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$\mathcal{D}$-elliptic sheaves and odd Jacobians

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ABSTRACT

We examine the existence of rational divisors on modular curves of $\mathcal{D}$-elliptic sheaves and on Atkin–Lehner quotients of these curves over local fields. Using a criterion of Poonen and Stoll, we show that in infinitely many cases the Tate–Shafarevich groups of the Jacobians of these Atkin–Lehner quotients have non-square orders.

1. Introduction

Let F be a global field. Let C be a smooth projective geometrically irreducible curve of genus g over F. Denote by $|F|$ the set of places of F. For $x \in |F|$, denote by F_x the completion of F at x. A place $x \in |F|$ is called *deficient* for C if $C_{F_x} := C \times_F F_x$ has no F_x-rational divisors of degree $g-1$, cf. [20]. It is known that the number of deficient places is finite. Let J be the Jacobian variety of C. Assume the Tate–Shafarevich group $Ш(J)$ is finite. In [20], Poonen and Stoll show that the order of $Ш(J)$ can be a square as well as twice a square. In the first case J is called *even*, and in the second case J is called *odd*. The parity of the number of deficient places is directly related to the parity of J [20, Section 8]:

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Theorem 1.1. *J is even if and only if the number of deficient places for C is even.*

Using this theorem, Poonen and Stoll show that infinitely many hyperelliptic Jacobians over $\mathbb{Q}$ are odd for every even genus. For function fields, Proposition 30 in [20] gives the following example: Let J be the Jacobian of the genus 2 curve

$$C: \quad y^2 = Tx^6 + x - aT$$

over $\mathbb{F}_q(T)$, where q is odd, and $a \in \mathbb{F}_q^\times$ is a non-square. One checks that only the place $\infty = 1/T$ is deficient for C. Since C defines a rational surface over $\mathbb{F}_q$, the Brauer group of that surface is finite. The main theorem in [7] then implies that $\text{Ш}(J)$ is also finite. Overall, $\text{Ш}(J)$ is finite and has non-square order. As far as I know, this is the only example in published literature of an odd Jacobian over a function field.

In this paper we adapt an idea of Jordan and Livné [10] to $F = \mathbb{F}_q(T)$, and exhibit infinitely many curves over F whose Jacobians are odd. These curves are obtained as quotients of modular curves of $\mathcal{D}$-elliptic sheaves. For this introduction we give an analytic description of these curves. Let D be a division quaternion algebra over F. Let R be the subset of places of F where D ramifies (see Section 2.2 for definitions). It is well known that R is a finite non-empty set of even cardinality, and for any choice of a finite non-empty set $R \subset |F|$ of even cardinality there is a unique, up to isomorphism, division quaternion algebra ramified exactly at the places in R. Assume the place $\infty := 1/T$ is not in R. Let $\mathscr{D}$ be a maximal $\mathbb{F}_q[T]$-order in D; all such orders are conjugate in D. Let $\Gamma = \mathscr{D}^\times$ be the group of units of $\mathscr{D}$. The group Γ is isomorphic to a discrete subgroup of $\mathrm{GL}_2(F_\infty)$. Hence Γ acts discontinuously on Drinfeld's upper-half plane $\Omega = \mathbb{C}_\infty - F_\infty$, where $\mathbb{C}_\infty$ is the completion of an algebraic closure of F_∞. The quotient $\Gamma \setminus \Omega$ is the rigid-analytic space corresponding to a smooth projective curve $X_{F_\infty}^{\mathscr{D}}$ over F_∞ (see Section 2.4). The curve $X_{F_\infty}^{\mathscr{D}}$ has a canonical model $X_F^{\mathscr{D}}$ defined over F (see Section 3.1). The automorphism group of $X_F^{\mathscr{D}}$ contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\#R}$, which is generated by involutions $\{w_x\}_{x\in R}$ naturally indexed by the places in R (see Section 3.2). Assume for simplicity that q is odd. The main results of the paper are the following two theorems:

Theorem 1.2. *There are no deficient places for $X_F^{\mathscr{D}}$ unless $R = \{x, y\}$ and both places have odd degrees. In this last case the deficient places for $X_F^{\mathscr{D}}$ are x and y.*

Theorem 1.3. *Assume $R = \{x, y\}$ and both places have even degrees. In addition, assume 4 does not divide $\deg(y)$ and the monic generator of the prime ideal of $\mathbb{F}_q[T]$ corresponding to y is not a square modulo the prime ideal corresponding to x. Then x is the only deficient place for the quotient curve $X_F^{\mathscr{D}}/w_y$.*

In fact, we prove stronger versions of these theorems (see Theorems 4.2 and 4.5). We also prove the following results which complement Theorem 1.3:

- There are infinitely many pairs $\{x, y\}$ satisfying the conditions of Theorem 1.3. Thus, there are infinitely many odd Jacobians over F.
- Only finitely many of the curves $X_F^{\mathscr{D}}/w_y$ are hyperelliptic.
- It is possible to choose $\{x, y\}$ so that both places have degree 2 and satisfy the congruence condition of Theorem 1.3. For such a choice the Tate–Shafarevich group of the Jacobian variety of $X_F^{\mathscr{D}}/w_y$ is finite, and therefore provably has non-square order. The dimension of this Jacobian is $(q^2 - 1)/2$.

Our proofs rely on the results in [19], where we have examined the existence of rational points on $X_F^{\mathscr{D}}$ over local fields.

2. Notation and terminology

2.1. Notation

$F = \mathbb{F}_q(T)$ is the field of rational functions on the projective line $\mathbb{P} := \mathbb{P}^1_{\mathbb{F}_q}$ over the finite field $\mathbb{F}_q$. Denote by $\mathcal{O}_x$ the ring of integers of F_x. The residue field of $\mathcal{O}_x$ will be denoted by $\mathbb{F}_x$. The degree of x is $\deg(x) := [\mathbb{F}_x : \mathbb{F}_q]$, so $q_x := \#\mathbb{F}_x = q^{\deg(x)}$. Let ϖ_x be a uniformizer of $\mathcal{O}_x$. We assume that the valuation $\mathrm{ord}_x : F_x \to \mathbb{Z}$ is normalized by $\mathrm{ord}_x(\varpi_x) = 1$. Let $A := \mathbb{F}_q[T]$ be the polynomial ring over $\mathbb{F}_q$; this is the subring of F consisting of functions which are regular away from $\infty = 1/T$. For a place $x \neq \infty$, let $\mathfrak{p}_x$ be the corresponding prime ideal of A, and $\wp_x \in A$ be the monic generator of $\mathfrak{p}_x$. For a ring H with a unit element, we denote by $H^\times$ the group of its invertible elements. For $S \subset |F|$, put

$$\mathrm{Odd}(S) = \begin{cases} 1, & \text{if all places in } S \text{ have odd degrees;} \\ 0, & \text{otherwise.} \end{cases}$$

2.2. Quaternion algebras

A *quaternion algebra* over a field F is a 4-dimensional associative F-algebra with center F which does not possess non-trivial two-sided ideals. It is known that a quaternion algebra is either a division algebra or is isomorphic to the algebra of 2×2 matrices $\mathbb{M}_2(F)$. If D is a quaternion algebra over a field F, and L is a field extension of F, then $D \otimes_F L$ is a quaternion algebra over L. Denote $D_x := D \otimes_F F_x$. We say that the algebra D *ramifies* (resp. *splits*) at $x \in |F|$ if D_x is a division algebra (resp. $D_x \cong \mathbb{M}_2(F_x)$). As we mentioned in the introduction, the number of places where D ramifies is even, and the set R of these places determines D up to isomorphism. (The empty set $R = \emptyset$ corresponds to $\mathbb{M}_2(F)$.)

There is a field extension L/F such that $D \otimes_F L \cong \mathbb{M}_2(L)$. Considering $\alpha \in D$ as an element of $\mathbb{M}_2(L)$ we can compute its determinant. The value $\mathrm{Nr}(\alpha)$ of this determinant is in F, and is independent of the choice of L; it is called the *reduced norm* of α.

An *$\mathcal{O}_\mathbb{P}$-order* in D is a sheaf of $\mathcal{O}_\mathbb{P}$-algebras with generic fibre D which is coherent and locally free as an $\mathcal{O}_\mathbb{P}$-module. A *$\mathcal{D}$-bimodule* for an $\mathcal{O}_\mathbb{P}$-order $\mathcal{D}$ in D is an $\mathcal{O}_\mathbb{P}$-module $\mathcal{I}$ with left and right $\mathcal{D}$-actions compatible with the $\mathcal{O}_\mathbb{P}$-action and such that

$$(\lambda i)\mu = \lambda(i\mu), \quad \text{for any } \lambda, \mu \in \mathcal{D} \text{ and } i \in \mathcal{I}.$$

A $\mathcal{D}$-bimodule $\mathcal{I}$ is *invertible* if there is another $\mathcal{D}$-bimodule $\mathcal{J}$ such that there are isomorphisms of $\mathcal{D}$-bimodules

$$\mathcal{I} \otimes_\mathcal{D} \mathcal{J} \cong \mathcal{D}, \qquad \mathcal{J} \otimes_\mathcal{D} \mathcal{I} \cong \mathcal{D}.$$

The group of isomorphism classes of invertible $\mathcal{D}$-bimodules will be denoted by $\mathrm{Pic}(\mathcal{D})$: the group operation is $\mathcal{I}_1 \otimes_\mathcal{D} \mathcal{I}_2$, cf. [21, (37.5)].

2.3. Graphs

We recall some of the terminology related to graphs, as presented in [27] and [12]. A graph $\mathcal{G}$ consists of a set of *vertices* $\mathrm{Ver}(\mathcal{G})$ and a set of *edges* $\mathrm{Ed}(\mathcal{G})$. Every edge y has *origin* $o(y) \in \mathrm{Ver}(\mathcal{G})$, *terminus* $t(y) \in \mathrm{Ver}(\mathcal{G})$, and *inverse* edge $\bar{y} \in \mathrm{Ed}(\mathcal{G})$ such that $\bar{\bar{y}} = y$ and $o(y) = t(\bar{y})$, $t(y) = o(\bar{y})$. The vertices $o(y)$ and $t(y)$ are the *extremities* of y. Note that it is allowed for distinct edges $y \neq z$ to have $o(y) = o(z)$ and $t(y) = t(z)$. We say that two vertices are *adjacent* if they are the extremities of some edge. The graph $\mathcal{G}$ is a *graph with lengths* if we are given a map

$$\ell = \ell_\mathcal{G} : \mathrm{Ed}(\mathcal{G}) \to \mathbb{N} = \{1, 2, 3, \ldots\}$$

such that $\ell(y) = \ell(\bar{y})$. An *automorphism* of $\mathcal{G}$ is a pair $\phi = (\phi_1, \phi_2)$ of bijections $\phi_1 : \mathrm{Ver}(\mathcal{G}) \to \mathrm{Ver}(\mathcal{G})$ and $\phi_2 : \mathrm{Ed}(\mathcal{G}) \to \mathrm{Ed}(\mathcal{G})$ such that $\phi_1(o(y)) = o(\phi_2(y))$, $\overline{\phi_2(y)} = \phi_2(\bar{y})$, and $\ell(y) = \ell(\phi_2(y))$.

Let Γ be a group acting on a graph $\mathcal{G}$ (i.e., Γ acts via automorphisms). For $v \in \mathrm{Ver}(\mathcal{G})$, denote by

$$\mathrm{Stab}_\Gamma(v) = \{\gamma \in \Gamma \mid \gamma v = v\}$$

the stabilizer of v in Γ. Similarly, let $\mathrm{Stab}_\Gamma(y) = \mathrm{Stab}_\Gamma(\bar{y})$ be the stabilizer of $y \in \mathrm{Ed}(\mathcal{G})$. There is a quotient graph $\Gamma \setminus \mathcal{G}$ such that $\mathrm{Ver}(\Gamma \setminus \mathcal{G}) = \Gamma \setminus \mathrm{Ver}(\mathcal{G})$ and $\mathrm{Ed}(\Gamma \setminus \mathcal{G}) = \Gamma \setminus \mathrm{Ed}(\mathcal{G})$.

2.4. Mumford uniformization

Let $\mathcal{O}$ be a complete discrete valuation ring with fraction field K, finite residue field k and a uniformizer π. Let Γ be a subgroup of $\mathrm{GL}_2(K)$ whose image $\overline{\Gamma}$ in $\mathrm{PGL}_2(K)$ is discrete with compact quotient. There is a formal scheme $\widehat{\Omega}$ over $\mathrm{Spf}(\mathcal{O})$ which is equipped with a natural action of $\mathrm{PGL}_2(K)$ and parametrizes certain formal groups. Raynaud's "generic fibre" of $\widehat{\Omega}$ is Drinfeld's non-archimedean half-plane $\Omega = \mathbb{P}^{1,\mathrm{an}}_K - \mathbb{P}^{1,\mathrm{an}}_K(K)$ over K. For the description of the rigid-analytic structure of Ω and the construction of $\widehat{\Omega}$ we refer to Chapter I in [2].

Kurihara in [12] extended Mumford's fundamental result [15] and proved the following: there is a normal, proper and flat scheme X^Γ over $\mathrm{Spec}(\mathcal{O})$ such that the formal completion of X^Γ along its closed fibre is isomorphic to the quotient $\Gamma \setminus \widehat{\Omega}$. The generic fibre X^Γ_K is a smooth, geometrically integral curve over K. The closed fibre X^Γ_k is reduced with normal crossing singularities, and every irreducible component is isomorphic to $\mathbb{P}^1_k$. If x is a double point on X^Γ_k, then there exists a unique integer m_x for which the completion of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}} \widehat{\mathcal{O}}^{\mathrm{ur}}$ is isomorphic to the completion of

$$\widehat{\mathcal{O}}^{\mathrm{ur}}[t,s]/\bigl(ts - \pi^{m_x}\bigr).$$

Here $\widehat{\mathcal{O}}^{\mathrm{ur}}$ denotes the completion of the maximal unramified extension of $\mathcal{O}$.

The *dual graph* $\mathcal{G}$ of X^Γ is the following graph with lengths. The vertices of $\mathcal{G}$ are the irreducible components of X^Γ_k. The edges of $\mathcal{G}$, ignoring the orientation, are the singular points of X^Γ_k. If x is a double point and $\{y, \bar{y}\}$ is the corresponding edge of $\mathcal{G}$, then the extremities of y and $\bar{y}$ are the irreducible components passing through x; choosing between y or $\bar{y}$ corresponds to choosing one of the branches through x. Finally, $\ell(y) = \ell(\bar{y}) = m_x$.

Let $\mathcal{T}$ be the graph whose vertices $\mathrm{Ver}(\mathcal{T}) = \{[\Lambda]\}$ are the homothety classes of $\mathcal{O}$-lattices in K^2, and two vertices $[\Lambda]$ and $[\Lambda']$ are adjacent if we can choose representatives $L \in [\Lambda]$ and $L' \in [\Lambda']$ such that $L' \subset L$ and $L/L' \cong k$. One shows that $\mathcal{T}$ is an infinite tree in which every vertex is adjacent to exactly $\#k + 1$ other vertices. This is the *Bruhat–Tits tree* of $\mathrm{PGL}_2(K)$, cf. [27, p. 70]. The group $\mathrm{GL}_2(K)$ acts on $\mathcal{T}$ as the group of linear automorphisms of K^2, so the group Γ also acts on $\mathcal{T}$. We assign lengths to the edges of the quotient graph $\Gamma \setminus \mathcal{T}$: for $y \in \mathrm{Ed}(\Gamma \setminus \mathcal{T})$ let $\ell(y) = \#\mathrm{Stab}_{\overline{\Gamma}}(\tilde{y})$, where $\tilde{y}$ is a preimage of y in $\mathcal{T}$. By Proposition 3.2 in [12], there is an isomorphism $\mathcal{G} \cong \Gamma \setminus \mathcal{T}$ of graphs with lengths.

Notation 2.1. For $x \in |F|$, we denote Mumford's formal scheme over $\mathrm{Spf}(\mathcal{O}_x)$ by $\widehat{\Omega}_x$, and the Bruhat–Tits tree of $\mathrm{PGL}_2(F_x)$ by $\mathcal{T}_x$.

3. Modular curves of $\mathcal{D}$-elliptic sheaves

3.1. $\mathcal{D}$-elliptic sheaves

The notion of $\mathcal{D}$-elliptic sheaves was introduced in [13]. Here we follow [28], which gives a somewhat different (but equivalent) definition of $\mathcal{D}$-elliptic sheaves that is more convenient for our purposes.

From now on we assume that D is a division quaternion algebra which is split at ∞. Let $\mathcal{D}$ be an $\mathcal{O}_\mathbb{P}$-order in D such that $\mathcal{D}_x := \mathcal{D} \otimes_{\mathcal{O}_\mathbb{P}} \mathcal{O}_x$ is a maximal order in D_x for any $x \neq \infty$, and $\mathcal{D}_\infty$ is isomorphic to the subring of $\mathbb{M}_2(\mathcal{O}_\infty)$ consisting of matrices which are upper triangular modulo ϖ_∞. Let

$\mathcal{D}(-\frac{1}{2}\infty)$ denote the two-sided ideal in $\mathcal{D}$ given by $\mathcal{D}(-\frac{1}{2}\infty)_x = \mathcal{D}_x$ for all $x \neq \infty$, and $\mathcal{D}(-\frac{1}{2}\infty)_\infty$ is the radical of $\mathcal{D}_x$. Concretely, $\mathcal{D}(-\frac{1}{2}\infty)_\infty$ is the ideal of $\mathcal{D}_\infty$ consisting of matrices which are upper triangular modulo ϖ_∞ with zeros on the diagonal.

For a scheme S over $\mathbb{F}_q$ denote by Frob_S its Frobenius endomorphism, which is the identity on the points and the qth power map on the functions. Denote by $\mathbb{P} \times S$ the fiber product $\mathbb{P} \times_{\mathrm{Spec}(\mathbb{F}_q)} S$. For a sheaf $\mathcal{F}$ on $\mathbb{P}$ and $\mathcal{G}$ on S, the sheaf $\mathrm{pr}_1^*(\mathcal{F}) \otimes \mathrm{pr}_2^*(\mathcal{G})$ on $\mathbb{P} \times S$ is denoted by $\mathcal{F} \boxtimes \mathcal{G}$.

Definition 3.1. Let $z : S \to \mathbb{P}$ be a scheme over $\mathbb{P}$, which we also consider as a scheme over $\mathbb{F}_q$ via the composition $S \to \mathbb{P} \to \mathrm{Spec}(\mathbb{F}_q)$. A *$\mathcal{D}$-elliptic sheaf with pole ∞* over S is a pair $E = (\mathcal{E}, t)$ consisting of a locally free right $\mathcal{D} \boxtimes \mathcal{O}_S$-module $\mathcal{E}$ of rank 1 and an injective homomorphism of $\mathcal{D} \boxtimes \mathcal{O}_S$-modules

$$t : (\mathrm{id}_{\mathbb{P}} \times \mathrm{Frob}_S)^*\left(\mathcal{E} \otimes_{\mathcal{D}} \mathcal{D}\left(-\frac{1}{2}\infty\right)\right) \to \mathcal{E}$$

such that the cokernel of t is supported on the graph $\Gamma_z \subset \mathbb{P} \times S$ of z and is a locally free $\mathcal{O}_S$-module of rank 2.

Definition 3.2. Let $\mathscr{X}$ be a stack over a scheme S. An S-scheme X is a *coarse moduli scheme* for $\mathscr{X}$ if there is an S-morphism $\pi : \mathscr{X} \to X$ such that:

(1) Every S-morphism from $\mathscr{X}$ to an S-scheme Y factors uniquely through π.
(2) If $\mathrm{Spec}(k) \to S$ is a geometric point (k is an algebraically closed field), then π induces a bijection between the set of isomorphism classes of objects in $\mathscr{X}$ over k and $X(k)$.

This is essentially Definition I.8.1 in [4]. In general, a stack need not have a coarse moduli scheme, but the universal property guarantees that if X exists then it is unique.

Theorem 3.3. *The moduli stack $\mathscr{X}^{\mathcal{D}}$ of $\mathcal{D}$-elliptic sheaves of fixed degree $\deg(\mathcal{E}) = -1$ is a Deligne–Mumford stack of finite type over $\mathbb{P}$. It admits a coarse moduli scheme which will be denoted by $X^{\mathcal{D}}$. The canonical morphism $X^{\mathcal{D}} \to \mathbb{P}$ is projective of pure relative dimension 1. This morphism has geometrically connected fibres and is smooth over $\mathbb{P} - R - \infty$.*

Proof. This is a special case of Theorem 4.11 in [28], or Theorems 4.1 and 5.1 in [13], except for the statement that $X^{\mathcal{D}} \to \mathbb{P}$ has geometrically connected fibres. To prove this last claim, by the Stein factorization theorem, it is enough to show that $X^{\mathcal{D}}_{\mathbb{C}_\infty} := X^{\mathcal{D}} \times_{\mathbb{P}} \mathrm{Spec}(\mathbb{C}_\infty)$ is connected. Using the uniformization theorem [1, Theorem 4.4.11] and the strong approximation theorem for $D^\times$, one can deduce that the number of connected components of $X^{\mathcal{D}}_{\mathbb{C}_\infty}$ is equal to the class number of A, which is 1. □

Notation 3.4. For a $\mathbb{P}$-scheme $S = \mathrm{Spec}(Q)$ we denote $X^{\mathcal{D}}_Q := X^{\mathcal{D}} \times_{\mathbb{P}} S$.

3.2. Atkin–Lehner involutions

Let $\mathfrak{P}_x$ be the radical of $\mathcal{D}_x$. By [21, (39.1)], $\mathfrak{P}_x$ is a two-sided ideal in $\mathcal{D}_x$, and every two-sided ideal of $\mathcal{D}_x$ is an integral power of $\mathfrak{P}_x$. It is known that there exists $\Pi_x \in \mathfrak{P}_x$ such that $\Pi_x\mathcal{D}_x = \mathcal{D}_x\Pi_x = \mathfrak{P}_x$. The positive integer e_x such that $\mathfrak{P}_x^{e_x} = \varpi_x\mathcal{D}_x$ is the *index* of $\mathcal{D}_x$. With this definition, $e_x = 2$ if $x \in R \cup \infty$, and $e_x = 1$, otherwise. Define the group of divisors

$$\mathrm{Div}(\mathcal{D}) := \left\{ \sum_{x \in |F|} n_x x \in \bigoplus_{x \in |F|} \mathbb{Q}x \;\middle|\; e_x n_x \in \mathbb{Z} \text{ for any } x \in |F| \right\}.$$

For a divisor $Z = \sum_{x \in |F|} n_x x \in \mathrm{Div}(\mathcal{D})$, let $\mathcal{D}(Z)$ be the invertible $\mathcal{D}$-bimodule given by $\mathcal{D}(Z)|_{\mathbb{P}-\mathrm{Supp}(Z)} = \mathcal{D}|_{\mathbb{P}-\mathrm{Supp}(Z)}$ and $\mathcal{D}(Z)_x = \mathfrak{P}_x^{-n_x e_x}$ for all $x \in \mathrm{Supp}(Z)$. For each $f \in F^\times$ there is an associated divisor $\mathrm{div}(f)=\sum_{x\in|F|}\mathrm{ord}_x(f)x$, which we consider as an element of $\mathrm{Div}(\mathcal{D})$. It follows from [21, (40.9)] that the sequence

$$0\to F^\times/\mathbb{F}_q^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\mathcal{D}) \xrightarrow{Z\mapsto\mathcal{D}(Z)} \mathrm{Pic}(\mathcal{D})\to 0 \tag{3.1}$$

is exact, cf. [28, Section 3.2]. Let $\mathrm{Div}^0(\mathcal{D})\subset\mathrm{Div}(\mathcal{D})$ be the subgroup of degree 0 divisors: $\sum_{x\in|F|}n_x x\in\mathrm{Div}^0(\mathcal{D})$ if $\sum_{x\in|F|}n_x\deg(x)=0$. Define $\mathrm{Pic}^0(\mathcal{D})$ to be the image of $\mathrm{Div}^0(\mathcal{D})$ in $\mathrm{Pic}(\mathcal{D})$. It is easy to check that $\mathrm{Pic}^0(\mathcal{D})\cong(\mathbb{Z}/2\mathbb{Z})^{\#R}$, and is generated by the divisors $(\frac{\deg(x)}{2}\infty-\frac{1}{2}x)$, $x\in R$.

If $\mathcal{L}\in\mathrm{Pic}(\mathcal{D})$, then

$$E=(\mathcal{E},t)\mapsto E\otimes\mathcal{L}:=(\mathcal{E}\otimes_{\mathcal{D}}\mathcal{L},t\otimes_{\mathcal{D}}\mathrm{id}_{\mathcal{L}})$$

defines an automorphism of the stack of $\mathcal{D}$-elliptic sheaves. Moreover, if $\mathcal{L}\in\mathrm{Pic}^0(\mathcal{D})$, then this action preserves the substack consisting of $(\mathcal{E},t)$ with $\deg(\mathcal{E})$ fixed, cf. [28, Section 4.1]. Hence $W:=\mathrm{Pic}^0(\mathcal{D})$ acts on $\mathscr{X}^{\mathcal{D}}$ by automorphisms. By the universal property of the coarse moduli scheme, W also acts on $X^{\mathcal{D}}$ by automorphisms.

Definition 3.5. We call the subgroup W of $\mathrm{Aut}(X^{\mathcal{D}})$ the *group of Atkin–Lehner involutions*, and denote by $w_x\in W$, $x\in R$, the automorphism induced by

$$\mathcal{D}\left(\frac{\deg(x)}{2}\infty-\frac{1}{2}x\right).$$

Remark 3.6. It follows from [17, Theorem 4.6] that if $\mathrm{Odd}(R)=0$, then $\mathrm{Aut}(X^{\mathcal{D}})=W$.

Definition 3.7. For $y\in R$, denote the quotient curve $X^{\mathcal{D}}/w_y$ by $X^{(y)}$. Since w_y is an automorphism of $X^{\mathcal{D}}$ as a $\mathbb{P}$-scheme, the quotient morphism $\pi: X^{\mathcal{D}}\to X^{(y)}$ is a morphism of $\mathbb{P}$-schemes. It is possible to define a quotient stack $\mathscr{X}^{\mathcal{D}}/w_y=:\mathscr{X}^{(y)}$, using the general machinery developed in [22]. Then $X^{(y)}$ can also be defined as the coarse moduli scheme of $\mathscr{X}^{(y)}$.

The *normalizer* of $\mathcal{D}_x$ in D_x is the subgroup of $D_x^\times$

$$N(\mathcal{D}_x)=\{g\in D_x^\times \mid g\mathcal{D}_x g^{-1}=\mathcal{D}_x\}.$$

If $g\in N(\mathcal{D}_x)$, then $g\mathcal{D}_x$ is a two-sided ideal of $\mathcal{D}_x$, so there exists $m\in\mathbb{Z}$ such that $g\mathcal{D}_x=\mathfrak{P}_x^m$ Define $v_{\mathcal{D}_x}(g)=\frac{m}{e_x}$. Note that for $g\in F_x\subset N(\mathcal{D}_x)$, we have $\mathrm{ord}_x(g)=v_{\mathcal{D}_x}(g)$.

Let $\mathscr{C}(\mathcal{D}):=\prod'_{x\in|F|}N(\mathcal{D}_x)/F^\times\prod_{x\in|F|}\mathcal{D}_x^\times$, where $\prod'_{x\in|F|}N(\mathcal{D}_x)$ denotes the restricted direct product of the groups $\{N(\mathcal{D}_x)\}_{x\in|F|}$ with respect to $\{\mathcal{D}_x^\times\}_{x\in|F|}$. Given $a=\{a_x\}_x\in\prod'_{x\in|F|}N(\mathcal{D}_x)$, we put $\mathrm{div}(a)=\sum_{x\in|F|}v_{\mathcal{D}_x}(a_x)x$. The assignment $a\mapsto\mathcal{D}(\mathrm{div}(a))$ induces an isomorphism [28, Corollary 3.4]:

$$\mathscr{C}(\mathcal{D})\cong\mathrm{Pic}(\mathcal{D}). \tag{3.2}$$

Let $\mathcal{D}^\infty:=H^0(\mathbb{P}-\infty,\mathcal{D})$; this is a maximal A-order in D. Let $\Gamma^\infty:=(\mathcal{D}^\infty)^\times$ be the units in $\mathcal{D}^\infty$. Define the *normalizer* of $\mathcal{D}^\infty$ in D as

$$N(\mathcal{D}^\infty):=\{g\in D^\times \mid g\mathcal{D}^\infty g^{-1}=\mathcal{D}^\infty\}.$$

Denote $\mathscr{C}(\mathcal{D}^\infty)=N(\mathcal{D}^\infty)/F^\times\Gamma^\infty$. Then (3.2) induces an isomorphism

$$\mathscr{C}(\mathcal{D}^\infty) \cong \mathrm{Pic}^0(\mathcal{D}). \tag{3.3}$$

By (37.25) and (37.28) in [21], the natural homomorphism

$$N(\mathcal{D}^\infty)/F^\times\Gamma^\infty \to \prod_{x\in|F|-\infty} N(\mathcal{D}_x)/F_x^\times\mathcal{D}_x^\times \tag{3.4}$$

is an isomorphism. Next, by (37.26) and (37.27) in [21],

$$N(\mathcal{D}_x)/F_x^\times\mathcal{D}_x^\times \cong \begin{cases} 1, & \text{if } x\notin R\cup\infty; \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } x\in R. \end{cases}$$

For $x\in R$, the non-trivial element of $N(\mathcal{D}_x)/F_x^\times\mathcal{D}_x^\times$ is the image of Π_x. According to [21, (34.8)], there exist elements $\{\lambda_x\in\mathcal{D}^\infty\}_{x\in R}$ such that $\mathrm{Nr}(\lambda_x)A=\mathfrak{p}_x$. The image of λ_x in $\mathcal{D}_x$ can be taken as Π_x. Overall, $\mathscr{C}(\mathcal{D}^\infty)\cong(\mathbb{Z}/2Z)^{\#R}$ is generated by λ_x's, and the isomorphism (3.3) is given by $w_x\mapsto\lambda_x$.

3.3. Uniformization theorems

Later in the paper we will need to know how the Atkin–Lehner involutions in Definition 3.5 act on $X^{\mathcal{D}}$ in terms of rigid-analytic uniformizations of these curves. Here we describe these analytic actions.

Since $D_\infty\cong\mathbb{M}_2(F_\infty)$, the group Γ^∞ can be considered as a discrete subgroup of $\mathrm{GL}_2(F_\infty)$ via an embedding

$$\Gamma^\infty \hookrightarrow D^\times(F_\infty)\cong \mathrm{GL}_2(F_\infty).$$

Let $\widehat{X}^{\mathcal{D}}_{\mathcal{O}_\infty}$ denote the completion of $X^{\mathcal{D}}_{\mathcal{O}_\infty}$ along its special fibre. By a theorem of Blum and Stuhler [1, Theorem 4.4.11], there is an isomorphism of formal $\mathcal{O}_\infty$-schemes

$$\Gamma^\infty\setminus\widehat{\Omega}_\infty\cong\widehat{X}^{\mathcal{D}}_{\mathcal{O}_\infty}, \tag{3.5}$$

which is compatible with the action of W; see [28, Section 4.6]. More precisely, the action of w_x on $\Gamma^\infty\setminus\widehat{\Omega}_\infty$ induced by (3.5) is given by the action of λ_x considered as an element of $\mathrm{GL}_2(F_\infty)$. Note that λ_x is in the normalizer of Γ^∞, so it acts on the quotient $\Gamma^\infty\setminus\widehat{\Omega}_\infty$ and this action does not depend on a particular choice of λ_x.

Now fix some $x\in R$. Let $\bar{D}$ be the quaternion algebra over F which is ramified exactly at $(R-x)\cup\infty$. Fix a maximal A-order $\mathfrak{D}$ in $\bar{D}(F)$, and denote

$$\begin{aligned} A^x &= A[\wp_x^{-1}]; \\ \mathfrak{D}^x &= \mathfrak{D}\otimes_A A^x; \\ \mathfrak{D}^{x,2} &= \{\gamma\in\mathfrak{D}^x \mid \mathrm{ord}_x(\mathrm{Nr}(\gamma))\in 2\mathbb{Z}\}; \\ \Gamma^x &= (\mathfrak{D}^{x,2})^\times. \end{aligned}$$

If we fix an identification of $\bar{D}_x$ with $\mathbb{M}_2(F_x)$, then Γ^x is a subgroup of $\mathrm{GL}_2(F_x)$ whose image in $\mathrm{PGL}_2(F_x)$ is discrete and cocompact. Let $\mathcal{O}_x^{(2)}$ be the unramified quadratic extension of $\mathcal{O}_x$. Let $\gamma_x\in\mathfrak{D}^x$ be an element such that $\mathrm{Nr}(\gamma_x)A=\mathfrak{p}_x$. Such γ_x exists by [21, (34.8)] and it normalizes Γ^x, hence acts on $\Gamma^x\setminus\widehat{\Omega}_x$. Let $\widehat{X}^{\mathcal{D}}_{\mathcal{O}_x}$ denote the completion of $X^{\mathcal{D}}_{\mathcal{O}_x}$ along its special fibre. By the analogue of the Cherednik–Drinfeld uniformization, proven in this context by Hausberger [8], there is an isomorphism of formal $\mathcal{O}_x$-schemes

$$\big[\big(\Gamma^x \setminus \widehat{\Omega}_x\big) \otimes \mathcal{O}_x^{(2)}\big]/\big(\gamma_x \otimes \mathrm{Frob}_x^{-1}\big) \cong \widehat{X}^{\mathcal{D}}_{\mathcal{O}_x}, \tag{3.6}$$

where $\mathrm{Frob}_x : \mathcal{O}_x^{(2)} \to \mathcal{O}_x^{(2)}$ denotes the lift of the Frobenius homomorphism $a \mapsto a^{q_x}$ on $\overline{\mathbb{F}}_x$ to an $\mathcal{O}_x$-homomorphism.

Let $N(\mathfrak{D}^{x,2})$ be the normalizer of $\mathfrak{D}^{x,2}$ in $\bar{D}$, and

$$\mathscr{C}\big(\mathfrak{D}^{x,2}\big) := N\big(\mathfrak{D}^{x,2}\big)/F^\times \Gamma^x.$$

As in (3.4), the natural homomorphism

$$N\big(\mathfrak{D}^{x,2}\big)/F^\times \Gamma^x \to \prod_{y \in |F| - \infty} N\big(\mathfrak{D}^{x,2}_y\big)/F_y^\times \big(\mathfrak{D}^{x,2}_y\big)^\times$$

is an isomorphism. The normalizer $N(\mathfrak{D}^{x,2}_x)$ is $F_x^\times (\mathfrak{D}^x_x)^\times$, so we have

$$N\big(\mathfrak{D}^{x,2}_x\big)/F_x^\times \big(\mathfrak{D}^{x,2}_x\big)^\times \cong \mathbb{Z}/2\mathbb{Z},$$

generated by γ_x. On the other hand, if $y \neq x$, then

$$N\big(\mathfrak{D}^{x,2}_y\big)/F_y^\times \big(\mathfrak{D}^{x,2}_y\big)^\times \cong N(\mathcal{D}_y)/F_y^\times \mathcal{D}_y^\times.$$

We see that

$$\mathscr{C}\big(\mathfrak{D}^{x,2}\big) \cong (\mathbb{Z}/2\mathbb{Z})^{\#R},$$

generated by a set of elements $\{\gamma_y \in \mathfrak{D}^x\}_{y \in R}$ such that $\mathrm{Nr}(\gamma_y)A = \mathfrak{p}_y$. The group W is canonically isomorphic with $\mathscr{C}(\mathfrak{D}^{x,2})$ via $w_y \mapsto \gamma_y$. The isomorphism (3.6) is compatible with the action of W: for $y \in R$, the action of w_y on the left-hand side of (3.6) is given by γ_y; see [28, Section 4.6].

4. Main results

Proposition 4.1. *Denote by $\mathrm{Div}^d_{F_x}(X^{\mathcal{D}})$ the set of Weil divisors on $X^{\mathcal{D}}_{F_x}$ which are rational over F_x and have degree d.*

(1) *If $x \notin R$, then $\mathrm{Div}^d_{F_x}(X^{\mathcal{D}}) \neq \emptyset$ for any d.*
(2) *If $x \in R$, then $\mathrm{Div}^d_{F_x}(X^{\mathcal{D}}) \neq \emptyset$ for even d, and $\mathrm{Div}^d_{F_v}(X^{\mathcal{D}}) = \emptyset$ for odd d.*

Proof. For $n \geqslant 1$, denote by $\mathbb{F}_x^{(n)}$ the degree n extension of $\mathbb{F}_x$, and by $F_x^{(n)}$ the degree n unramified extension of F_x.

First, suppose $x \notin R \cup \infty$. By Theorem 3.3, $X^{\mathcal{D}}_{\mathbb{F}_x}$ is a smooth projective curve. Weil's bound on the number of rational points on a curve over a finite field guarantees the existence of an integer $N \geqslant 1$ such that $X^{\mathcal{D}}_{\mathbb{F}_x}(\mathbb{F}_x^{(n)}) \neq \emptyset$ for any $n \geqslant N$. The geometric version of Hensel's lemma [9, Lemma 1.1] implies that $X^{\mathcal{D}}_{F_x}(F_x^{(n)}) \neq \emptyset$. Let $P \in X^{\mathcal{D}}_{F_x}(F_x^{(N+1)})$ and $Q \in X^{\mathcal{D}}_{F_x}(F_x^{(N)})$. The divisor $d \cdot Z$, where

$$Z = \sum_{\sigma \in \mathrm{Gal}(F_x^{(N+1)}/F_x)} P^\sigma - \sum_{\tau \in \mathrm{Gal}(F_x^{(N)}/F_x)} Q^\tau,$$

is F_x-rational and has degree d.

Next, suppose $x=\infty$. By (3.5), $X^{\mathcal{D}}_{F_\infty}$ is Mumford uniformizable. This implies that $X^{\mathcal{D}}_{F_\infty}$ has a regular model over $\mathcal{O}_\infty$ whose special fibre consists of $\mathbb{F}_\infty$-rational $\mathbb{P}^1$'s crossing at $\mathbb{F}_\infty$-rational points. In particular, over any extension $\mathbb{F}_\infty^{(n)}$, $n\geqslant 2$, there are smooth $\mathbb{F}_\infty^{(n)}$-rational points. Again by Hensel's lemma [9, Lemma 1.1], there are $F_\infty^{(n)}$-rational points on $X^{\mathcal{D}}_{F_\infty}$ for any $n\geqslant 2$. The trace to F_∞ of such a point is in $\mathrm{Div}^n_{F_\infty}(X^{\mathcal{D}})$. One obtains a rational divisor of degree 1 by taking the difference of degree 3 and 2 rational divisors. This proves (1).

Finally, suppose $x\in R$. By [19, Theorem 4.1], $X^{\mathcal{D}}_{F_x}(F_x^{(2)})\neq\emptyset$. Taking the trace of an $F_x^{(2)}$-rational point and multiplying the resulting divisor by n, we see that $\mathrm{Div}^{2n}_{F_x}(X^{\mathcal{D}})\neq\emptyset$ for any n. Now suppose d is odd but $\mathrm{Div}^d_{F_x}(X^{\mathcal{D}})\neq\emptyset$. Let $Z\in\mathrm{Div}^d_{F_x}(X^{\mathcal{D}})$. Write $Z=Z_1-Z_2$, where Z_1 and Z_2 are effective divisors. Since $\deg(Z)=\deg(Z_1)-\deg(Z_2)$ is odd, exactly one of these divisors has odd degree. Denote by F_x^{alg} the algebraic closure of F_x, F_x^{sep} the separable closure of F_x, and let $G:=\mathrm{Gal}(F_x^{\mathrm{sep}}/F_x)$. Since Z is F_x-rational, both Z_1 and Z_2 are G-invariant. Assume without loss of generality that $\deg(Z_1)$ is odd. Write $Z_1=Z_o+Z_e$, where $Z_o=\sum_{P\in X^{\mathcal{D}}_{F_x}(F_x^{\mathrm{alg}})} n_P P$, $n_P\in\mathbb{Z}$ are odd, and $Z_e=\sum_{Q\in X^{\mathcal{D}}_{F_x}(F_x^{\mathrm{alg}})} n_Q Q$, $n_Q\in\mathbb{Z}$ are even. Again Z_o and Z_e are G-invariant. Since $\deg(Z_e)$ is even, Z_o is non-zero. Since $\deg(Z_o)$ is necessarily odd, the support of Z_o must consist of an odd number of points. This set of points is G-invariant. We have a finite set of odd cardinality with an action of G, so one of the orbits necessarily has odd length. Thus, there is a point P in the support of Z such that the set of Galois conjugates of P has odd cardinality. This implies that the separable degree $[F_x(P):F_x]_s$ is odd. If P is not separable, then the degree of inseparability of $F_x(P)$ over F_x divides the weight n_P of P in Z (as Z is F_x-rational). Since n_P is odd by assumption, the inseparable degree $[F_x(P):F_x]_i$ is also odd. Overall, the degree of the extension $F_x(P)/F_x$ is odd. We conclude that there is a finite extension K/F_x of odd degree such that $X^{\mathcal{D}}_{F_x}(K)\neq\emptyset$. This contradicts [19, Theorem 4.1], so $\mathrm{Div}^d_{F_x}(X^{\mathcal{D}})$ must be empty. $\square$

Theorem 4.2. *Consider the following two conditions*:

(1) *q is even*;
(2) *q is odd, $\#R=2$, and* $\mathrm{Odd}(R)=1$.

If one of these conditions holds, then the deficient places for $X^{\mathcal{D}}_F$ are the places in R. Otherwise, there are no deficient places for $X^{\mathcal{D}}_F$. In either case, by Theorem 1.1, the Jacobian variety of $X^{\mathcal{D}}_F$ is even.

Proof. Using Proposition 4.1, it is enough to show that the genus of $X^{\mathcal{D}}_F$ is even if and only if one of the above conditions holds. The genus $g(X^{\mathcal{D}}_F)$ of $X^{\mathcal{D}}_F$ is given by the formula (see [16])

$$g\bigl(X^{\mathcal{D}}_F\bigr)=1+\frac{1}{q^2-1}\prod_{x\in R}(q_x-1)-\frac{q}{q+1}\cdot 2^{\#R-1}\cdot\mathrm{Odd}(R). \tag{4.1}$$

Note that modulo q the genus is congruent to $1+(-1)^{\#R+1}$. Since $\#R$ is even, q divides $g(X^{\mathcal{D}}_F)$. Hence the genus is even if q is even. From now on we assume that q is odd.

First, assume $\mathrm{Odd}(R)=0$. Let $x\in R$ be a place of even degree, and $y\neq x$ be another place. Since $(q_x-1)(q_y-1)/(q^2-1)$ is an even integer, $g(X^{\mathcal{D}}_F)$ is odd. Now assume $\mathrm{Odd}(R)=1$. Denote $r=\#R$. Let $\{n_1,n_2,\dots,n_r\}$ be the degrees of places in R. These are odd integers by assumption. For odd integers n and q we can write

$$q^n-1=(q-1)q^{n-1}+(q+1)M,$$

where M is even. Thus,

$$\frac{1}{q^2-1}\prod_{i=1}^{r}\bigl(q^{n_i}-1\bigr)=\frac{S}{q+1}\prod_{i=2}^{r}(q-1)q^{n_i-1}+M',$$

where $S = q^{n_1-1} + q^{n_1-2} + \cdots + q + 1$ and M' is an even integer. This implies that

$$g\big(X_F^{\mathcal{D}}\big) - 1 = q\frac{Sq^{m-1}(q-1)^{r-1} - 2^{r-1}}{q+1} + M',$$

where $m > 1$ is an even integer. Let $a \geqslant 1$ and $c \geqslant 1$ be odd integers, and $b \geqslant 0$ be even. Let

$$B = q^a\big(q^b + q^{b-1} + \cdots + 1\big).$$

We are reduced to proving that

$$\Delta = \frac{B(q-1)^c - 2^c}{q+1}$$

is odd if and only if $c = 1$. If $b \geqslant 2$, then by writing $q^b + q^{b-1} + \cdots + 1 = q^b + (q+1)(q^{b-2} + \cdots + 1)$ we see that the parity of Δ coincides with the parity of

$$\frac{q^{a+b}(q-1)^c - 2^c}{q+1}.$$

Thus, we can assume $b = 0$. Expand

$$q^a(q-1)^c - 2^c = \sum_{i=0}^{c}(-1)^i\binom{c}{i}\big(q^{a+c-i} + (-1)^{i+1}\big).$$

Note that $(q^{a+c-i} + (-1)^{i+1})/(q+1)$ is an integer which is even if and only if i is even. Therefore, the parity of Δ coincides with the parity of

$$\binom{c}{1} + \binom{c}{3} + \cdots + \binom{c}{c} = 2^{c-1},$$

which is obviously odd if and only if $c = 1$. $\square$

Proposition 4.3. *Denote by* $\mathrm{Div}^d_{F_x}(X^{(y)})$ *the set of Weil divisors on* $X^{(y)}_{F_x}$ *which are rational over* F_x *and have degree* d.

(1) *If* $x \notin R$ *or* $x = y$, *then* $\mathrm{Div}^d_{F_x}(X^{(y)}) \neq \emptyset$ *for any* d.
(2) *If* $x \in R - y$ *and* d *is even, then* $\mathrm{Div}^d_{F_x}(X^{(y)}) \neq \emptyset$.
(3) *If* $x \in R - y$ *and* $\mathrm{Div}^d_{F_x}(X^{(y)}) \neq \emptyset$ *for an odd* d, *then there is an extension* K/F_x *of odd degree such that* $X^{(y)}_{F_x}(K) \neq \emptyset$.

Proof. Let $\pi : X^{\mathcal{D}}_{F_x} \to X^{(y)}_{F_x}$ be the quotient morphism. If $Z \in \mathrm{Div}^d_{F_x}(X^R)$, then the pushforward $\pi_*(Z)$ is in $\mathrm{Div}^d_{F_x}(X^{(y)})$, so Proposition 4.1 implies (2) and (1) for $x \notin R$. Part (3) follows from the argument in the proof of Proposition 4.1. It remains to prove that $\mathrm{Div}^d_{F_y}(X^{(y)}) \neq \emptyset$ for any d. By (3.6) and ensuing discussion, $X^R_{F_y}$ is the $w_y \otimes \mathrm{Frob}_y^{-1}$ quadratic twist of the Mumford curve $\Gamma^y \setminus \widehat{\Omega}_y$. Hence the quotient $X^{(y)}_{F_y}$ of $X^{\mathcal{D}}_{F_y}$ by w_y is Mumford uniformizable (without a twist) and one can argue as in the proof of Proposition 4.1 in the case when $x = \infty$. $\square$

Proposition 4.4. *Assume q is odd, and $x, y \in R$ are two distinct places of even degrees. If d is odd, then* $\mathrm{Div}^{(d)}_{F_x}(X^{(y)}) = \emptyset$.

Proof. Suppose d is odd and $\mathrm{Div}^{(d)}_{F_x}(X^{(y)}) \neq \emptyset$. Then by Proposition 4.3 there is an extension K/F_x of odd degree such that $X^{(y)}_{F_x}(K) \neq \emptyset$. The graph $G := \Gamma^x \setminus \mathcal{T}_x$ is the dual graph of the Mumford curve uniformized by Γ^x. From (3.6) we get an action of W on G. The same argument as in [10, p. 683] shows that if $X^{(y)}_{F_x}(K) \neq \emptyset$, then there is an edge s in G such that the following two conditions hold:

(1) either $\ell(s)$ is even or $w_y(s) = s$;
(2) either $w_x(s) = \bar{s}$ or $w_x w_y(s) = \bar{s}$.

By [19, Lemma 4.4], an edge of G has length 1 or $q+1$, and the number of edges of length $q+1$ is equal to

$$2^{\#R-1}\,\mathrm{Odd}(R-x)\big(1-\mathrm{Odd}(x)\big).$$

From the assumption that y has even degree we get that all edges of G have length 1. Thus, for the existence of K-rational points on $X^{(y)}_{F_x}$ we must have $w_y(s) = s$, and either $w_x(s) = \bar{s}$ or $w_x w_y(s) = \bar{s}$. Obviously $w_y(s) = s$ and $w_x w_y(s) = \bar{s}$ imply $w_x(s) = \bar{s}$. Therefore, the considerations reduce to a single case

$$w_x(s) = \bar{s} \quad \text{and} \quad w_y(s) = s.$$

Let $\tilde{s}$ be an edge of $\mathcal{T}_x$ lying above s. Modifying γ_x by an element of Γ^x, we may assume that $\gamma_x \tilde{s} = \bar{\tilde{s}}$. Let v be one of the extremities of $\tilde{s}$. Then γ_x^2 fixes v and $\mathrm{Nr}(\gamma_x^2)$ generates $\mathfrak{p}_x^2$. Thus, $\gamma_x^2 \in F_x^\times \mu\, \mathrm{GL}_2(\mathcal{O}_x)\mu^{-1}$ for some $\mu \in \mathrm{GL}_2(F_x)$. By the norm condition, we get $\gamma_x^2 = \wp_x c$, where $c \in \mu\, \mathrm{GL}_2(\mathcal{O}_x)\mu^{-1}$. Hence $\mathrm{ord}_x(\mathrm{Nr}(\gamma_x^2/\wp_x)) = 0$. On the other hand, since $\gamma_x^2/\wp_x$ also belongs to $\mathfrak{D}^x$, c belongs to a maximal A-order $\mathfrak{D}'$ in $\bar{D}$ (in fact, $\mathfrak{D}' = \mu\, \mathrm{GL}_2(\mathcal{O}_x)\mu^{-1} \cap \mathfrak{D}^x$). Since $\mathrm{Nr}(c)$ has zero valuation at every $v \in |F| - \infty$, $c \in (\mathfrak{D}')^\times$. By our assumption, $\deg(y)$ is even and $\bar{D}$ is ramified at y and ∞, so $(\mathfrak{D}')^\times \cong \mathbb{F}_q^\times$; cf. [5, Lemma 1]. Hence $\gamma_x^2 = c\wp_x$, where $c \in \mathbb{F}_q^\times$. Since $\deg(x)$ is even, c must be a non-square, as otherwise ∞ splits in $F(\sqrt{c\wp_x})$, which contradicts the fact that this is a subfield of the quaternion algebra $\bar{D}$ ramified at ∞. Fix a non-square $\xi \in \mathbb{F}_q^\times$. Overall, we conclude that the condition $w_x(s) = \bar{s}$ translates into

$$\gamma_x^2 = \xi \wp_x,$$

for an appropriate choice of γ_x.

Modifying γ_y by an element of Γ^x, we can further assume that $\gamma_y(\tilde{s}) = \tilde{s}$. Next, note that γ_y belongs to some maximal A-order $\mathfrak{D}''$ in $\bar{D}$. Since $\bar{D}$ is ramified at y and $\mathrm{Nr}(\gamma_y)A = \mathfrak{p}_y$, the element γ_y generates the radical of $\mathfrak{D}^x_y$. Hence $\gamma_y^2 = c \cdot \wp_y$, where $c \in \mathfrak{D}''$. Comparing the norms of both sides, we see that c must be a unit in $\mathfrak{D}''$. The same argument as with $\mathfrak{D}'$ shows that $(\mathfrak{D}'')^\times \cong \mathbb{F}_q^\times$, so after possibly scaling γ_y by a constant in $\mathbb{F}_q^\times$, we get

$$\gamma_y^2 = \xi \wp_y.$$

Let $\langle \Gamma^x, \gamma_y\rangle$ be the subgroup of $\mathrm{GL}_2(F_x)$ generated by Γ^x and γ_y. By construction, the element γ_y fixes $\tilde{s}$. Since the edges of G have length 1, the stabilizer of $\tilde{s}$ in Γ^x is $(A^x)^\times$. Therefore,

$$\mathrm{Stab}_{\langle \Gamma^x, \gamma_y\rangle}(\tilde{s})/\big(A^x\big)^\times \subset \mathbb{F}_q(\gamma_y)^\times.$$

On the other hand, $\gamma_x^{-1}\gamma_y\gamma_x(\tilde{s}) = \tilde{s}$. We conclude that there is $n \in \mathbb{Z}$ and $a, b \in \mathbb{F}_q$ (a, b are not both zero) such that

$$\gamma_y\gamma_x = \wp_x^n\gamma_x(a + b\gamma_y).$$

Now the same argument as in the proof of part (3) of Theorem 4.1 in [19] shows that for such an equality to be true we must have $n = 0$, $a = 0$ and $b = -1$, i.e.,

$$\gamma_y\gamma_x = -\gamma_x\gamma_y.$$

The quadratic extensions $F(\gamma_x)$ and $F(\gamma_y)$ of F are obviously linearly disjoint. Therefore, $\bar{D}$ is isomorphic to the quaternion algebra $H(\xi\wp_x, \xi\wp_y)$ over F having the presentation:

$$i^2 = \xi\wp_x, \qquad j^2 = \xi\wp_y, \qquad ij = -ji.$$

As is well known, the algebra $H(\xi\wp_x, \xi\wp_y)$ ramifies (resp. splits) at $v \in |F|$ if and only if the local symbol $(\xi\wp_x, \xi\wp_y)_v = -1$ (resp. $= 1$); cf. [30, p. 32]. On the other hand, by [26, p. 210]

$$(\xi\wp_x, \xi\wp_y)_x = \left(\frac{\xi\wp_y}{\mathfrak{p}_x}\right) \quad \text{and} \quad (\xi\wp_x, \xi\wp_y)_y = \left(\frac{\xi\wp_x}{\mathfrak{p}_y}\right),$$

where $(\frac{\cdot}{\cdot})$ is the Legendre symbol. Since x and y have even degree, ξ is a square modulo $\mathfrak{p}_x$ and $\mathfrak{p}_y$. Thus, $(\frac{\xi\wp_y}{\mathfrak{p}_x}) = (\frac{\wp_y}{\mathfrak{p}_x})$ and $(\frac{\xi\wp_x}{\mathfrak{p}_y}) = (\frac{\wp_x}{\mathfrak{p}_y})$. The algebra $\bar{D}$ splits at x and ramifies at y, so we must have

$$\left(\frac{\wp_y}{\mathfrak{p}_x}\right) = 1 \quad \text{and} \quad \left(\frac{\wp_x}{\mathfrak{p}_y}\right) = -1.$$

But the quadratic reciprocity [23, Theorem 3.5] says that

$$\left(\frac{\wp_y}{\mathfrak{p}_x}\right)\left(\frac{\wp_x}{\mathfrak{p}_y}\right) = (-1)^{\frac{q-1}{2}\deg(x)\deg(y)} = 1.$$

This leads to a contradiction, so $\mathrm{Div}^d_{F_x}(X^{(y)}) = \emptyset$. $\square$

Theorem 4.5. *Assume q is odd and all places in R have even degrees. Consider the following three conditions*:

(1) $R = \{x, y\}$, *i.e.*, $\#R = 2$;
(2) $(\frac{\wp_y}{\mathfrak{p}_x}) = -1$;
(3) $\deg(y)$ *is not divisible by* 4.

If one of these conditions fails, then there are no deficient places for $X_F^{(y)}$. If all three conditions hold, then x is the only deficient place for $X_F^{(y)}$. In the first case the Jacobian of $X_F^{(y)}$ is even and in the second case it is odd.

Proof. Let $\mathrm{Fix}(w_y)$ be the number of fixed points of w_y acting on $X_F^{\mathcal{D}}$. By the Hurwitz genus formula applied to the quotient map $\pi : X_F^{\mathcal{D}} \to X_F^{(y)}$, the genus of $X_F^{(y)}$ is equal to

$$g\big(X^{(y)}\big) = \frac{g(X^{\mathcal{D}}) + 1}{2} - \frac{\mathrm{Fix}(w_y)}{4}$$

(note that π has only tame ramification). The genus $g(X^{\mathcal{D}})$ is given by the formula (4.1). On the other hand, by [17, Proposition 4.12]

$$\mathrm{Fix}(w_y) = h(\xi\wp_y)\prod_{x\in R}\left(1-\left(\frac{\xi\wp_y}{\mathfrak{p}_x}\right)\right),$$

where $\xi \in \mathbb{F}_q^\times$ is a fixed non-square, and $h(\xi\wp_y)$ denotes the ideal class number of the Dedekind ring $\mathbb{F}_q[T, \sqrt{\xi\wp_y}]$. (A remark is in order: In [17], w_y is defined analytically as the involution of $\Gamma^\infty \setminus \Omega_\infty$ induced by λ_y, hence here we are implicitly using the fact that (3.5) is compatible with the action of W.) Combining these formulas, we get

$$g\left(X^{(y)}\right) = 1 + \frac{1}{2(q^2-1)}\prod_{x\in R}(q_x-1) - \frac{h(\xi\wp_y)}{4}\prod_{x\in R}\left(1-\left(\frac{\xi\wp_y}{\mathfrak{p}_x}\right)\right). \tag{4.2}$$

It is easy to see that the middle summand is always an even integer. Hence $g(X^{(y)})$ is even if and only if the last summand is odd. According to [3, Theorem 1], the class number $h(\xi\wp_y)$ is always even and it is divisible by 4 if and only if $\deg(y)$ is divisible by 4. Using this fact, one easily checks that the last summand is odd if and only if the three conditions are satisfied. The theorem now follows from Propositions 4.3 and 4.4. □

Proposition 4.6. *There are infinitely many pairs $R = \{x, y\}$ for which the conditions in Theorem 4.5 are satisfied. Hence there are infinitely many $X_F^{(y)}$ with odd Jacobians.*

Proof. Fix an arbitrary y such that $\deg(y) \equiv 2 \pmod 4$. By the function field analogue of Dirichlet's theorem [23, Theorem 4.7], there are infinitely many places $x \in |F|$ of even degree such that $(\frac{\wp_x}{\mathfrak{p}_y}) = -1$. The quadratic reciprocity implies that for such places $(\frac{\wp_y}{\mathfrak{p}_x}) = -1$. □

Proposition 4.7. *For a fixed q there are only finitely many R such that $X_F^{(y)}$ is hyperelliptic.*

Proof. Fix some $x \notin R \cup \infty$. Corollary 4.8 in [16] gives a lower bound on the number of $\mathbb{F}_x^{(2)}$-rational points on $X_{\mathbb{F}_x}^{\mathcal{D}}$. Since the quotient map $X_{\mathbb{F}_x}^{\mathcal{D}} \to X_{\mathbb{F}_x}^{(y)}$ is defined over $\mathbb{F}_x$ and has degree 2, from this bound we get

$$\#X_{\mathbb{F}_x}^{(y)}\left(\mathbb{F}_x^{(2)}\right) \geqslant \frac{1}{2}\#X_{\mathbb{F}_x}^{\mathcal{D}}\left(\mathbb{F}_x^{(2)}\right) \geqslant \frac{1}{2(q^2-1)}\prod_{z\in R\cup x}(q_z-1).$$

By [14, Proposition 5.14], if $X_F^{(y)}$ is hyperelliptic, then $X_{\mathbb{F}_x}^{(y)}$ is also hyperelliptic. Hence there is a degree-2 morphism $X_{\mathbb{F}_x}^{(y)} \to \mathbb{P}_{\mathbb{F}_x}^1$ defined over $\mathbb{F}_x$. This implies

$$\#X_{\mathbb{F}_x}^{(y)}\left(\mathbb{F}_x^{(2)}\right) \leqslant 2\#\mathbb{P}_{\mathbb{F}_x}^1\left(\mathbb{F}_x^{(2)}\right) = 2\left(q_x^2+1\right).$$

Comparing with the earlier lower bound on $\#X_{\mathbb{F}_x}^{(y)}(\mathbb{F}_x^{(2)})$, we get

$$\prod_{z\in R\cup x}(q_z-1) \leqslant 4\left(q_x^2+1\right)\left(q^2-1\right). \tag{4.3}$$

Let $r = \sum_{z \in R} \deg(z)$. By [19, Lemma 7.7], we can choose $x \notin R \cup \infty$ such that $\deg(x) \leqslant \log_q(r+1)+1$. Since $\prod_{z \in R}(q_z - 1) \geqslant q^{r/2}$, the inequality (4.3) implies $q^{r/2} < 32q^3 r$, which obviously is possible only for finitely many R. Therefore, only finitely many $X_F^{(y)}$ are hyperelliptic. $\square$

Proposition 4.8. *Assume q is odd, $R = \{x, y\}$, and $\deg(x) = \deg(y) = 2$. Denote by $J^{(y)}$ the Jacobian variety of $X_F^{(y)}$. The Tate–Shafarevich group $\text{Ш}(J^{(y)})$ is finite.*

Proof. The definitions of the concepts discussed in this paragraph can be found in [6]. Let $\mathfrak{n} \lhd A$ be an ideal. Let $X_0(\mathfrak{n})$ be the compactified Drinfeld modular curve classifying pairs $(\phi, C_\mathfrak{n})$, where ϕ is a rank-2 Drinfeld A-modules and $C_\mathfrak{n} \cong A/\mathfrak{n}$ is a cyclic subgroup of ϕ. Let $J_0(\mathfrak{n})$ denote the Jacobian of $X_0(\mathfrak{n})_F$. Let $\Gamma_0(\mathfrak{n})$ be the level-$\mathfrak{n}$ Hecke congruence subgroup of $\mathrm{GL}_2(A)$. Let $S_0(\mathfrak{n})$ be the $\mathbb{C}$-vector space of automorphic cusp forms of Drinfeld type on $\Gamma_0(\mathfrak{n})$. Let $\mathbb{T}(\mathfrak{n})$ be the commutative $\mathbb{Z}$-algebra generated by the Hecke operators acting on $S_0(\mathfrak{n})$. The Hecke algebra $\mathbb{T}(\mathfrak{n})$ is a finitely generated free $\mathbb{Z}$-module which also naturally acts on $J_0(\mathfrak{n})$. Let $f \in S_0(\mathfrak{n})$ be a newform which is an eigenform for all $t \in \mathbb{T}(\mathfrak{n})$. Denote by $\lambda_f(t)$ the eigenvalue of t acting on f. The map $\mathbb{T}(\mathfrak{n}) \to \mathbb{C}$, $t \mapsto \lambda_f(t)$, is an algebra homomorphism; denote its kernel by I_f. The image $I_f(J_0(\mathfrak{n}))$ is an abelian subvariety of $J_0(\mathfrak{n})$ defined over F. Let $A_f := J_0(\mathfrak{n})/I_f(J_0(\mathfrak{n}))$. Similar to the case of classical modular Jacobians over $\mathbb{Q}$, the Jacobian $J_0(\mathfrak{n})$ is isogenous over F to a direct product of abelian varieties A_f, where each f is a newform of some level $\mathfrak{m}|\mathfrak{n}$ (a given A_f can appear more than once in the decomposition of $J_0(\mathfrak{n})$). This implies that $\text{Ш}(J_0(\mathfrak{n}))$ is finite if and only if $\text{Ш}(A_f)$ is finite for all such A_f. On the other hand, by the main theorem of [11], $\text{Ш}(A_f)$ is finite if and only if

$$\mathrm{ord}_{s=1} L(A_f, s) = \mathrm{rank}_\mathbb{Z} A_f(F),$$

where $L(A_f, s)$ denotes the L-function of A_f; see [11] or [25] for the definition.

Let $J^\mathcal{D}$ denote the Jacobian of $X_F^\mathcal{D}$. Let $\mathfrak{r} := \prod_{x \in R} \mathfrak{p}_x$. The Jacquet–Langlands correspondence over F in combination with some other deep results implies that there is a surjective homomorphism $J_0(\mathfrak{r}) \to J^\mathcal{D}$ defined over F; see [18, Theorem 7.1]. Since by construction $X^{(y)}$ is a quotient of $X^\mathcal{D}$, there is also a surjective homomorphism $J^\mathcal{D} \to J^{(y)}$ defined over F. Thus, there is a surjective homomorphism $J_0(\mathfrak{r}) \to J^{(y)}$ defined over F. This implies that if $\text{Ш}(J_0(\mathfrak{r}))$ is finite, then $\text{Ш}(J^{(y)})$ is also finite.

Now assume q is odd, $R = \{x, y\}$, and $\deg(x) = \deg(y) = 2$. In this case $J_0(\mathfrak{r})$ is isogenous to $J^\mathcal{D}$ as both have dimension q^2. There are no old forms of level $\mathfrak{r}$, since $S_0(1)$, $S_0(\mathfrak{p}_x)$ and $S_0(\mathfrak{p}_y)$ are zero-dimensional. Let $f \in S_0(\mathfrak{r})$ be a Hecke eigenform. The L-function $L(f, s)$ of f is a polynomial in q^{-s} of degree $\deg(x) + \deg(y) - 3 = 1$, cf. [29, p. 227]. Hence $\mathrm{ord}_{s=0} L(f, s) \leqslant 1$. Using the analogue of the Gross–Zagier formula over F [24, p. 440], one concludes that $\mathrm{ord}_{s=1} L(A_f, s) \leqslant \mathrm{rank}_\mathbb{Z} A_f(F)$. The converse inequality is known to hold for any abelian variety over F; see the main theorem of [25]. Hence $\text{Ш}(A_f)$ is finite, which, as we explained, implies that $\text{Ш}(J^{(y)})$ is also finite. $\square$

Corollary 4.9. *Assume q is odd. Fix $x \in |F|$ with $\deg(x) = 2$. There are $(q^2 - 1)/4$ places $y \in |F|$ such that $\deg(y) = 2$ and $(\frac{\wp_y}{\mathfrak{p}_x}) = -1$. For $R = \{x, y\}$, the Tate–Shafarevich group $\text{Ш}(J^{(y)})$ is finite and has non-square order. The dimension of $J^{(y)}$ is $(q^2 - 1)/2$.*

Proof. If $R = \{x, y\}$ is such that $\deg(x) = \deg(y) = 2$ and $(\frac{\wp_y}{\mathfrak{p}_x}) = -1$, then $\text{Ш}(J^{(y)})$ is finite by Proposition 4.8, and has non-square order by Theorem 4.5. In this case $\dim J^{(y)} = (q^2 - 1)/2$ by (4.2).

It remains to count the number of places y. Consider the geometric quadratic extension $K := F(\sqrt{\wp_x})$ of F, and let C be the corresponding smooth projective curve over $\mathbb{F}_q$. Since $\deg(x) = 2$, the genus of this curve is zero, so $C \cong \mathbb{P}^1_{\mathbb{F}_q}$. Let $z \in |F|$ be a place of degree 1. Either z remains inert in K, in which case it produces a degree-2 place of K, or z splits in K and produces two places of degree 1. Let N be the number of degree-1 places of F which split in K. Since C is a projective line, the number of degree-1 places of K is $q + 1$. Hence $2N = q + 1$. This implies that the number of degree-1 places of F which remain inert in K is $(q+1)-N=(q+1)/2$. Next, K has $q(q-1)/2$ places of degree 2 (this is just the number of monic irreducible quadratic polynomials in A). As we saw, $(q+1)/2$ of these places come from degree-1 places of F. One place comes from x, which ramifies, so the remaining $(q^2-2q-3)/2$ degree-2 places of K must come from degree-2 places of F which split in K. Therefore, the number of degree-2 places of F which remain inert in K is

$$\frac{q(q-1)}{2}-\frac{q^2-2q-3}{4}-1=\frac{q^2-1}{4}. \qquad \square$$

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