Mihran Papikian

On eigenvalues of p-adic curvature

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Abstract. We study the eigenvalues of the p-adic curvature transformations on buildings. In particular, we determine the maximal eigenvalues of these transformations.

1. Introduction

Let $\mathcal{K}$ be a non-archimedean locally compact field with finite residue field of order q. Let G be an almost simple linear algebraic group defined over $\mathcal{K}$ of $\mathcal{K}$-rank $\ell + 1$. Let $\mathfrak{T}$ be the Bruhat-Tits building associated with $G(\mathcal{K})$ [7]. This is an infinite, locally finite, contractible simplicial complex of dimension $\ell + 1$. Let X be the link of a vertex of $\mathfrak{T}$. X is a finite simplicial complex of dimension ℓ, which is a building in the sense of Tits [6]. In [12], Garland defined a certain combinatorial Laplace operator Δ acting on the i-cochains $C^i(X)$, $0 \leq i \leq \ell - 1$; see Definition 3.3. $\mathfrak{T}$ can be realized as the skeleton of a non-archimedean symmetric space [3, Chap. 5], and from this point of view the operators Δ are the non-archimedean analogues of curvature transformations of riemannian symmetric spaces. Denote by $m^i(X)$ the minimal non-zero eigenvalue of Δ acting on $C^i(X)$. By a rather ingenious argument, Garland proved that for any $\varepsilon > 0$ there is a constant $q(\varepsilon, \ell)$ depending only on ε and ℓ such that if $q > q(\varepsilon, \ell)$ then $m^i(X) \geq \ell - i - \varepsilon$. The main application of Garland's estimate on $m^i(X)$ is a vanishing result for the cohomology groups of discrete cocompact subgroups of $G(\mathcal{K})$; see Theorem 6.2. This vanishing theorem plays an important role in many problems arising in representation theory and arithmetic geometry.

The ideas in [12], especially the relationship between the vanishing of group cohomology and lower estimates on minimal non-zero eigenvalues of combinatorial Laplacians, have since been generalized to other contexts, cf. [10,15]. In this paper we are interested in the spectrum of Δ in the original set-up of [12], and especially in its maximal eigenvalue.

Let X be an arbitrary finite building of dimension ℓ. Denote by $M^i(X)$ the maximal eigenvalue of Δ acting on $C^i(X)$, $0 \leq i \leq \ell - 1$. The main result of this paper is the following (Theorem 5.4):

M. Papikian: Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA. e-mail: papikian@math.psu.edu

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Theorem 1.1. *For any* $0 \leq i \leq \ell - 1$ *there is an equality* $M^i(X) = \ell + 1$.

Using this result, we show that $m^0(X) \leq \ell$ (Theorem 5.5). Due to Garland's lower estimate, this is the best possible upper bound on $m^0(X)$ which does not depend on q. Based on some explicit calculations, we also propose a conjectural description of the behavior of all eigenvalues of Δ acting on $C^0(X)$ as $q \to \infty$ (Conjecture 5.7). Incidentally, our explicit calculations of the eigenvalues of Laplace operators indicate that, despite the hope expressed in [12], Garland's method is not powerful enough to prove the vanishing of cohomology groups unconditionally, i.e., without a restriction on q being sufficiently large; see Remark 6.1.

Our proof is based on a modification of Garland's original arguments. The results in [12] are stated for buildings. On the other hand, as is nicely explained in Borel's exposé [5], part of the argument in [12] works for quite general simplicial complexes. We follow [5] for the most part of the paper.

Note that questions about eigenvalues of Δ acting on $C^0(X)$ can be reinterpreted as problems in Spectral Graph Theory, which is a rather active area of combinatorics, cf. [9,4]. (The graph here is the 1-skeleton $X^{(1)}$ of X.) Nevertheless, the general results of this theory do not seem to provide any significant information about the spectrum of p-adic curvature. One reason is probably the fact that when $\dim(X) > 1$ the action of Δ on a function on the vertices of X is given by a formula which takes into account not only the structure of the graph $X^{(1)}$, but also the simplicial structure of X.

Finally, we should mention that there is a recent renewed interest in Laplacians acting on buildings in connection with constructing Ramanujan complexes—a higher dimensional analogue of Ramanujan graphs; cf. [1,2,13,14].

2. Simplicial complexes

We start by fixing the terminology and notation related to simplicial complexes.

A *simplicial complex* is a collection X of finite nonempty sets, such that if s is an element of X, so is every nonempty subset of s. The element s of X is called a *simplex* of X; its *dimension* is $|s|-1$. Each nonempty subset of s is called a *face* of s. A simplex of dimension i will usually be referred to as i-simplex. The *dimension* $\dim(X)$ of X is the largest dimension of one of its simplices (or is infinite if there is no such largest dimension). A subcollection of X that is itself a complex is called a *subcomplex* of X. The *vertices* of the simplex s are the one-point elements of the set s.

Let s be a simplex of X. The *star* of s in X, denoted $\mathrm{St}(s)$, is the subcomplex of X consisting of the union of all simplices of X having s as a face. The *link* of s, denoted $\mathrm{Lk}(s)$, is the subcomplex of $\mathrm{St}(s)$ consisting of the simplices which are disjoint from s. If one thinks of $\mathrm{St}(s)$ as the "unit ball" around s in X, then $\mathrm{Lk}(v)$ is the "unit sphere" around s.

A specific ordering of the vertices of s up to an even permutation is called an *orientation* of s. An *oriented* simplex is a simplex s together with an orientation of s. Denote the set of i-simplices by $\widehat{S}_i(X)$, and the set of oriented i-simplices by $S_i(X)$. We will denote the vertices $\widehat{S}_0(X) = S_0(X)$ of X also by $\mathrm{Ver}(X)$. For

$s \in S_i(X)$, $\bar{s} \in S_i(X)$ denotes the same simplex but with opposite orientation. An $\mathbb{R}$-valued *i-cochain* on X is a function f from the set of oriented i-simplices of X to $\mathbb{R}$, such that $f(s) = -f(\bar{s})$. Such functions are also called *alternating*. The i-cochains naturally form a $\mathbb{R}$-vector space which is denoted $C^i(X)$. If $i < 0$ or $i > \dim(X)$, we let $C^i(X) = 0$.

3. Laplace operators

From now on we assume that X is a finite n-dimensional complex such that

($\star$) Each simplex of X is a face of some n-simplex.

For $s \in S_i(X)$, let $w(s)$ be the number of (non-oriented) n-simplices containing s. In view of ($\star$), $w(s) \neq 0$ for any s.

Lemma 3.1. *Let $\sigma \in S_i(X)$ be fixed. Then*

$$\sum_{\substack{s \in \widehat{S}_{i+1}(X) \\ \sigma \subset s}} w(s) = (n-i) \cdot w(\sigma).$$

Proof. Given a n-simplex t such that $\sigma \subset t$ there are exactly $(n-i)$ simplices s of dimension $(i+1)$ such that $\sigma \subset s \subset t$. Hence in the sum of the lemma we count every n-simplex containing σ exactly $(n-i)$ times. □

Define a positive-definite pairing on $C^i(X)$ by

$$(f, g) := \sum_{s \in \widehat{S}_i(X)} w(s) \cdot f(s) \cdot g(s), \tag{3.1}$$

where $f, g \in C^i(X)$ and in $w(s) \cdot f(s) \cdot g(s)$ we choose some orientation of s. (This is well-defined since both f and g are alternating).

Define the *coboundary*, a linear transformation $d : C^i(X) \to C^{i+1}(X)$, by

$$(df)([v_0, \ldots, v_{i+1}]) = \sum_{j=0}^{i+1} (-1)^j f([v_0, \ldots, \hat{v}_j, \ldots, v_{i+1}]), \tag{3.2}$$

where $[v_0, \ldots, v_{i+1}] \in S_{i+1}(X)$ and the symbol $\hat{v}_j$ means that the vertex v_j is to be deleted from the array.

Let $s = [v_0, \ldots, v_i] \in S_i(X)$ and $v \in \mathrm{Ver}(X)$. If the set $\{v, v_0, \ldots, v_i\}$ is an $(i+1)$-simplex of X, then we denote by $[v, s] \in S_{i+1}(X)$ the oriented simplex $[v, v_0, \ldots, v_i]$. Define a linear transformation $\delta : C^i(X) \to C^{i-1}(X)$ by

$$(\delta f)(s) = \sum_{\substack{v \in \mathrm{Ver}(X) \\ [v,s] \in S_i(X)}} \frac{w([v, s])}{w(s)} f([v, s]). \tag{3.3}$$

In (3.2) and (3.3), by convention, an empty sum is assumed to be 0. δ is the adjoint of d with respect to (3.1):

Lemma 3.2. *If $f \in C^i(X)$ and $g \in C^{i+1}(X)$, then $(df, g) = (f, \delta g)$.*

Proof.

$$\begin{aligned}(df, g) = &\sum_{\substack{s\in\widehat{S}_{i+1}(X)\\ s=[v_0,\dots,v_{i+1}]}} w(s)\sum_{j=0}^{i+1} f([v_0,\dots,\hat{v}_j,\dots,v_{i+1}])\\ &\times g([v_j, v_0,\dots,\hat{v}_j,\dots,v_{i+1}])\\ = &\sum_{\sigma\in\widehat{S}_i(X)} w(\sigma)f(\sigma)\sum_{[v,\sigma]\in\widehat{S}_{i+1}(X)}\frac{w([v,\sigma])}{w(\sigma)}g([v,\sigma]) = (f,\delta g).\end{aligned}$$

□

Definition 3.3. The *Laplace operator* on $C^i(X)$ is the linear operator $\Delta = \delta d$.

By Lemma 3.2, Δ is self-adjoint with respect to the pairing (3.1), and for any $f \in C^i(X)$

$$(\Delta f, f) = (df, df) \geq 0.$$

Hence Δ is diagonalizable and its eigenvalues are non-negative real numbers. From the previous equation it is clear that the eigenspace of Δ corresponding to 0 is exactly the subspace of i-cocycles $Z^i(X) := \ker(d)$ in $C^i(X)$.

Remark 3.4. When X is 1-dimensional, i.e., is a graph, Δ acts on a function on the vertices of X by

$$(\Delta f)(v) = f(v) - \frac{1}{\deg(v)}\sum_{x\sim v} f(x),$$

where $x \sim v$ means x is adjacent to v, and $\deg(v)$ is the degree of the vertex v. This is not what is usually called a graph laplacian, although this version also arises in graph theory, cf. [4, pp. 3–7]. Note that when $\dim(X) > 1$, the action of Δ on a function on the vertices of X is not given by the formula above.

Remark 3.5. The Laplace operator in [12, Def. 3.15] is defined as $\delta d + d\delta$. What we denote by Δ in this paper is denoted by Δ^+ in *loc. cit.* When X is the link of a vertex in a Bruhat-Tits building, Garland calls Δ^+ the *p-adic curvature*; see [12, p. 400].

4. Fundamental inequality

For $v \in \mathrm{Ver}(X)$ let ρ_v be the linear transformation on $C^i(X)$ defined by:

$$(\rho_v f)(s) = \begin{cases} f(s) & \text{if } v \in s;\\ 0 & \text{otherwise.}\end{cases}$$

Since any i-simplex has $(i+1)$-vertices, for $f \in C^i(X)$ we have the obvious equality

$$\sum_{v \in \mathrm{Ver}(X)} \rho_v f = (i+1) f. \tag{4.1}$$

We also have the following obvious lemma:

Lemma 4.1. (1) $\rho_v \rho_v = \rho_v$;
(2) *For* $f \in C^i(X)$ *and* $g \in C^i(X)$, $(\rho_v f, g) = (f, \rho_v g)$.

Let d_v and δ_v be the linear operators d and δ acting on the cochains of the finite simplicial complex $\mathrm{Lk}(v)$, and let $\Delta_v := \delta_v d_v$. Note that $\mathrm{Lk}(v)$ is a $(n-1)$-dimensional complex satisfying condition $(\star)$. For $f, g \in C^i(X)$ define their inner product on $\mathrm{Lk}(v)$ by

$$(f, g)_v := \sum_{s \in \widehat{S}_i(\mathrm{Lk}(v))} w_v(s) \cdot f(s) \cdot g(s), \tag{4.2}$$

where $w_v(s)$ is the number of $(n-1)$-simplices in $\mathrm{Lk}(v)$ containing s. This is simply the pairing (3.1) for $C^i(\mathrm{Lk}(v))$ computed on the restrictions of f and g to $\mathrm{Lk}(v)$.

Lemma 4.2. *If* $f \in C^i(X)$, *then*

$$i \cdot (\Delta f, f) + (n-i)(f, f) = \sum_{v \in \mathrm{Ver}(X)} (\Delta \rho_v f, \rho_v f).$$

Proof. See [5, Lem. 1.3]. In the proof it is crucial that the inner product $(\cdot, \cdot)$ on $C^i(X)$ is defined using the weights $w(s)$. □

Corollary 4.3. *Let* $f \in C^i(X)$. *If there is a positive real number* Λ *such that*

$$(\Delta \rho_v f, \rho_v f) \leq \Lambda \cdot (\rho_v f, f)$$

for all $v \in \mathrm{Ver}(X)$, *then*

$$i \cdot (\Delta f, f) \leq (\Lambda \cdot (i+1) - (n-i))\,(f, f).$$

Proof. This follows from Lemma 4.1, Lemma 4.2 and (4.1). □

From now on we assume that $i \geq 1$. Define a linear transformation $\tau_v : C^i(X) \to C^{i-1}(X)$ by

$$(\tau_v f)(s) = \begin{cases} f([v, s]) & \text{if } s \in S_{i-1}(\mathrm{Lk}(v)); \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.4. *For* $f, g \in C^i(X)$, *we have* $(\tau_v f, \tau_v g)_v = (\rho_v f, \rho_v g)$.

Proof. We have

$$(\tau_v f, \tau_v g)_v = \sum_{\sigma \in \widehat{S}_{i-1}(\mathrm{Lk}(v))} w_v(\sigma) \cdot \tau_v f(\sigma) \cdot \tau_v g(\sigma).$$

It is easy to see that there is a one-to-one correspondence between the n-simplices of X containing $[v, \sigma]$ and the $(n-1)$-simplices of $\mathrm{Lk}(v)$ containing σ, so $w_v(\sigma) = w([v, \sigma])$. Hence the above sum can be rewritten as

$$\sum_{s \in \widehat{S}_i(\mathrm{St}(v))} w(s) \cdot (\rho_v f)(s) \cdot (\rho_v g)(s).$$

Since $\rho_v f$ is zero away from $\mathrm{St}(v)$, the sum can be extended to the whole $\widehat{S}_i(X)$, so the lemma follows. □

Lemma 4.5. *For $f \in C^i(X)$, we have $(\Delta\rho_v f, \rho_v f) = (\Delta_v \tau_v f, \tau_v f)_v$.*

Proof. By Lemma 4.1 and Lemma 4.4,

$$(\Delta\rho_v f, \rho_v f) = (\rho_v \Delta\rho_v f, \rho_v f) = (\tau_v \Delta\rho_v f, \tau_v f)_v.$$

Next, we show that $\tau_v \Delta\rho_v f = \Delta_v \tau_v f$, which implies the claim. For $s \in S_{i-1}(\mathrm{Lk}(v))$, we have

$$\tau_v \Delta\rho_v f(s) = \delta d\rho_v f([v, s]) = \sum_{\substack{x \in \mathrm{Ver}(\mathrm{Lk}(v)) \\ [x,s] \in S_i(\mathrm{Lk}(v))}} \frac{w([x, v, s])}{w([v, s])} d\rho_v f([x, v, s])$$

$$= \sum_{\substack{x \in \mathrm{Ver}(\mathrm{Lk}(v)) \\ [x,s] \in S_i(\mathrm{Lk}(v))}} \frac{w_v([x, s])}{w_v(s)} (\rho_v f([v, s]) - \rho_v f([x, s]) + \rho_v f([x, v, ds])).$$

In the last term ds denotes is the image of s under the boundary operator and $[\cdot]$ is extended linearly to $\mathbb{Z}[S_i(\mathrm{Lk}(v))]$. Continuing our calculation

$$= \sum_{\substack{x \in \mathrm{Ver}(\mathrm{Lk}(v)) \\ [x,s] \in S_i(\mathrm{Lk}(v))}} \frac{w_v([x, s])}{w_v(s)} (f([v, s]) - f([v, x, ds]))$$

$$= \sum_{\substack{x \in \mathrm{Ver}(\mathrm{Lk}(v)) \\ [x,s] \in S_i(\mathrm{Lk}(v))}} \frac{w_v([x, s])}{w_v(s)} (\tau_v f(s) - \tau_v f([x, ds]))$$

$$= \sum_{\substack{x \in \mathrm{Ver}(\mathrm{Lk}(v)) \\ [x,s] \in S_i(\mathrm{Lk}(v))}} \frac{w_v([x, s])}{w_v(s)} d_v \tau_v f([x, s]) = \delta_v d_v \tau_v f(s) = \Delta_v \tau_v f(s).$$

□

Notation 4.6. Given a finite simplicial complex Y satisfying $(\star)$, let $M^i(Y)$ and $m^i(Y)$ be the maximal and minimal non-zero eigenvalues of Δ acting on $C^i(Y)$, respectively. Denote

$$\lambda^i_{\max}(Y) := \max_{v \in \mathrm{Ver}(Y)} M^i(\mathrm{Lk}(v)) \quad \text{and} \quad \lambda^i_{\min}(Y) := \min_{v \in \mathrm{Ver}(Y)} m^i(\mathrm{Lk}(v)).$$

Proposition 4.7. *For $f \in C^i(X)$, we have*

$$(\Delta\rho_v f, \rho_v f) \leq \lambda_{\max}^{i-1}(X) \cdot (\rho_v f, f).$$

Proof. By Lemma 4.5, $(\Delta\rho_v f, \rho_v f) = (\Delta_v \tau_v f, \tau_v f)_v$. Let $\{e_1, \ldots, e_h\}$ be an orthogonal basis of $C^{i-1}(\mathrm{Lk}(v))$ with respect to $(\cdot,\cdot)_v$ which consists of Δ_v-eigenvectors. Write $\tau_v f = \sum_j a_j e_j$. Then

$$(\Delta_v \tau_v f, \tau_v f)_v \leq M^{i-1}(\mathrm{Lk}(v)) \sum_{j=1}^{h} a_j^2 (e_j, e_j)_v \leq \lambda_{\max}^{i-1}(X) \cdot (\tau_v f, \tau_v f)_v.$$

On the other hand, by Lemma 4.1 and Lemma 4.4, $(\tau_v f, \tau_v f)_v = (\rho_v f, \rho_v f) = (\rho_v f, f)$. ◻

Denote by $\tilde{H}^i(\mathrm{Lk}(v), \mathbb{R})$ the ith reduced simplicial cohomology group of $\mathrm{Lk}(v)$.

Theorem 4.8. (Fundamental inequality) *For $1 \leq i \leq n-1$, we have*

$$i \cdot M^i(X) \leq (i+1) \cdot \lambda_{\max}^{i-1}(X) - (n-i).$$

If $\tilde{H}^{i-1}(\mathrm{Lk}(v), \mathbb{R}) = 0$ for every $v \in \mathrm{Ver}(X)$, then

$$i \cdot m^i(X) \geq (i+1) \cdot \lambda_{\min}^{i-1}(X) - (n-i).$$

Proof. Let $f \in C^i(X)$ be such that $\Delta f = M^i(X) \cdot f$. Proposition 4.7 implies that the assumption of Corollary 4.3 is satisfied with $\Lambda = \lambda_{\max}^{i-1}(X)$. This proves the first part. The second part is Garland's original fundamental estimate [12, Sect. 5]. ◻

Notation 4.9. For $m \geq 1$, let I_m denote the $m \times m$ identity matrix and let J_m denote the $m \times m$ matrix whose entries are all equal to 1. The minimal polynomial of J_m is $x(x-m)$.

Example 4.10. Let X be a n-simplex. We claim that the eigenvalues of Δ acting on $C^i(X)$ are 0 and $(n+1)$ for any $0 \leq i \leq n-1$. Since $Z^i(X) \neq 0$, 0 is an eigenvalue, so we need to show that the only non-zero eigenvalue of Δ is $(n+1)$, or equivalently, $m^i(X) = M^i(X) = n+1$. First, suppose $i = 0$. Since for any simplex of X there is a unique n-simplex containing it, one easily checks that Δ acts on $C^0(X)$ as the matrix $(n+1)I_{n+1} - J_{n+1}$. The only eigenvalues of this matrix are 0 and $(n+1)$. Now let $i \geq 1$. The link of any vertex is a $(n-1)$-simplex, so by induction $\lambda_{\min}^{i-1}(X) = \lambda_{\max}^{i-1}(X) = n$. Since the reduced cohomology groups of a simplex vanish, the Fundamental Inequality implies

$$i \cdot M^i(X) \leq (i+1)n - (n-i) = i(n+1)$$

and

$$i \cdot m^i(X) \geq (i+1)n - (n-i) = i(n+1).$$

Hence $(n+1) \leq m^i(X) \leq M^i(X) \leq (n+1)$, which implies the claim.

5. Proof of the main result

Let G be a group equipped with a Tits system (G, B, N, S) of rank $\ell + 1$. To every Tits system, there is an associated simplicial complex $\mathfrak{B}$ of dimension ℓ, called the *building* of (G, B, N, S). For the definitions and basic properties of buildings we refer to Chaps. IV and V in [6]. The simplices of $\mathfrak{B}$ are in one-to-one correspondence with proper parabolic subgroups of G. Assume from now on that G is finite. Then $\mathfrak{B}$ is a finite simplicial complex satisfying $(\star)$. Given a simplex s of $\mathfrak{B}$, it is known that $\mathrm{Lk}(s)$ is again a building corresponding to a Tits system of rank $\ell - \dim(s)$.

We would like to determine $M^i(\mathfrak{B})$ for $0 \leq i \leq \ell - 1$. This will be done inductively, using induction on i and ℓ. The base of induction is the following lemma:

Lemma 5.1. *If* $\ell = 1$ *then* $M^0(\mathfrak{B}) = 2$.

Proof. When $\ell = 1$, the eigenvalues of Δ acting on $C^0(\mathfrak{B})$ were calculated by Feit and Higman in [11]. The claim follows from these calculations. See also Proposition 7.10 in [12] when $\mathfrak{B}$ is of Lie type. □

Let K be the fundamental chamber of $\mathfrak{B}$, i.e., the ℓ-simplex of $\mathfrak{B}$ corresponding to the Borel subgroup B of the given Tits system. Every simplex s of $\mathfrak{B}$ can be transformed to a unique face s' of K under the action of G. Label the vertices of K by the elements of $I_\ell := \{0, 1, \ldots, \ell\}$, and define $\mathrm{Type}(s)$ to be the subset of I_ℓ corresponding to the vertices of s'. G naturally acts on $\mathfrak{B}$ and this action is type-preserving and strongly transitive; see [6, Sect. V.3]. From this perspective one can think of K as the quotient $\mathfrak{B}/G$.

Lemma 5.2. $\ell + 1$ *and* 0 *are eigenvalues of* Δ *acting on* $C^i(\mathfrak{B})$. *In particular,* $M^i(\mathfrak{B}) \geq \ell + 1$.

Proof. Given a function $f \in C^i(K)$, we can lift it (uniquely) to a G-invariant function $\tilde{f} \in C^i(\mathfrak{B})$ defined by $\tilde{f}(\tilde{s}) := f(s)$, where $\tilde{s}$ is any preimage of s in $\mathfrak{B}$. As is explained in [5, Sect. 4.2], we have $\widetilde{\Delta f} = \Delta \tilde{f}$. Hence the claim follows from Example 4.10. □

Let $f \in C^0(\mathfrak{B})$, and let R be a fixed constant. For $\alpha \in I_\ell$ define a function f_α on the vertices of $\mathfrak{B}$ by

$$f_\alpha(v) = \begin{cases} R \cdot f(v) & \text{if } \mathrm{Type}(v) = \alpha; \\ f(v) & \text{if } \mathrm{Type}(v) \neq \alpha. \end{cases}$$

Lemma 5.3. *Let* $f \in C^0(\mathfrak{B})$ *and suppose* $\Delta f = c \cdot f$. *Then*

$$\sum_{\alpha \in I_\ell} (\Delta f_\alpha, f_\alpha) = \Big[(\ell - c)(R - 1)^2 + c(R^2 + \ell)\Big] \cdot (f, f).$$

Proof. Fix some type α and let $g \in C^0(\mathfrak{B})$ be a function such that $g(v) = 0$ if $\mathrm{Type}(v) \neq \alpha$. Then $(\Delta g, g) = \ell \cdot (g, g)$. Indeed,

$$\begin{aligned}(\Delta g, g) = (dg, dg) &= \sum_{[x,v]\in \widehat{S}_1(\mathfrak{B})} w([x, v])(g(v) - g(x))^2 \\ &= \sum_{\mathrm{Type}(v)=\alpha} g(v)^2 \sum_{x\in \mathrm{Ver}(\mathrm{Lk}(v))} w([x, v]) \\ &= \ell \sum_{\mathrm{Type}(v)=\alpha} w(v) \cdot g(v)^2 = \ell \cdot (g, g).\end{aligned}$$

(The middle equality on the previous line follows from Lemma 3.1.) If we apply this to $g = f_\alpha - f$, then we get

$$(\Delta f_\alpha, f_\alpha) = \ell \cdot (f_\alpha, f_\alpha) - 2(\ell - c)(f_\alpha, f) + (\ell - c)(f, f). \tag{5.1}$$

We clearly have

$$\sum_{\alpha\in I_\ell} f_\alpha = (\ell + R) \cdot f \quad \text{and} \quad \sum_{\alpha\in I_\ell} (f_\alpha, f_\alpha) = (\ell + R^2) \cdot (f, f).$$

Summing (5.1) over all types and using the previous two equalities, we get the claim. □

For a fixed $\alpha \in I_\ell$ define a linear transformation ρ_α on $C^0(\mathfrak{B})$ by

$$\rho_\alpha = \sum_{\mathrm{Type}(v)=\alpha} \rho_v.$$

For $f \in C^0(\mathfrak{B})$ and any α, we have

$$(\rho_\alpha df_\alpha, df_\alpha) = (df_\alpha, df_\alpha) - ((1 - \rho_\alpha)df, df), \tag{5.2}$$

and

$$(\Delta\rho_\alpha df_\alpha, \rho_\alpha df_\alpha) = ((1 - \rho_\alpha)df, df) \tag{5.3}$$

The Eqs. (5.2) and (5.3) are the Eqs. (3) and (6) in [5, Sect. 4.5], respectively.

Theorem 5.4. *For any* $0 \leq i \leq \ell - 1$ *there is an equality* $M^i(\mathfrak{B}) = \ell + 1$.

Proof. By Lemma 5.2, it is enough to show that $M^i(\mathfrak{B}) \leq \ell + 1$. We start with $M^0(\mathfrak{B})$. Let $f \in C^0(\mathfrak{B})$. Since the vertices of any simplex in $\mathfrak{B}$ have distinct types, one easily checks that

$$\sum_{\mathrm{Type}(v)=\alpha} (\Delta\rho_v df_\alpha, \rho_v df_\alpha) = (\Delta\rho_\alpha df_\alpha, \rho_\alpha df_\alpha),$$

so by Proposition 4.7

$$(\Delta\rho_\alpha df_\alpha, \rho_\alpha df_\alpha) \leq \lambda^0_{\max}(\mathfrak{B}) \cdot (\rho_\alpha df_\alpha, \rho_\alpha df_\alpha). \tag{5.4}$$

Since for any $v \in \mathrm{Ver}(\mathfrak{B})$, $\mathrm{Lk}(v)$ is a building of dimension $\ell - 1$, the induction on ℓ gives $\lambda^0_{\max}(\mathfrak{B}) = \ell$. Combining this with (5.4), (5.2) and (5.3), we get

$$(1+\ell)\cdot((1-\rho_\alpha)df, df) \le \ell \cdot (df_\alpha, df_\alpha). \tag{5.5}$$

Now assume $\Delta f = c \cdot f$. Note that

$$\begin{aligned}\sum_{\alpha\in I_\ell}(1-\rho_\alpha)df &= (\ell+1)df - \sum_{v\in \mathrm{Ver}(\mathfrak{B})}\rho_v df\\ &= (\ell+1)df - 2df = (\ell-1)df,\end{aligned} \tag{5.6}$$

so summing the inequalities (5.5) over all types and using Lemma 5.3, we get

$$(\ell+1)(\ell-1)c\cdot(f,f) \le \ell\cdot\left[(\ell-c)(R-1)^2 + c(R^2+\ell)\right]\cdot(f,f). \tag{5.7}$$

If we put $R = (\ell - c)/\ell$, then (5.7) forces $c \le \ell+1$. In particular, $M^0(\mathfrak{B}) \le \ell+1$.

Now let $i \ge 1$. The induction on i and ℓ implies that $\lambda^{i-1}_{\max}(\mathfrak{B}) = \ell$. From the Fundamental Inequality 4.8 we get

$$i \cdot M^i(\mathfrak{B}) \le (i+1)\cdot \ell - (\ell - i),$$

which implies $M^i(\mathfrak{B}) \le \ell + 1$. □

Theorem 5.5. *$m^0(\mathfrak{B}) \le \ell$.*

Proof. Denote $c := m^0(\mathfrak{B})$. First we claim that $c \ne \ell+1$. If $c = \ell+1$ then by Theorem 5.4 Δ has only two distinct eigenvalues, namely 0 and $\ell+1$. Since Δ is a semi-simple operator, this implies that $\Delta^2 = (\ell+1)\Delta$. In $\mathfrak{B}$ we can find two vertices x and y which are not adjacent but such that there is another vertex v which is adjacent to both x and y. Let $g \in C^0(\mathfrak{B})$ be a function supported at x, i.e., $g(x) \ne 0$ and $g(w) = 0$ if $w \ne x$. Now $(\Delta g)(y) = 0$ because this is a sum of the values of g at y and the vertices adjacent to y, and x is not one of them. On the other hand, $(\Delta^2 g)(y) \ne 0$ since this is a sum which involves $g(x)$ with a non-zero coefficient. This contradicts the equality $\Delta^2 = (\ell+1)\Delta$.

Let $f \in C^0(\mathfrak{B})$ be a Δ-eigenfunction with eigenvalue c. Since $c \ne \ell+1$, Eq. (1) in [5, Sect. 4.6] gives

$$(\Delta f_\alpha, f_\alpha) \ge c\cdot(f_\alpha, f_\alpha).$$

Summing over all types,

$$\sum_{\alpha\in I_\ell}(\Delta f_\alpha, f_\alpha) \ge c(\ell + R^2)\cdot(f,f).$$

Comparing this inequality with the expression in Lemma 5.3, we conclude that

$$(\ell - c)(R-1)^2 \ge 0.$$

Since R is arbitrary, we must have $c \le \ell$. □

Theorem 5.6. (Garland) *Assume that G is the group of $\mathbb{F}_q$-valued points of a simple, simply connected Chevalley group. For any $\varepsilon > 0$ there is a constant $q(\varepsilon, \ell)$ depending only on ε and ℓ, such that if $q > q(\varepsilon, \ell)$ then $m^i(\mathfrak{B}) > \ell - i - \varepsilon$.*

Proof. For the proof see Sections 6, 7, 8 in [12], or Proposition 5.4 in [5]. □

Theorems 5.4–5.6, combined with the calculations in Sect. 6, suggest the following possibility:

Conjecture 5.7. *In the situation of Theorem 5.6, let $c \neq 0, \ell + 1$ be an eigenvalue of Δ acting on $C^0(\mathfrak{B})$. For any $\varepsilon > 0$ there is a constant $q(\varepsilon, \ell)$ depending only on ℓ and ε such that if $q > q(\varepsilon, \ell)$, then $\ell - \varepsilon < c < \ell + \varepsilon$.*

6. Examples

In this section we compute explicitly in some cases the eigenvalues of Δ acting on $C^i(\mathfrak{B})$. We concentrate on $G = \mathrm{SL}_{\ell+2}(\mathbb{F}_q)$ for small ℓ, with $B \subset G$ being the upper triangular group and N being the monomial group, cf. [6, Sect. V.5]. Denote the corresponding building by $\mathfrak{B}_{\ell,q}$. The dimension of $\mathfrak{B}_{\ell,q}$ is ℓ. Denote by $\mathfrak{m}_\ell^i(q; x)$ the minimal polynomial of Δ acting on $C^i(\mathfrak{B}_{\ell,q})$, $0 \leq i \leq \ell - 1$.

First, we recall an elementary description of $\mathfrak{B}_{\ell,q}$ which is convenient for actual calculations. Let V be a linear space over $\mathbb{F}_q$ of dimension $\ell + 2$. A *flag* in V is a nested sequence $\mathcal{F} : F_0 \subset F_1 \subset \cdots \subset F_i$ of distinct linear subspaces $F_0, \ldots, F_i$ of V such that $F_0 \neq 0$ and $F_i \neq V$. $\mathfrak{B}_{\ell,q}$ is isomorphic to the simplicial complex whose vertices correspond to the non-zero linear subspaces of V distinct from V; the vertices $v_0, \ldots, v_i$ form an i-simplex if the corresponding subspaces form a flag.

Now assume $\ell = 1$. In this case $\mathfrak{B}_{\ell,q}$ is isomorphic to the 1-dimensional complex whose vertices correspond to 1 and 2-dimensional subspaces of a 3-dimensional vector space V over $\mathbb{F}_q$, two vertices being adjacent if one of the corresponding subspaces is contained in the other. With a slight abuse of terminology, we will call 1 and 2 dimensional subspaces lines and planes, respectively. The number of lines and planes in V is $m = q^2 + q + 1$ each. Let $A = (a_{ij})$ be the $m \times m$ matrix whose rows are enumerated by the lines in V and columns by the planes, and $a_{ij} = -1$ if the ith line lies in the jth plane, and is 0 otherwise. We can choose a basis of $C^0(\mathfrak{B}_{\ell,q})$ so that $(q + 1)\Delta$ acts as the matrix

$$(q + 1)I_{2m} + \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}.$$

Let $M = \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$. Since any two distinct lines lie in a unique plane and any line lies in $(q + 1)$ planes, $AA^t = qI_m + J_m$. By a similar argument, $A^t A = qI_m + J_m$. Hence

$$M^2 = qI_{2m} + \begin{pmatrix} J_m & 0 \\ 0 & J_m \end{pmatrix}.$$

This implies that $(M^2 - qI_{2m})(M^2 - (q+1)^2 I_{2m}) = 0$. Since $(q+1)\Delta - (q+1)I_{2m} = M$, we conclude that $(q+1)\Delta$ satisfies the polynomial equation

$$x(x-(2q+2))(x^2-(2q+2)x+(q^2+q+1)) = 0.$$

It is not hard to see that this is in fact the minimal polynomial of $(q+1)\Delta$. Hence

$$\mathfrak{m}_1^0(q;x) = x(x-2)\left(x^2 - 2x + \frac{q^2+q+1}{q^2+2q+1}\right).$$

The minimal non-zero root is $1 - \sqrt{q}/(q+1)$. The smallest possible value of this expression is approximately 0.53, which occurs at $q = 2$. The coefficients of $\mathfrak{m}_1^0(q;x)$ converge to the coefficients of $x(x-2)(x-1)^2$ as $q \to \infty$.

The very next case $\ell = 2$ is already considerably harder to compute by hand. With the help of a computer, we deduced that

$$\begin{aligned}\mathfrak{m}_2^0(q;x) = x(x-2)\,(x-3)&\left(x - \frac{2q^2+3q+2}{q^2+q+1}\right)\\ &\times\left(x^2 - \frac{4q^2+3q+4}{q^2+q+1}x + \frac{4q^2+4}{q^2+q+1}\right).\end{aligned}$$

The minimal non-zero root is

$$\frac{1}{2(q^2+q+1)}\left(4q^2+3q+4-\sqrt{8q^3+9q^2+8q}\right),$$

which is at least 1.08 and tends to 2 from below as $q \to \infty$. The whole polynomial tends coefficientwise to the polynomial $x(x-3)(x-2)^4$ as $q \to \infty$. Next

$$\begin{aligned}\mathfrak{m}_2^1(q;x) = x(x-1)(x-2)(x-3)&\\ \times&\left(x^2-2x+\frac{q^2+1}{q^2+2q+1}\right)\left(x^2-3x+\frac{2q^2+2q+2}{q^2+2q+1}\right)\\ \times&\left(x^2-4x+\frac{4q^2+6q+4}{q^2+2q+1}\right).\end{aligned}$$

The minimal non-zero root is $1 - \sqrt{2q}/(q+1)$. This is always in the interval $[1/3, 1)$. Moreover, this eigenvalue is strictly larger than $1/3$ for $q > 2$ and tends to 1 as $q \to \infty$; the whole polynomial tends to $x(x-3)(x-1)^4(x-2)^4$.

The formulae for $\mathfrak{m}_2^0(q;x)$ and $\mathfrak{m}_2^1(q;x)$ are partly conjectural, although almost certainly correct. We computed these polynomials for $q = 2, 3, 4, 5, 7$ using computer calculations with concrete finite fields, and then came up with a formula which recovers all the previous polynomials when we specialize q.

The complexity of calculations grows exponentially with i, ℓ and q, so for $\ell = 3$ my computer was able to handle only $i = 0$ for $q = 2$ and 3:

$$\mathfrak{m}_3^0(2; x) = x(x-4)\left(x - \frac{23}{7}\right)\left(x - \frac{19}{7}\right) \times \left(x^4 - 12x^3 + \frac{581528}{11025}x^2 - \frac{220232}{2205}x + \frac{6734719}{99225}\right),$$

$$\mathfrak{m}_3^0(3; x) = x(x-4)\left(x - \frac{42}{13}\right)\left(x - \frac{36}{13}\right) \times \left(x^4 - 12x^3 + \frac{14350977}{270400}x^2 - \frac{2760633}{27040}x + \frac{309843369}{4326400}\right).$$

The minimal non-zero roots of these polynomials are approximately 1.68 and 1.89, respectively. To have a reasonable guess for the coefficients of $\mathfrak{m}_3^0(q; x)$, one needs to compute these polynomials for at least the next few values of q. Nevertheless, note that the coefficients of above polynomials are close to the coefficients of $x(x-4)(x-3)^6$.

The final example we have is

$$\mathfrak{m}_4^0(2; x) = x(x-4)(x-5)\left(x - \frac{144}{35}\right)\left(x^2 - \frac{1322}{155}x + \frac{2798}{155}\right) \times \left(x^2 - \frac{276}{35}x + \frac{536}{35}\right)\left(x^3 - \frac{1778}{155}x^2 + \frac{1306}{31}x - \frac{7512}{155}\right).$$

The minimal non-zero root is approximately 2.32, and the coefficients of $\mathfrak{m}_4^0(2; x)$ are close to the coefficients of $x(x-5)(x-4)^9$.

Remark 6.1. Let $\mathcal{K}$ be a field complete with respect to a non-trivial discrete valuation and which is locally compact. Let $\mathbb{F}_q$ be the residue field of $\mathcal{K}$. Let $\mathcal{G}$ be an almost simple linear algebraic group over $\mathcal{K}$. Suppose $\mathcal{G}$ has $\mathcal{K}$-rank $\ell + 1$. Let $\mathfrak{T}$ be the Bruhat-Tits building associated with $\mathcal{G}(\mathcal{K})$. The link of a simplex s in $\mathfrak{T}$ is a finite building of dimension $\ell - \dim(s)$. Using a discrete analogue of Hodge decomposition and the Fundamental Inequality one proves the following theorem (see [5, Thm. 3.3]):

Theorem 6.2. *If* $\lambda_{\min}^{i-1}(\mathfrak{T}) > \frac{\ell+1-i}{i+1}$, *then* $H^i(\Gamma, \mathbb{R}) = 0$ *for any discrete cocompact subgroup* Γ *of* $\mathcal{G}(\mathcal{K})$.

Combining this with Theorem 5.6, one concludes that there is a constant $q(\ell)$ depending only on ℓ such that if $q > q(\ell)$ then $H^i(\Gamma, \mathbb{R}) = 0$ for $1 \leq i \leq \ell$. This is the main result of [12]. It is natural to ask whether the restriction on q being sufficiently large is redundant. This is indeed the case, as was shown by Casselman [8], who proved the vanishing of the middle cohomology groups by an entirely different argument.

Now let $\mathcal{G} = \mathrm{SL}_{\ell+2}$. Then $\lambda_{\min}^{i-1}(\mathfrak{T}) = m^{i-1}(\mathfrak{B}_{\ell,q})$ In all examples discussed above $m^0(\mathfrak{B}_{\ell,q}) > \ell/2$, so in these cases Garland's method proves the vanishing of $H^1(\Gamma, \mathbb{R})$ without any assumptions on q. On the other hand, $m^1(\mathfrak{B}_{2,2}) = 1/3$.

But to apply Theorem 6.2 to show that $H^2(\Gamma, \mathbb{R}) = 0$ we need $\lambda^1_{\min}(\mathfrak{T}) > 1/3$. Hence when $\ell = 2$ we need to assume $q > 2$ to conclude $H^2(\Gamma, \mathbb{R}) = 0$ from Garland's method.

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