.) The rest is a trivial consequence of Corollary 3.17 and the first part of Harder's theorem. ■

4 Drinfeld modular varieties

In this section we recall the definition of Drinfeld modules and Drinfeld modular schemes, and then compare the number of $\mathbb{F}_q^{(n)}$ -rational points on Drinfeld modular varieties over \mathbb{F}_q to their ℓ -adic Betti numbers.

4.1 Preliminaries

Let S be a scheme over A . Denote by γ the canonical ring homomorphism $\gamma : A \rightarrow H^0(S, \mathcal{O}_S)$. Fix some $n \in \mathbb{Z}_{>0}$. A pair $D = (\mathcal{G}, \varphi)$ consisting of an \mathbb{F}_q -vector space scheme \mathcal{G} over S and an \mathbb{F}_q -algebra homomorphism

$$\varphi : A \longrightarrow \text{End}_S(\mathcal{G}), \quad a \longmapsto \varphi_a \quad (4.1)$$

from A into the ring of \mathbb{F}_q -linear S -endomorphisms of \mathcal{G} is called a *Drinfeld module of rank n over S* if the following conditions are satisfied:

- (1) the group scheme \mathcal{G} is Zariski-locally isomorphic to the additive group scheme $\mathbb{G}_{a,S}$ over S ;
- (2) for each nonzero $a \in A$, φ_a is finite flat of degree $|a|_\infty^n$;
- (3) the induced action on the tangent space at the identity is via the structure map γ .

The *characteristic* of D is the image of S in $\text{Spec}(A)$ under $\gamma^* : S \rightarrow \text{Spec}(A)$.

Example 4.1. When S is the spectrum of a field K , the definition of a Drinfeld module over S can be reformulated as follows. Let $K\{\tau\}$ be the noncommutative ring of polynomials in τ with coefficients in K , and the commutation rule $\tau\alpha = \alpha^q\tau$ for all $\alpha \in K$. Let $\gamma : A \rightarrow K$ be

the structure homomorphism. A Drinfeld module D over K of rank n is an \mathbb{F}_q -linear ring homomorphism $\varphi : A \rightarrow K\{\tau\}$, such that

$$\varphi_T = \gamma(T) + \alpha_1\tau + \cdots + \alpha_n\tau^n, \quad (4.2)$$

where $\alpha_1, \dots, \alpha_n \in K$ and $\alpha_n \neq 0$.

Over a field we will usually denote the Drinfeld module $D = (\mathcal{G}, \varphi)$ simply by φ .

For $\alpha \in A$, we denote by ${}_\alpha\varphi := \ker(\varphi_\alpha)$ the finite flat group scheme over S of α -division points of D ; ${}_\alpha\varphi$ is a subscheme of A -modules of $\mathbb{G}_{\alpha, S}$ via φ . For an ideal $\mathfrak{n} \triangleleft A$, let

$${}_n\varphi = \bigcap_{\alpha \in \mathfrak{n}} {}_\alpha\varphi \quad (4.3)$$

be the n -torsion subgroup of D . Since A is a principal ideal domain, ${}_n\varphi = {}_\alpha\varphi$ for any generator α of \mathfrak{n} . If \mathfrak{n} is disjoint from the characteristic of D , then ${}_n\varphi$ is locally constant with value $(A/\mathfrak{n})^n$ for the étale topology on S .

Definition 4.2. Assume \mathfrak{n} is disjoint from the characteristic of D . A level n -structure on D is an isomorphism of schemes of (A/\mathfrak{n}) -modules over S :

$$\lambda : (A/\mathfrak{n})_S^n \longrightarrow {}_n\varphi, \quad (4.4)$$

where $(A/\mathfrak{n})_S^n$ is the constant scheme of (A/\mathfrak{n}) -modules over S with value $(A/\mathfrak{n})^n$.

Theorem 4.3 (Drinfeld). Let $\mathcal{M}^n(n)$ be the functor which to each $A[n^{-1}]$ -scheme S associates the set of isomorphism classes $(D, \lambda)_S$ of Drinfeld A -modules D of rank n over S with level n -structure λ . If $\mathfrak{n} \neq A$, then $\mathcal{M}^n(n)$ is representable by a smooth affine $A[n^{-1}]$ -scheme $M^n(n)$ of relative dimension $(n-1)$. \square

Proof. See [7, Section 5] or [24, Chapter 1]. \blacksquare

Remark 4.4. On the contrary, $M^n(1)$ is not smooth if $n \geq 3$. Its compactification is the weighted projective space $\mathbb{P}_A(q-1, q^2-1, \dots, q^n-1)$.

The group $G(A/\mathfrak{n})$ acts on the right of $M^n(n)$. In terms of the moduli problem the action of $g \in G(A/\mathfrak{n})$ is given by

$$g : (D, \lambda) \longmapsto (D, \lambda \circ g). \quad (4.5)$$

From now on we assume that $\mathfrak{n} \neq A$ and $n \geq 2$.

Theorem 4.5. There is an $A[n^{-1}]$ -morphism

$$w_n : M^n(n) \longrightarrow M^1(n), \quad (4.6)$$

which is $G(A/n)$ -equivariant, in the sense that for $g \in G(A/n)$,

$$w_n \circ g = \det(g) \circ w_n. \quad (4.7)$$

□

Proof. See [36, Theorem 4.1]. The proof uses Anderson's theory of t-motives. The morphism w_n is induced by an analogue of the Weil pairing for Drinfeld modules.

We sketch an alternative construction. By a deep observation of Drinfeld [8], one knows that the category of rank- n Drinfeld modules (with level structures) is equivalent to the category of rank- n elliptic sheaves (with level structures). A rank- n elliptic sheaf \mathcal{E} is essentially a rank- n vector bundle equipped with certain modifications. It is easy to check that the determinant $\det(\mathcal{E})$ is a rank-1 elliptic sheaf. The functor $\mathcal{E} \mapsto \det(\mathcal{E})$ induces the desired morphism w_n . ■

Let F_n be the function field of $M^1(n)$. From class field theory it is known that F_n is the maximal abelian extension of F with conductor n which is completely split at ∞ ; moreover, $\text{Gal}(F_n/F) \cong (A/n)^\times / \mathbb{F}_q^\times$; see [7, Theorem 1] and [21].

Corollary 4.6. The fibres of $w_n : M^n(n) \rightarrow M^1(n)$ are smooth and geometrically irreducible. □

Proof. From Theorem 4.5 and the above paragraph it is clear that w_n is surjective, and moreover, $M^1(n)$ has $[(A/n)^\times : \mathbb{F}_q^\times]$ geometrically connected components. The theory of rigid-analytic uniformization implies that $M^n(n)$, $n \geq 2$, also has $[(A/n)^\times : \mathbb{F}_q^\times]$ geometrically irreducible components, each (as an analytic variety) isomorphic to $\Omega^n / \Gamma(n)$; see [7, Section 6]. (Here Ω^n is Drinfeld's symmetric space.) The claim follows since w_n commutes with base change to any $A[n^{-1}]$ -field. ■

Recall from Definition 1.3 that a prime $p \triangleleft A$ is admissible if $x \mapsto x^n$ is an automorphism of $\mathbb{F}_p^\times / \mathbb{F}_q^\times$.

Lemma 4.7. There are infinitely many admissible primes. □

Proof. Let $d := \deg(p)$. We need to show that there are infinitely many d such that $N_d := (q^d - 1)/(q - 1)$ is coprime to n . Let ℓ be a prime divisor of n . If $\ell = p$, then N_d is not divisible by ℓ for any $d \neq 0$. From now on assume $\ell \neq p$. If $q \equiv 1 \pmod{\ell}$, then N_d is divisible by ℓ if and only if $d \equiv 0 \pmod{\ell}$. If $q \not\equiv 1 \pmod{\ell}$, then N_d is divisible by ℓ if and only if $q^d \equiv 1 \pmod{\ell}$. This last congruence holds only if $(d, \ell - 1) \neq 1$.

Let ℓ_1, \dots, ℓ_s be the prime divisors of n which divide $(q-1)$. Let p_1, \dots, p_r be the prime divisors of n which do not divide $(q-1)$ and are not equal to p . From the previous paragraph, we conclude that any p which has degree coprime to $\ell_1 \cdots \ell_s \cdot (p_1-1) \cdots (p_r-1)$ is admissible. ■

Let $Z(A/n) \triangleleft G(A/n)$ be the subgroup of scalar matrices. For a prime $q \triangleleft A[n^{-1}]$, let $M_q^n(n) := M^n(n) \otimes_{A[n^{-1}]} \mathbb{F}_q$, and

$$X_{n,q}^n := (M_q^n(n))/Z(A/n). \quad (4.8)$$

Proposition 4.8. Assume p is an admissible prime not equal to q . Then, $X_{p,q}^n$ is a smooth, absolutely irreducible, $(n-1)$ -dimensional affine variety defined over \mathbb{F}_q , which is a form of one of the components of $M_q^n(p)$. □

Proof. By Corollary 4.6, the fibres of $M_q^n(p) \rightarrow M_q^1(p)$ are smooth and geometrically irreducible. Hence by Theorem 4.5 and [36, Lemma 4.2], the fibres of $X_{p,q}^n \rightarrow M_q^1(p)/\det(Z(\mathbb{F}_p))$ are absolutely irreducible. On the other hand, since p is an admissible prime, $\det(Z(\mathbb{F}_p))$ surjects onto $(A/p)^\times/\mathbb{F}_q^\times$. Therefore, $M_q^1(p)/\det(Z(\mathbb{F}_p))$ is $\text{Spec}(\mathbb{F}_q)$. The claim of the proposition follows. ■

4.2 Rational points

Let K be any A -field, and φ be a rank- n Drinfeld module over K . Let L be a field extension of K . Denote by $\text{End}_L(\varphi)$ the centralizer of $A \xrightarrow{\cong} \varphi(A)$ in $L\{\tau\}$. More concretely, $\text{End}_L(\varphi)$ consists of all $u \in L\{\tau\}$ such that $u \cdot \varphi_a = \varphi_a \cdot u$ for all $a \in A$. Let $\text{Aut}_L(\varphi) := \text{End}_L(\varphi)^\times$. If φ has rank n then, as is easy to check, $\text{Aut}_L(\varphi)$ is the subgroup $\mathbb{F}_{q^s}^\times$ of $\mathbb{F}_{q^n}^\times$ for some s dividing n . We denote $\text{End}_{\overline{K}}(\varphi)$ by $\text{End}(\varphi)$, and similarly for $\text{Aut}_{\overline{K}}(\varphi)$. It is known that $\text{End}(\varphi)$ is a free A -module of rank less than or equal to n^2 ; see [7, Section 2].

Fix some prime q . Denote the degree m extension of \mathbb{F}_q by $\mathbb{F}_q^{(m)}$. Let K be a finite extension of \mathbb{F}_q , and let $\text{Fr}_K : x \mapsto x^{\#K}$ be the associated (arithmetic) Frobenius morphism. It is clear that $\text{Fr}_K \in \text{End}_K(\varphi)$. If $K = \mathbb{F}_q$, then we write Fr_q for Fr_K , so the Frobenius of $\mathbb{F}_q^{(m)}$ is $\text{Fr}_q^m = \tau^{m \cdot \deg(q)}$.

Let $l \triangleleft A$ be a prime different from q . One defines the *l-adic Tate module* of φ to be

$$T_l(\varphi) := \varprojlim_{\frac{s}{l}} \varphi(\overline{\mathbb{F}_q}), \quad (4.9)$$

where the transition morphisms are given by φ_a for any generator a of l . It is easy to see that $T_l(\varphi)$ is a free A_l -module of rank n . On $T_l(\varphi)$ we have a natural action of $\text{End}(\varphi)$. The

associated homomorphism

$$i_l : \text{End}(\varphi) \otimes_A A_l \longrightarrow \text{End}_{A_l}(\Gamma_l(\varphi)) \quad (4.10)$$

is injective; see [7, Proposition 2.4]. Denote by $P_\varphi(X)$ the characteristic polynomial of $i_l(\text{Fr}_K)$.

Proposition 4.9. The characteristic polynomial $P_\varphi(X)$ is a monic degree- n polynomial in $A[X]$ whose coefficients are independent of l . The zeros x_i of $P_\varphi(X)$ in \bar{F}_∞ satisfy $|x_i|_\infty = (\#K)^{1/n}$, where $|\cdot|_\infty$ is the extension of the normalized absolute value on F_∞ . \square

Proof. See [13, Corollary 3.4 and Theorem 5.1]. \blacksquare

Proposition 4.10. Some power of Fr_K lies in $\varphi(A)$ if and only if ${}_q\varphi(\bar{\mathbb{F}}_q) = 0$. \square

Proof. See [13, Proposition 4.1]. \blacksquare

A Drinfeld module φ in characteristic q with ${}_q\varphi(\bar{\mathbb{F}}_q) = 0$ is called *supersingular*.

Proposition 4.11. Any supersingular Drinfeld module φ of rank n and characteristic q is isomorphic to some supersingular $\tilde{\varphi}$ for which $\text{Fr}_q^n \in \tilde{\varphi}(A)$. Such $\tilde{\varphi}$ is necessarily defined over $\mathbb{F}_q^{(n)}$. \square

Proof. See Proposition 4.2 and its proof in [13]. \blacksquare

Proposition 4.12. Let $\mathfrak{S}(n, q)$ be the set of isomorphism classes of supersingular Drinfeld modules of rank n and characteristic q , and let $d := \deg(q)$. Then,

$$\sum_{\varphi \in \mathfrak{S}(n, q)} \frac{(q-1)}{\#\text{Aut}(\varphi)} = \prod_{i=1}^{n-1} \frac{(q^{di} - 1)}{(q^{i+1} - 1)}. \quad (4.11)$$

\square

Proof. This is the mass-formula in [12, Theorem 1]. \blacksquare

Example 4.13. Consider the Drinfeld module $\tilde{\varphi}$ over \mathbb{F}_T , given by $\tilde{\varphi}_T = \tau^n$. Since $\tilde{\varphi}_T$ is purely inseparable, $\tilde{\varphi}$ is supersingular of rank n . In fact, this is the only supersingular module in characteristic T (up to an isomorphism). In this case, $\text{Fr}_T = \tau$ and $\text{Fr}_T^n \in \tilde{\varphi}(A)$. It is easy to see that $\text{Aut}(\tilde{\varphi}) \cong \mathbb{F}_{q^n}^\times$. Hence, $\mathbb{F}_{q^n}\{\tau\} \subset \text{End}(\tilde{\varphi})$. Since $\mathbb{F}_{q^n}\{\tau\}$ is free of rank n^2 over A , we conclude that $\text{End}(\tilde{\varphi}) = \mathbb{F}_{q^n}\{\tau\}$. It is clear that the center of $\text{End}(\tilde{\varphi})$ is $\mathbb{F}_q\{\tau^n\}$, the submodule generated by A .

Lemma 4.14. There is a natural number $N(K, n)$, which depends only on $\#K$ and n , with the following property: if φ is a rank- n Drinfeld module defined over K and Fr_K acts on

${}_l\varphi(\bar{K})$ as an element of A for some prime ideal $l \triangleleft A$ with $\deg(l) > N(K, n)$, then φ is supersingular. \square

Proof. It is known that $i_l(\text{Fr}_K)$ is semisimple; see [34]. Denote by $\tilde{P}_\varphi \in A[X]$ the monic square-free divisor of P_φ having the largest degree. Denote by $\mathcal{D}_\varphi \in R$ the discriminant of \tilde{P}_φ ; see [23, Chapter IV, Section 6]. As is clear from the definition, since \tilde{P}_φ is square-free, $\mathcal{D}_\varphi \neq 0$. Let $l \neq q$ be a prime. If Fr_K acts as a scalar on ${}_l\varphi$, then $P_\varphi(X) \bmod l$ is equal to the n th power of a linear polynomial. Hence, $\tilde{P}_\varphi \bmod l$ is also equal to a power of a linear polynomial. If \tilde{P}_φ is not a linear polynomial, then this implies that $\mathcal{D}_\varphi \equiv 0 \bmod l$. By Proposition 4.9, $|\mathcal{D}_\varphi|_\infty \leq (\#K)^{(n-1)}$. Hence, we must have $\deg(l) \leq (n-1) \log_q(\#K) =: N(K, n)$. We conclude that if $\deg(l) > N(K, n)$ and Fr_K acts as an element of A on ${}_l\varphi$, then \tilde{P}_φ is a linear polynomial and P_φ is a power of a linear polynomial. Therefore, $\text{Fr}_K \in \varphi(A)$ which by Proposition 4.10 implies that φ is supersingular. \blacksquare

Proposition 4.15. Let $q \triangleleft A$ be a fixed prime, and let $d := \deg(q)$. Let $p \triangleleft A$ be a prime different from q . Then, for all but finitely many p ,

$$\#\mathcal{X}_{p,q}^n(\mathbb{F}_q^{(n)}) = \frac{[\Gamma : \Gamma(p)]}{(q-1)} \prod_{i=1}^{n-1} \frac{(q^{di} - 1)}{(q^{i+1} - 1)}. \quad (4.12)$$

\square

Proof. The points on $M_q^n(p)$ corresponding to the isomorphism classes of supersingular Drinfeld modules with a level p -structure will be called supersingular. A point on $\mathcal{X}_{p,q}^n$ will be called supersingular if it is the image of a supersingular point on $M_q^n(p)$ under the quotient map $M_q^n(p) \rightarrow \mathcal{X}_{p,q}^n$.

We will prove that all supersingular points on $\mathcal{X}_{p,q}^n$ are $\mathbb{F}_q^{(n)}$ -rational, and if $\deg(p)$ is large enough, then these are the only $\mathbb{F}_q^{(n)}$ -rational points. We start by counting the number of supersingular points on $\mathcal{X}_{p,q}^n$.

Consider the finite flat covering $\pi : M_q^n(p) \rightarrow M_q^n(1)$. Generically, its degree is $\#(G(A/p)/Z(\mathbb{F}_q))$ (cf. [24, 25, Lemma 1.4.2]). This induces a covering

$$\pi' : \mathcal{X}_{p,q}^n \longrightarrow M_q^n(1). \quad (4.13)$$

The degree of π' , generically, is $\# \text{PGL}_n(\mathbb{F}_p)$. Let φ be a supersingular Drinfeld module. The points corresponding to φ are branch points for π' with indices $\#(\text{Aut}(\varphi)/\mathbb{F}_q^\times)$ (a generic Drinfeld module in any characteristic has automorphism group isomorphic to \mathbb{F}_q^\times). Hence the number of such points on $\mathcal{X}_{p,q}^n$ is $\# \text{PGL}_n(\mathbb{F}_p)(q-1)/\# \text{Aut}(\varphi)$. We conclude

that the number of supersingular points on $X_{p,q}^n$ is

$$\#\mathrm{PGL}_n(\mathbb{F}_p) \sum_{\varphi \in \mathfrak{S}(n,q)} \frac{(q-1)}{\#\mathrm{Aut}(\varphi)} = \frac{[\Gamma : \Gamma(\mathfrak{p})]}{(q-1)} \prod_{i=1}^{n-1} \frac{(q^{di} - 1)}{(q^{i+1} - 1)}, \quad (4.14)$$

where the last equality follows from Proposition 4.12 and the observation that

$$(q-1)\#\mathrm{PGL}_n(\mathbb{F}_p) = [\Gamma : \Gamma(\mathfrak{p})]. \quad (4.15)$$

Now we show that the image on $X_{p,q}^n$ of any supersingular point $(\varphi, \lambda) \in M_q^n(\mathfrak{p})$ is rational over $\mathbb{F}_q^{(n)}$. Using Proposition 4.11, we can assume that φ is defined over $\mathbb{F}_q^{(n)}$, and we need to show that any level- p structure λ on the module φ under the action of Fr_q^n gives a structure lying over the same point in $X_{p,q}^n$ as the original one. The action of Fr_q^n on λ is via its image under the composition

$$\mathrm{End}(\varphi) \longrightarrow (\mathrm{End}(\varphi) \otimes_A (A/\mathfrak{p}))^\times \longrightarrow G(\mathbb{F}_p), \quad (4.16)$$

where the last map is (A/\mathfrak{p}) -linear, and in fact can be shown to be an isomorphism (cf. [13, Theorem 4.3]). On the other hand, by Proposition 4.11, $\mathrm{Fr}_q^n \in \varphi(A)$, so its image in $G(\mathbb{F}_p)$ lies in $Z(\mathbb{F}_p)$. Hence, (φ, λ) and $\mathrm{Fr}_q^n(\varphi, \lambda)$ have the same image in $X_{p,q}^n$, and this point is $\mathbb{F}_q^{(n)}$ -rational.

Conversely, suppose the image of some point $(\varphi, \lambda) \in M_q^n(\mathfrak{p})$ is in $X_{p,q}^n(\mathbb{F}_q^{(n)})$. Then,

$$\mathrm{Fr}_q^n(\varphi, \lambda) = (\mathrm{Fr}_q^n(\varphi), \mathrm{Fr}_q^n(\lambda)) = (\varphi, \lambda \circ g) \quad (4.17)$$

for some $g \in Z(\mathbb{F}_p)$. Therefore, φ can be defined over $\mathbb{F}_q^{(n)}$ and Fr_q^n acts as a scalar on ${}_p\varphi$. Lemma 4.14 implies that φ must be supersingular if $\deg(\mathfrak{p}) > (n-1)n \deg(q)$. \blacksquare

4.3 Asymptotic bounds

Recall from the introduction the ℓ -adic cohomology groups with compact supports $H_c^*(M_\eta^n(n) \otimes_{\mathbb{F}} \overline{\mathbb{F}}, \overline{\mathbb{Q}}_\ell)$, which from now on we denote by $H_\eta^*(n)$. Similarly, for a proper prime ideal \mathfrak{q} of $A[n^{-1}]$, denote $H_{\mathfrak{q}}^*(n) := H_c^*(M_{\mathfrak{q}}^n(n) \otimes_{\mathbb{F}_{\mathfrak{q}}} \overline{\mathbb{F}}_{\mathfrak{q}}, \overline{\mathbb{Q}}_\ell)$.

$H_\eta^*(n)$ is endowed with commuting actions of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ and a certain Hecke algebra \mathbb{T}_n . Since $M_\eta^n(n)$ is a smooth affine scheme of pure relative dimension $(n-1)$ over \mathbb{F} , the cohomology groups $H_\eta^i(n)$ are finite dimensional and vanish for $i \notin [n-1, 2(n-1)]$ (cf. [25, Section 12.2]). Denote by $h_\eta^i(n) = \dim_{\overline{\mathbb{Q}}_\ell} H_\eta^i(n)$, $i \geq 0$, the (compact) ℓ -adic Betti numbers of $M_\eta^n(n) \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Similarly, denote by $h_{\mathfrak{q}}^i(n)$ the (compact) ℓ -adic Betti numbers of $M_{\mathfrak{q}}^n(n) \otimes_{\mathbb{F}_{\mathfrak{q}}} \overline{\mathbb{F}}_{\mathfrak{q}}$. Let $h_\eta(n) = \sum_{i \geq 0} h_\eta^i(n)$, and $h_{\mathfrak{q}}(n) = \sum_{i \geq 0} h_{\mathfrak{q}}^i(n)$.

Proposition 4.16. Fix a proper prime ideal $\mathfrak{q} \triangleleft A$. Let $\mathfrak{n} \triangleleft A$ be a nontrivial ideal coprime to \mathfrak{q} . Then,

$$h_{\mathfrak{q}}(\mathfrak{n}) \geq \mathfrak{n} \cdot \dim_{\mathbb{C}} W_{\text{sp}}(\mathfrak{n}). \quad (4.18)$$

If (1.9) is true, then

$$\lim_{\deg(\mathfrak{n}) \rightarrow \infty} \left(\frac{h_{\mathfrak{q}}(\mathfrak{n})}{\dim_{\mathbb{C}} W_{\text{sp}}(\mathfrak{n})} \right) = \mathfrak{n}. \quad (4.19)$$

□

Proof. Denote by $W_{\text{sp}}^m(\mathfrak{n})$ the space of cusp forms on $GL_m(\mathbb{F}_{\infty})$ of level \mathfrak{n} which are special at ∞ (in particular, $W_{\text{sp}}^n(\mathfrak{n}) = W_{\text{sp}}(\mathfrak{n})$ in our earlier notation). We claim that for any $(p_1, \dots, p_h) \neq (\mathfrak{n}) \in \mathbf{P}(\mathfrak{n})$.

$$\lim_{\deg(\mathfrak{n}) \rightarrow \infty} \frac{\dim_{\mathbb{C}} (W_{\text{sp}}^{p_1}(\mathfrak{n}) \otimes \dots \otimes W_{\text{sp}}^{p_h}(\mathfrak{n}))}{\dim_{\mathbb{C}} W_{\text{sp}}^{\mathfrak{n}}(\mathfrak{n})} = 0. \quad (4.20)$$

Indeed, using Corollary 3.20, one checks that $\dim_{\mathbb{C}}(W_{\text{sp}}^m(\mathfrak{n}))$ is $O(q^{m^2 \deg(\mathfrak{n})})$ for $\deg(\mathfrak{n}) \gg 0$. On the other hand, $p_1^2 + \dots + p_h^2 < \mathfrak{n}^2$ if $(p_1, \dots, p_h) \neq (\mathfrak{n})$.

To proceed further, we appeal to the main result of [24, 25]. The central goal of [24, 25] is to describe the virtual $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \times \mathbb{T}_{\mathfrak{n}}$ -module $\mathcal{H} := \sum_{i \geq 0} (-1)^i H_{\mathfrak{n}}^i(\mathfrak{n})$. This description is essentially the Langlands conjecture for cuspidal automorphic irreducible representations of $G(\mathbb{A})$ which are special at ∞ . Laumon shows [25, Theorem 12.5.1] that \mathcal{H} , as a sum of irreducible $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F}) \times \mathbb{T}_{\mathfrak{n}}$ -modules, is equal to a sum of cuspidal representations of $G(\mathbb{A})$ induced from the representations on $W_{\text{sp}}^{p_1}(\mathfrak{n}) \otimes \dots \otimes W_{\text{sp}}^{p_h}(\mathfrak{n})$, $(p_1, \dots, p_h) \in \mathbf{P}(\mathfrak{n})$, tensored with the Galois representations attached to these cuspidal representations by the Langlands correspondence (the same theorem for $\mathfrak{n} = 2$ is due to Drinfeld [7]). This result immediately implies that

$$h_{\mathfrak{n}}(\mathfrak{n}) \geq \mathfrak{n} \cdot \dim_{\mathbb{C}} W_{\text{sp}}(\mathfrak{n}). \quad (4.21)$$

If we moreover assume (1.9), then Laumon's theorem and (4.20) imply

$$\lim_{\deg(\mathfrak{n}) \rightarrow \infty} \left(\frac{h_{\mathfrak{n}}(\mathfrak{n})}{\dim_{\mathbb{C}} W_{\text{sp}}^{\mathfrak{n}}(\mathfrak{n})} \right) = \mathfrak{n}. \quad (4.22)$$

Now, for any i the $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ -module $H_{\mathfrak{n}}^i$ is unramified away from $\text{supp}(\mathfrak{n}) \cup \{\infty\}$, and moreover, for each proper ideal \mathfrak{q} of $A[\mathfrak{n}^{-1}]$ there is a $\text{Gal}(\overline{\mathbb{F}}_{\mathfrak{q}}/\mathbb{F}_{\mathfrak{q}})$ -equivariant isomorphism $H_{\mathfrak{n}}^i(\mathfrak{n}) \cong H_{\mathfrak{q}}^i(\mathfrak{n})$. Besides some theorems from SGA, the proof of this fact uses Pink's construction of toroidal compactifications of $M^{\mathfrak{n}}(\mathfrak{n})$; see [25, equations (12.2.2.1), (12.2.2.2), and remark (12.2.7)]. We conclude that $h_{\mathfrak{n}}(\mathfrak{n}) = h_{\mathfrak{q}}(\mathfrak{n})$, and the theorem follows. ■

Remark 4.17. One possible approach to proving Conjecture 1.5, and hence also (1.9), can be through understanding of the individual cohomology groups $H_{\mathbb{F}_q}^i(n)$ as $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \times \mathbb{T}_n$ -modules. For example, most likely the Galois representations which correspond to $W_{\text{sp}}^n(n)$ occur only in $H_{\mathbb{F}_q}^{n-1}(n)$ and $h_{\mathbb{F}_q}^{n-1}(n) \sim n \dim_{\mathbb{C}} W_{\text{sp}}^n(n)$. When $n = 2$, this follows from Drinfeld's theorem [7].

Notation 4.18. Let $\text{Frob}_q = \text{Fr}_q^{-1} \in \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the geometric Frobenius element. If $q \notin \text{supp}(n)$, then Frob_q defines an automorphism of $H_q^i(n)$. Assume $H_q^i(n) \neq 0$. Denote the eigenvalues of Frob_q acting on this finite-dimensional $\overline{\mathbb{Q}}_\ell$ -vector space by $\alpha_{q,i,1}(n), \alpha_{q,i,2}(n), \dots, \alpha_{q,i,s}(n)$ (here $s = h_q^i(n)$).

Theorem 4.19. Fix a proper prime $q \triangleleft A$. Let $d := \deg(q)$. Let \mathfrak{p} be an admissible prime ideal not equal to q . Then, $X_{\mathfrak{p},q}^n$ is a smooth, geometrically connected, $(n-1)$ -dimensional affine variety defined over \mathbb{F}_q and

$$\limsup_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{\#X_{\mathfrak{p},q}^n(\mathbb{F}_q^{(n)})}{h(X_{\mathfrak{p},q}^n)} \right) \leq \frac{1}{n} \prod_{i=1}^{n-1} (q^{di} - 1). \quad (4.23)$$

If (1.9) is true, then

$$\lim_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{\#X_{\mathfrak{p},q}^n(\mathbb{F}_q^{(n)})}{h(X_{\mathfrak{p},q}^n)} \right) = \frac{1}{n} \prod_{i=1}^{n-1} (q^{di} - 1). \quad (4.24)$$

□

Proof. $M_q^n(\mathfrak{p}) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is a disjoint union of $[\mathbb{F}_{\mathfrak{p}}^\times : \mathbb{F}_q^\times]$ copies of $X_{\mathfrak{p},q}^n \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Hence,

$$h_q(\mathfrak{p}) = [\mathbb{F}_{\mathfrak{p}}^\times : \mathbb{F}_q^\times] \cdot h(X_{\mathfrak{p},q}^n). \quad (4.25)$$

This equality, combined with Corollary 3.20 and Proposition 4.16, implies

$$\liminf_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{h(X_{\mathfrak{p},q}^n)}{[\Gamma : \Gamma(\mathfrak{p})]} \right) \geq \frac{n}{(q^n - 1) \cdots (q - 1)}, \quad (4.26)$$

and, if (1.9) is true,

$$\lim_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{h(X_{\mathfrak{p},q}^n)}{[\Gamma : \Gamma(\mathfrak{p})]} \right) = \frac{n}{(q^n - 1) \cdots (q - 1)}. \quad (4.27)$$

Now the theorem follows from Proposition 4.15. ■

Theorem 4.20. With notation of Theorem 4.19, suppose (1.9) is true. Let $m \geq 1$. Then,

$$\lim_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{\text{WD}(X_{\mathfrak{p},q}^n, \mathbb{F}_q^{(m)})}{h(X_{\mathfrak{p},q}^n)} \right) = q^{d m(n-1)/2}. \quad (4.28)$$

Proof. Let $\sum_{\text{sp}} |\alpha_{q,i,j}(\mathfrak{p})^m|$ be the sum over the eigenvalues $\alpha_{q,i,j}$ corresponding to $W_{\text{sp}}^n(\mathfrak{p})$ under the Langlands conjecture; see [25, Theorems 12.4.1, 12.5.1, and Corollary 12.4.9]. By the Ramanujan-Petersson conjecture [25, Theorem 12.4.1]

$$\sum_{\text{sp}} |\alpha_{q,i,j}(\mathfrak{p})^m| = q^{\deg(q)m((n-1)/2)} n \dim_{\mathbb{C}}(W_{\text{sp}}^n(\mathfrak{p})). \quad (4.29)$$

On the other hand, by Deligne's theorem [6, Theorem 3.3.1] all $|\alpha_{q,i,j}(\mathfrak{p})^m|$ are bounded by $q^{\deg(q)m(n-1)}$, which is independent of \mathfrak{p} . Hence, by Proposition 4.16 and its proof

$$\lim_{\deg(\mathfrak{p}) \rightarrow \infty} \left(\frac{\sum_{i,j} |\alpha_{q,i,j}(\mathfrak{p})^m|}{h_q(\mathfrak{p})} \right) = q^{d m(n-1)/2}. \quad (4.30)$$

Since $M_{\mathfrak{p},q}^n \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ is a disjoint union of copies of $X_{\mathfrak{p},q}^n \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, we have

$$\frac{\sum_{i,j} |\alpha_{q,i,j}(\mathfrak{p})^m|}{h_q(\mathfrak{p})} = \frac{\text{WD}(X_{\mathfrak{p},q}^n, \mathbb{F}_q^{(m)})}{h(X_{\mathfrak{p},q}^n)}, \quad (4.31)$$

and the theorem follows. ■

Appendix

Proof of Proposition 3.8

Throughout this appendix we use the notation of Section 3. First we need to prove an identity for q -multinomial coefficients. Denote

$$(a)_k := \prod_{i=0}^{k-1} (1 - aq^i), \quad (a)_\infty := \prod_{i=0}^{\infty} (1 - aq^i). \quad (A.1)$$

(For now, q can be thought of just as a fixed parameter.) To each ordered partition $\mathfrak{p} = (p_1, \dots, p_h)$ of $m \geq 1$ corresponds the q -multinomial coefficient:

$$\left[\begin{matrix} m \\ \mathfrak{p} \end{matrix} \right] = \frac{(q)_m}{(q)_{p_1} (q)_{p_2} \cdots (q)_{p_h}}. \quad (A.2)$$

It is well known that the q -multinomial coefficients are polynomials in q .

Remark A.1. Let $\mathbf{p} = (p_1, \dots, p_h) \in \mathbf{P}(m)$. Let $\mathbb{C}\langle x_1, x_2, \dots, x_h \rangle$ be the noncommutative polynomial ring where the constants commute with all x_i 's and $x_j x_i = q x_i x_j$ for any $i < j$. Then, $\begin{bmatrix} m \\ \mathbf{p} \end{bmatrix}$ is the coefficient of $x_1^{p_1} \cdots x_h^{p_h}$ in $(x_1 + \cdots + x_h)^m$, which explains the terminology.

Lemma A.2.

$$\sum_{\mathbf{p} \in \mathbf{P}(m)} (-1)^{\ell(\mathbf{p})} \begin{bmatrix} m \\ \mathbf{p} \end{bmatrix} = (-1)^m q^{m(m-1)/2}. \quad (\text{A.3})$$

Proof. In the proof we will use two formulae of Euler [1, Corollary 2.2]:

$$1 + \sum_{i=1}^{\infty} \frac{x^i}{(q)_i} = \frac{1}{(x)_{\infty}}, \quad 1 + \sum_{i=1}^{\infty} \frac{(-1)^i x^i q^{i(i-1)/2}}{(q)_i} = (x)_{\infty}. \quad (\text{A.4})$$

It is easy to see that the left-hand side of the desired identity is the coefficient of $x^m/(q)_m$ in

$$\begin{aligned} 1 + \sum_{h=1}^{\infty} \sum_{m=1}^{\infty} \frac{x^m}{(q)_m} (-1)^h \sum_{\substack{p_1 + \cdots + p_h = m \\ p_1, \dots, p_h \geq 1}} \frac{(q)_m}{(q)_{p_1} (q)_{p_2} \cdots (q)_{p_h}} \\ = 1 + \sum_{h=1}^{\infty} (-1)^h \sum_{p_1, \dots, p_h \geq 1} \frac{x^{p_1 + p_2 + \cdots + p_h}}{(q)_{p_1} (q)_{p_2} \cdots (q)_{p_h}}. \end{aligned} \quad (\text{A.5})$$

By the first formula in (A.4) this last expression is equal to

$$1 + \sum_{h=1}^{\infty} (-1)^h \left(\frac{1}{(x)_{\infty}} - 1 \right)^h = \frac{1}{1 - \left(1 - \frac{1}{(x)_{\infty}} \right)} = (x)_{\infty}. \quad (\text{A.6})$$

Now the claim follows from the second formula in (A.4), as the coefficients of $x^m/(q)_m$ in the left-hand side of that formula is $(-1)^m q^{m(m-1)/2}$. \blacksquare

Remark A.3. By taking $q \rightarrow 1$ in Lemma A.2, we get the following identity for the usual multinomial coefficients:

$$\sum_{(p_1, \dots, p_h) \in \mathbf{P}(m)} (-1)^h \binom{m}{p_1, \dots, p_h} = (-1)^m. \quad (\text{A.7})$$

Recall that for $m \geq 1$ we were denoting $\phi(m) = -(q)_m^{-1}$.

Corollary A.4.

$$-\phi(m) = \sum_{\mathbf{p} \in \mathbf{P}(m)} (-1)^{\ell(\mathbf{p})} (\#P_{\mathbf{p}}(\mathbb{F}_q))^{-1}. \quad (\text{A.8})$$

Proof. Let $g_k = \# \mathrm{GL}_k(\mathbb{F}_q)$. Then,

$$g_k = \prod_{i=0}^{k-1} (q^k - q^i) = q^{k(k-1)/2} \prod_{i=1}^k (q^i - 1) = (-1)^k q^{k(k-1)/2} (q)_k. \quad (\text{A.9})$$

Let $\mathbf{p} = (p_1, \dots, p_h) \in \mathbf{P}(m)$. It is easy to check that

$$\#P_{\mathbf{p}}(\mathbb{F}_q) = g_{p_1} \cdot g_{p_2} \cdots g_{p_h} \cdot q^{\theta(\mathbf{p})}, \quad (\text{A.10})$$

where $\theta(\mathbf{p}) = m^2 - \sum_{i=1}^h \sum_{j=i}^h p_i p_j$. Plugging in the expression (A.9) and simplifying, we get

$$\#P_{\mathbf{p}}(\mathbb{F}_q) = (-1)^m q^{m(m-1)/2} \prod_{i=1}^h (q)_{p_i}. \quad (\text{A.11})$$

Hence,

$$\sum_{\mathbf{p} \in \mathbf{P}(m)} (-1)^{\ell(\mathbf{p})} (\#P_{\mathbf{p}}(\mathbb{F}_q))^{-1} = \frac{(-1)^m q^{-m(m-1)/2}}{(q)_m} \sum_{\mathbf{p} \in \mathbf{P}(m)} (-1)^{\ell(\mathbf{p})} \begin{bmatrix} m \\ \mathbf{p} \end{bmatrix} = \frac{1}{(q)_m}, \quad (\text{A.12})$$

where the last equality is due to Lemma A.2. ■

There is a partial ordering on the vertices of \mathcal{W} . If $v = [0, a_2, a_3, \dots, a_n]$ and $v' = [0, b_2, b_3, \dots, b_n]$ are in \mathcal{W} , then we put $v \preceq v'$ if $a_j \leq b_j$ for all $2 \leq j \leq n$, and $v \prec v'$ if at least one of the inequalities is strict. From the definitions, it is easy to see that the vertices $\{v_0, \dots, v_i\}$ of \mathcal{W} form an i -simplex if and only if, up to reindexing, $v_0 \prec v_1 \prec \dots \prec v_i \preceq d_2(v_0)$. We call v_0 the *smallest vertex* of σ .

Proof of Proposition 3.8. First we prove the proposition assuming $n = 2$, as this case is somewhat degenerate. If $n = 2$, then \mathcal{W} is the infinite half-line:

$$[0, 0] - [0, 1] - [0, 2] - \dots - [0, m] - \dots. \quad (\text{A.13})$$

Lemma 3.2 implies that $\mathrm{Stab}_{\Gamma}(0) = \mathrm{GL}_2(\mathbb{F}_q)$, and $\mathrm{Stab}_{\Gamma}([0, m])$ is the group of the upper-triangular matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, with $a, d \in \mathbb{F}_q^{\times}$ and $b \in \mathcal{A}$, $\deg(b) \leq m$. Now $K^0([0, m])$ consists of one 0-dimensional simplex, namely, $[0, m]$, and one 1-dimensional simplex, namely, the edge joining $[0, m]$ to $[0, m+1]$. The stabilizer of this latter edge is $\mathrm{Stab}_{\Gamma}([0, m]) \cap \mathrm{Stab}_{\Gamma}([0, m+1])$, which is equal to $\mathrm{Stab}_{\Gamma}([0, m])$ when $m > 0$, and is the Borel subgroup of upper-triangular matrices in $\mathrm{GL}_2(\mathbb{F}_q)$ when $m = 0$. From this the claim of the proposition easily follows.

Now assume $n \geq 3$. Let $K^{00}(v)$ be the subset of $K^0(v)$ consisting of simplices not containing v . First, we show that $\tilde{\chi}(K^{00}(v)) = 0$ for any v . In fact, we will prove a stronger statement: the set $K^{00}(v)$ can be divided into pairs of simplices (s, σ) such that s is a face of σ , the smallest vertices of s and σ are the same, $\dim(\sigma) = \dim(s) + 1$, $\Gamma_s = \Gamma_\sigma$, and each simplex of $K^{00}(v)$ appears exactly once in some pair. (This clearly implies $\tilde{\chi}(K^{00}(v)) = 0$ as the summands corresponding to s and σ cancel each other.) We proceed by induction on n .

When $n = 3$, $K^{00}(v)$ consists of the 1-simplex $s = \{d_2(v), d_3(v)\}$, and the 2-simplex $\sigma = \{d_2(v), d_3(v), d_2d_3(v)\}$. The smallest vertex of both σ and s is $d_3(v)$. Using Lemmas 3.1 and 3.2, one easily checks that $\Gamma_s = \Gamma_\sigma$.

Assume we have proven the claim for $n - 1$. Let $v' \in \text{Ver}(K(v))$ be $v' = d_2^{s_2} \cdots d_n^{s_n}(v)$. Define $\pi_2(v') := d_3^{s_3} \cdots d_n^{s_n}(v)$. This gives a map

$$\pi_2 : \text{Ver}(K(v)) \longrightarrow \text{Ver}(K_2^0(v)). \quad (\text{A.14})$$

We claim that π_2 is a simplicial map from $K(v)$ onto $K_2^0(v)$, that is, if $\sigma = \{v_0, \dots, v_i\}$ is an i -simplex in $K(v)$, then the vertices (modulo repetitions) $\{\pi_2(v_0), \dots, \pi_2(v_i)\}$ form a simplex in $K_2^0(v)$. Since σ is a simplex, we can assume $v_0 \prec v_1 \prec \cdots \prec v_i \preceq d_2(v_0)$. We need to show that the vertices in $\pi_2(\sigma)$ can be arranged to satisfy similar inequalities. We can assume there is v_j with $\deg_2(v_j) = 1$; otherwise, π_2 is the identity on σ and the claim is trivial. Let j be the smallest index for which $\deg_2(v_j) = 1$. Then, $\deg_2(v_k) = 1$ for any $k \geq j$. If $j = 0$, then clearly

$$\pi_2(v_0) \prec \pi_2(v_1) \cdots \prec \pi_2(v_i) \preceq d_2\pi_2(v_0). \quad (\text{A.15})$$

Now assume $j > 0$. We claim that

$$\begin{aligned} \pi_2(v_j) \prec \pi_2(v_{j+1}) \prec \cdots \prec \pi_2(v_i) &\preceq \pi_2(v_0) \\ &\prec \pi_2(v_1) \cdots \prec \pi_2(v_{j-1}) \preceq d_2\pi_2(v_j) = v_j. \end{aligned} \quad (\text{A.16})$$

Since $\pi_2(v_k) = v_k$ for $k \leq j - 1$, all the inequalities are obvious except possibly for $\pi_2(v_i) \preceq \pi_2(v_0) = v_0$, which is true since $v_i \preceq d_2(v_0)$. (Note that π_3, π_4 , etc. defined similarly to π_2 are not necessarily simplicial maps. Take, e.g., $n = 4$ and consider the edge $\{[0, 0, 1, 1], [0, 1, 1, 2]\}$ in $K(0)$. Then, $\pi_3([0, 0, 1, 1]) = 0$, $\pi_3([0, 1, 1, 2]) = [0, 1, 1, 2]$ which are not adjacent.)

Suppose σ is an i -simplex in $K(v)$. By the previous paragraph $\pi_2(\sigma)$ is a simplex. Moreover, since the kernel of π_2 extended to the ambient \mathbb{R} -vector space containing $K(v)$ is 1-dimensional, $\pi_2(\sigma)$ is either i or $(i - 1)$ -dimensional. If $\dim(\pi_2(\sigma)) = i - 1$, then σ

has an $(i - 1)$ -dimensional face s such that $\sigma = \{s, d_2(v')\}$ for some $v' \in \text{Ver}(s)$. It is easy to check that this face can be uniquely characterized as follows: if $\sigma = \{v_0, \dots, v_i\}$ with $v_0 \prec \dots \prec v_i$, then $s = \{v_0, \dots, v_{i-1}\}$ and $v_i = d_2(v_0)$. We call s the d_2 -*bottom* of σ . On the other hand, if s is an i -simplex in $K^0(v)$ and $\dim(\pi_2(s)) = i$, then s is the d_2 -bottom of a unique σ in $K^0(v)$. Indeed, let $s = \{v_0, \dots, v_i\}$ with $v_0 \prec \dots \prec v_i$. The vertex v_0 is not in $K_2^1(v)$ as otherwise $s \in K_2^1(v)$, which contradicts the assumption $s \in K^0(v)$. Hence $\pi_2(v_0) = v_0$. Now the assumption that $\dim(\pi_2(s)) = i$ implies $v_i \not\prec d_2(v_0)$. The set of vertices $\{v_0, \dots, v_i, d_2(v_0)\}$ forms an $(i + 1)$ -simplex σ , with d_2 -bottom s . The previous arguments also show that σ is the only $(i + 1)$ -simplex having s as its d_2 -bottom.

Let (s, σ) be a pair of simplices in $K(v)$ such that s is the d_2 -bottom of σ . As is easy to see, if one of these simplices lies in $K^0(v)$, then so does the other one. Combining this with the previous paragraph, we conclude that all simplices in $K^0(v)$ can be divided into disjoint pairs (s, σ) , s is the d_2 -bottom of σ .

Let (s, σ) be as above. Assume either s does not lie in $K_2^0(v)$, or $\deg_2(v) \geq 1$. We claim that under one of these assumptions $\Gamma_s = \Gamma_\sigma$. Let $s = \{v_0, \dots, v_i\}$ with $v_0 \prec \dots \prec v_i$. Then, $\sigma = \{v_0, \dots, v_i, d_2(v_0)\}$. Using Lemma 3.1, we can assume $i = 1$, and need to show

$$\text{Stab}_\Gamma(v_0) \cap \text{Stab}_\Gamma(v_1) \subset \text{Stab}_\Gamma(d_2(v_0)). \quad (\text{A.17})$$

Since $\deg_2(v_1) \geq 1$, this follows from Lemma 3.2.

Now let $\deg_2(v) = 0$, and $s \in K_2^0(v)$. In this case we do not necessarily have $\Gamma_s = \Gamma_\sigma$. We will pair all simplices of $K^{00}(v)$ lying or having a codimension one face in $K_2^0(v)$ in a different way. Write $v = [0, 0, i_3, \dots, i_n]$. Let $v' = [0, i_3, \dots, i_n]$. We have a canonical isomorphism of simplicial complexes $K_2^0(v) \cong K(v')$, where the second complex is in the building of $\text{PGL}_{n-1}(\mathbb{F}_\infty)$, which preserves the partial ordering \prec on the vertices. Denote $\Gamma' := \text{GL}_{n-1}(A)$. By induction hypothesis, all simplices in $K^{00}(v')$ can be divided into disjoint pairs (s', σ') where s' is a face of σ' of codimension one, s' contains the smallest vertex of σ' , and $\Gamma_{s'} = \Gamma_{\sigma'}$. It is easy to check that if we consider s', σ' as simplices of $K^{00}(v)$, we still have $\Gamma_{s'} = \Gamma_{\sigma'}$. Let $\sigma' = \{v_0, \dots, v_i\}$, $v_0 \prec \dots \prec v_i$. The set of vertices $\sigma'' = \{v_0, \dots, v_i, d_2(v_0)\}$ forms an $(i + 1)$ -simplex in $K^{00}(v)$. Consider its codimension one face $s'' = \{s', d_2(v_0)\}$. If $\Gamma_{s'} = \Gamma_{\sigma'}$, then by Lemma 3.1 we also have $\Gamma_{s''} = \Gamma_{\sigma''}$. By the induction hypothesis, v_0 is a vertex of s' , so neither σ'' nor s'' is the d_2 -bottom of another simplex. The pair (s'', σ'') is uniquely characterized by (s', σ') . Using the induction hypothesis again, we conclude that any simplex whose d_2 -bottom lies in $K_2^0(v)$ occurs in some unique pair (s'', σ'') . Since the union of the simplices in all pairs (s', σ') , (s'', σ'') is equal to the set of simplices of $K^{00}(v)$ lying or having a codimension one face in $K_2^0(v)$, this finishes the induction step.

Next, let $K^{01}(v)$ be the subset of $K^0(v)$ consisting of simplices containing v . Since $K^0(v)$ is the disjoint union of $K^{00}(v)$ and $K^{01}(v)$,

$$\tilde{\chi}(K^0(v)) = \tilde{\chi}(K^{00}(v)) + \tilde{\chi}(K^{01}(v)) = \tilde{\chi}(K^{01}(v)). \quad (\text{A.18})$$

Denote $v_2 := d_2(v)$, $v_3 := d_3(v)$, \dots , $v_n := d_n(v)$. First, assume $v \neq \mathbf{0}$. Then, $v = [0, 0, \dots, 0, i_h, \dots, i_n]$, where $i_h \neq 0$, for some $h \geq 2$. By Lemma 3.2, $\text{Stab}_\Gamma(v) \subset \text{Stab}_\Gamma(v_h)$. Let $s \in K^{01}(v)$ be a simplex which does not have v_h as one its vertices. Then, $\{s, v_h\}$ is also a simplex in $K^{01}(v)$ and $\text{Stab}_\Gamma(\{s, v_h\}) = \text{Stab}_\Gamma(s)$. On the other hand, if σ is a simplex in $K^{01}(v)$ which has v_h as a vertex, then the unique codimension one face s of σ which does not contain v_h is also in $K^{01}(v)$ (as $v \neq v_h$). Again we have $\text{Stab}_\Gamma(\sigma) = \text{Stab}_\Gamma(s)$. Summarizing, the set $K^{01}(v)$ can be divided into pairs of simplices (σ, s) such that s is a codimension one face of σ , $\Gamma_s = \Gamma_\sigma$, and each simplex appears exactly once in some pair. This implies $\tilde{\chi}(K^{01}(v)) = 0$.

Now let $v = \mathbf{0}$. Let $\{e_1, \dots, e_n\}$ be our fixed basis of \mathcal{V} . The vertex $\mathbf{0}$ corresponds to the lattice $L = \mathcal{O}_\infty e_1 \oplus \dots \oplus \mathcal{O}_\infty e_n$ in \mathcal{V} , and v_j corresponds to the sublattice of L :

$$\mathcal{O}_\infty e_1 \oplus \dots \oplus \mathcal{O}_\infty e_{j-1} \oplus \pi_\infty \mathcal{O}_\infty e_j \oplus \dots \oplus \pi_\infty \mathcal{O}_\infty e_n. \quad (\text{A.19})$$

Hence, in the \mathbb{F}_q -vector space $V := L/\pi_\infty L$, v corresponds to V , and v_j corresponds to the subspace V_j spanned by $\{e_1, \dots, e_{j-1}\}$. The stabilizer Γ_σ of the i -simplex $\sigma = \{v, v_{j_1}, \dots, v_{j_i}\}$, where $i \geq 1$, $v \prec v_{j_1} \prec \dots \prec v_{j_i}$, is the stabilizer in $\text{GL}_n(\mathbb{F}_q)$ of the flag $V_{j_i} \subset V_{j_{i-1}} \subset \dots \subset V_{j_1} \subset V$ in V . This subgroup is $P_{\mathbf{p}}(\mathbb{F}_q)$, where

$$\mathbf{p} = (j_i - 1, j_{i-1} - j_i, j_{i-2} - j_{i-1}, \dots, j_1 - j_2, n - j_1 + 1). \quad (\text{A.20})$$

Note that $\ell(\mathbf{p}) = i + 1$. The stabilizer of the 0 -simplex $\sigma = \{v\}$ is $P_{(n)}(\mathbb{F}_q) = \text{GL}_n(\mathbb{F}_q)$. Since we assume $\deg(n) \geq 1$, $\Gamma_\sigma = \overline{\Gamma}_\sigma$. We conclude

$$\tilde{\chi}(K^{01}(v)) = \sum_{\mathbf{p} \in P(n)} (-1)^{\ell(\mathbf{p})-1} (\#P_{\mathbf{p}}(\mathbb{F}_q))^{-1} = \phi(n), \quad (\text{A.21})$$

where the last equality is due to Corollary A.4. This finishes the proof of the proposition. \blacksquare

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