MASS FORMULA FOR CENTRAL DIVISION ALGEBRAS OVER FUNCTION FIELDS

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Abstract. We prove a mass formula for certain central division algebras over global fields of positive characteristic.

1. Introduction

Let B be the unique quaternion algebra over $\mathbb{Q}$ which ramifies exactly at p and ∞. Let $I_1, I_2, \dots, I_h$ be the left ideal classes of a fixed maximal order R in B, and let $R_i = O_r(I_i)$ be the right order of I_i, $i = 1, \dots, h$. It is known that each conjugacy class of maximal orders in B is represented (once or twice) in the set $\{R_1, R_2, \dots, R_h\}$. Let $R_i^\times$ denote the group of units in R_i for all $i = 1, \dots, h$. The *Eichler mass formula* states that

$$\sum_{i=1}^{h} \frac{1}{\# R_i^\times} = \frac{p-1}{24}. \tag{1.1}$$

There are (at least) two different ways to prove this. The first (due to Deuring and Igusa) goes as follows. Let $\mathbb{F}_p$ be the finite field of p elements and let $E_1, E_2, \dots, E_h$ be the isomorphism classes of super-singular elliptic curves over $\overline{\mathbb{F}}_p$. One establishes a one-to-one correspondence between the isomorphism classes of super-singular elliptic curves $E_1, E_2, \dots, E_h$ and the left ideal classes $I_1, I_2, \dots, I_h$ above, with the additional property $\mathrm{End}(E_i) \cong R_i$. Then $R_i^\times \cong \mathrm{Aut}(E_i)$, and the problem reduces to analyzing the possibilities for the automorphism groups of super-singular elliptic curves and computing the number of such curves with a given automorphism group. This all can be done independently of the theory of quaternion algebras; see [Sil86, Ch.3,5]. This proof has a clear advantage of giving more information than presumably is needed for the proof of the mass formula, i.e., it gives a complete description of each summand on the left. The disadvantage is that it is somewhat mysterious why the answer is the same $(p-1)/24$ for all p, even though the possible automorphism groups vary considerably in different characteristics (especially $p = 2, 3$ compared to $p > 3$; see [Sil86, Thm.III.10.1]).

The second (Eichler's original) proof relates the calculation of the sum $\sum_{i=1}^{h} \frac{1}{\# R_i^\times}$ to a calculation of a certain volume. In the course of the proof one does not use anything specific about p, except that it is a ramification prime of the quaternion algebra, and the sum in question (via mentioned volume calculation) turns out to be equal to $\zeta_{\mathbb{Q}}^{(p)}(-1)/2$, where $\zeta_{\mathbb{Q}}^{(p)}(s)$ is the Riemann zeta function without the Euler factor at p.

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Now let K be the field of rational functions on a smooth proper geometrically connected curve X over a finite field $\mathbb{F}_q$, and let D_K be a central division algebra over K of dimension n^2. We make an extra assumption that at the places where D_K is ramified $D_v := D_K \otimes K_v$ is a division algebra (this is automatic if $n = 2$). Choose an arbitrary point of X which we denote by ∞. Let $A = \mathrm{H}^0(X - \infty, \mathcal{O})$ be the ring of rational functions on X which are regular away from ∞. This is a Dedekind domain with fraction field K. Let R be a fixed maximal A-order in D_K.

Let h be the class number of D_K, that is, the number of left R-ideal classes. It is known [Rei75, §26] that h is finite and is independent of the choice of R (but depends on the choice of ∞). Let $I_1, I_2, \dots, I_h$ be different representatives of left ideal classes with $I_1 = R$, and let $R_i = O_r(I_i)$, $i = 1, \dots, h$. Then the groups of units $R_i^\times$ are all finite. Indeed, let Z be the center of D as an algebraic group. Then $G = D^\times/Z^\times$ as an algebraic variety over K is projective. Consider $G(K_\infty)$. This is compact in ∞-adic topology, and contains $R_i^\times/A^\times$ as a discrete group. Hence $R_i^\times/A^\times$ is finite, and as $A^\times$ is finite, $R_i^\times$ must be finite.

Let $\zeta_K(s) = \prod_v \zeta_v(s)$ be the zeta function of the function field K; here the product is over all places v of K, $\zeta_v(s) = (1 - q_v^{-s})^{-1}$, and $q_v = q^{\deg(v)}$ is the order of the residue field at v. It is known that

$$\zeta_K(s) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

where $P(X) \in \mathbb{Z}[X]$ is a polynomial of degree $2g$ with the constant coefficient equal to 1. (Here g is the genus of X.) For a finite set S of places of K, we put

$$\zeta_{K,S}(s) = \zeta_K(s) \prod_{v \in S} (1 - q_v^{-s}).$$

The main result of this paper is the following

Theorem 1.1. *With previous notations, let S be the finite set of places where D_K is ramified, and let $h(A) := \#\mathrm{Pic}(A)$ be the class number of A. Then*

$$\sum_{i=1}^{h} \frac{1}{\# R_i^\times} = \frac{h(A)}{(q-1)} \prod_{i=1}^{n-1} \zeta_{K,S}(-i).$$

Certain cases of this theorem are due to Gekeler. Namely, when $S = \{\mathfrak{p}, \infty\}$ consists of two places one of which is ∞, and one of the following holds: either $n = 2$, or n is arbitrary but $A = \mathbb{F}_q[T]$. Similar to Deuring's proof of (1.1), Gekeler's proof consists in interpreting the maximal orders of D_K as the endomorphism rings of super-singular Drinfeld modules, then using properties of Drinfeld modular schemes for the latter; see [Gek91] and [Gek92].

2. Proof of the theorem

2.1. **Preliminaries.** We keep the notations of the previous section. We let $\mathcal{N}(\cdot)$ to denote the reduced norm from D_K to K or from $D_v = D_K \otimes_K K_v$ to K_v. Let R be any maximal A-order of D_K. For any place $v \in \mathrm{Spec}(A)$ where D_K is split, D_v is isomorphic to the central simple algebra $M_n(K_v)$ of $n \times n$ matrices over K_v. It is known [Rei75, (11.2),(11.6)] that $R_v = R \otimes_A A_v$ is a maximal A_v-order. Since all maximal orders in D_v are the conjugates of $M_n(A_v)$ by invertible elements in D_v [Rei75, (17.3)], by choosing the isomorphism $D_v \cong M_n(K_v)$ appropriately, we can assume $R_v = M_n(A_v)$. Thus we assume from now on that split D_v, for $v \neq \infty$, is identified with $n \times n$ matrices over K_v in such a way that there exists a maximal order R of D_K with $R_v = M_n(A_v)$ for all split v. The *idele* group $D_{\mathbb{A}}^{\times}$ of D is

$$D_{\mathbb{A}}^{\times} = \{\tilde{a} = (a_v) \in \prod_v D_v^{\times} \mid a_v \in R_v^{\times} \text{ for almost all } v\}.$$

Here the product is over all places v of K. This definition is independent of the choice of ∞, and also independent of choice of the maximal order R as for any two maximal A-orders R and Q in D_K, $R_v = Q_v$ for almost all v.

There is a natural norm defined on $D_{\mathbb{A}}^{\times}$ as

$$\|\tilde{a}\| = \prod_v |\mathcal{N}(a_v)|_v,$$

where $|\cdot|_v$ is the canonical v-adic norm, and the product is over all places of K. The norm gives a surjective homomorphism $\|\cdot\| : D_{\mathbb{A}}^{\times} \to q^{\mathbb{Z}}$, and we will denote by $D_{\mathbb{A}}^{(d)}$, for $d \geq 1$, the open subset of $D_{\mathbb{A}}^{\times}$ consisting of elements $\tilde{a}$ with $0 \leq \log_q(\|\tilde{a}\|) < d$. The group of non-zero elements $D_K^{\times}$ embeds diagonally into $D_{\mathbb{A}}^{(1)}$ thanks to the product formula. Finally if R is a maximal A-order of D_K define

$$R_{\mathbb{A}}^{\times} := \{\tilde{a} = (a_v) \in D_{\mathbb{A}}^{(1)} \mid a_v \in R_v^{\times} \text{ for all } v \neq \infty\}.$$

Definition 2.1. Two left R-ideals I and J are said to be *equivalent* if there is $a \in D_K^{\times}$ with $J = Ia$. The number of equivalence classes of left R-ideals is called the *class number of* D_K (with respect to ∞).

Proposition 2.2. *With ∞ being fixed, the class number of D_K is finite and independent of the choice of maximal order R.*

Proof. Let $d_\infty = \deg(\infty)$. Every left R-ideal I is locally principal, i.e., $I_v = R_v a_v$ with $a_v \in D_v^{\times}$ for all $v \neq \infty$; see [Rei75, §17]. Moreover, for almost all places $I_v = R_v$, that is, $a_v \in R_v^{\times}$. Hence there is an element $\tilde{b} \in D_{\mathbb{A}}^{\times}$ with $b_v = a_v$ for all $v \neq \infty$, and we can take $\tilde{b} \in D_{\mathbb{A}}^{(d_\infty)}$ by choosing b_∞ appropriately. Conversely, if $\tilde{b} = (b_v) \in D_{\mathbb{A}}^{(d_\infty)}$, then it is easy to check that there is a unique left R-ideal I such that $I_v = R_v b_v$ for all $v \neq \infty$.

Since $D_{\mathbb{A}}^{(d_\infty)}$ is a disjoint union of d_∞ cosets of $D_{\mathbb{A}}^{(1)}$ in $D_{\mathbb{A}}^{\times}$, and $R_{\mathbb{A}}^{\times}$ is the isotropy subgroup of R under the action of $D_{\mathbb{A}}^{(1)}$ on the left R-ideals, we get that the number of left R-ideal classes is equal to the number of double cosets in $R_{\mathbb{A}}^{\times} \setminus D_{\mathbb{A}}^{(1)} / D_K^{\times}$ multiplied by d_∞. By a standard argument $D_{\mathbb{A}}^{(1)} / D_K^{\times}$ is compact and $R_{\mathbb{A}}^{\times}$ is open in $D_{\mathbb{A}}^{(1)}$. So the class number is finite.

As for the independence of class number from a particular choice of R, one observes that any two maximal A-orders are conjugate by an element of $D_{\mathbb{A}}^{(d)}$. Now the independence is clear. □

2.2. **First expression for mass formula.** In this subsection we explain how mass formula can be reduced to a certain volume calculation. With notations as in previous sections, let h be the class number of D_K, with ∞ being fixed, and let $I_1, I_2, \ldots, I_h$ be different representatives of left ideal classes of a maximal A-order R. Let $R_i = O_r(I_i)$, $i = 1, \ldots, h$. Then all the factors in the sum $\sum_{i=1}^{h} (\# R_i^{\times})^{-1}$ are well-defined. We want an alternative expression for it. To do this we first fix a Haar measure on $D_{\mathbb{A}}^{\times}$. Normalize the local multiplicative Haar measure $d^{\times} x_v$ on $D_v^{\times}$ so that $\int_{\Gamma_v} d^{\times} x_v = 1$ for $\Gamma_v = \mathrm{GL}_n(\mathcal{O}_v)$ at the split places, and Γ_v being the units in the unique maximal order in D_v for the ramified places (recall that we are assuming that at the places where D_K ramified, D_v is a division algebra). Define the measure $d^\times x$ on $D_\mathbb{A}^\times$ to be the restricted product measure (as $D_\mathbb{A}^\times$ and $D_v^\times$ are unimodular we do not distinguish between left and right Haar measures). The group $D_\mathbb{A}^{(1)}$ is the kernel of the composition of the idelic norm and the reduced norm on $D_\mathbb{A}^\times$. The group $D_K^\times$ embeds diagonally into $D_\mathbb{A}^{(1)}$ and is discrete. The quotient $D_\mathbb{A}^{(1)}/D_K^\times$ is compact, hence has a finite volume with respect to the push-forward measure of $d^\times x$.

Proposition 2.3. *Let $d_\infty = \deg(\infty)$. The volume of $D_\mathbb{A}^{(1)}/D_K^\times$ with respect to the push-forward measure of the Haar measure on $D_\mathbb{A}^\times$ normalized as above is given by the following expression:*

$$\mathrm{Vol}(D_\mathbb{A}^{(1)}/D_K^\times)\cdot d_\infty = \sum_{i=1}^{h} \frac{1}{\# R_i^\times}.$$

Proof. Write $D_\mathbb{A}^{(d_\infty)} = \coprod_{i=1}^h R_\mathbb{A}^\times \tilde{a}_i D_K^\times$ as in the proof of Proposition 2.2, where $R\tilde{a}_i$, $i=1,\dots,h$, represent all the left R-ideal classes. Since $D_\mathbb{A}^{(d_\infty)}$ is essentially a d_∞ disjoint copies of $D_\mathbb{A}^{(1)}$, we have

$$\begin{aligned}
\mathrm{Vol}(D_\mathbb{A}^{(1)}/D_K^\times)\cdot d_\infty &= \sum_i \mathrm{Vol}(R_\mathbb{A}^\times \tilde{a}_i D_K^\times/D_K^\times)\\
&= \sum_i \mathrm{Vol}(\tilde{a}_i^{-1} R_\mathbb{A}^\times \tilde{a}_i D_K^\times/D_K^\times)\\
&= \sum_i \mathrm{Vol}((\tilde{a}_i^{-1} R_\mathbb{A} \tilde{a}_i)^\times D_K^\times/D_K^\times)\\
&= \sum_i \mathrm{Vol}((\tilde{a}_i^{-1} R_\mathbb{A} \tilde{a}_i)^\times/(\tilde{a}_i^{-1} R_\mathbb{A} \tilde{a}_i)^\times \cap D_K^\times)\\
&= \mathrm{Vol}(R_\mathbb{A}^\times)\sum_i \frac{1}{\#(\tilde{a}_i^{-1} R\tilde{a}_i)^\times}\\
&= \mathrm{Vol}(R_\mathbb{A}^\times)\sum_i \frac{1}{\# R_i^\times}.
\end{aligned}$$

It remains to show that with our choice of the measure $\mathrm{Vol}(R_\mathbb{A}^\times)=1$. By definition

$$R_\mathbb{A}^\times = \{\tilde{a}=(a_v)\in D_\mathbb{A}^{(1)} \mid a_v \in R_v^\times \text{ for all } v\neq\infty\}.$$

So for $\tilde{a}\in D_\mathbb{A}^{(1)}$ we also have $|\mathrm{Nr}(a_\infty)|_\infty = 1$, i.e., a_∞ is a unit in a maximal order of D_∞. In particular, $(R_\mathbb{A}^\times)_v \cong \Gamma_v$ for all places v, and $\mathrm{Vol}(R_\mathbb{A}^\times)=1$ as required. □

2.3. **Second expression for mass formula.** The question has been reduced to computing $\mathrm{Vol}(D_\mathbb{A}^{(1)}/D_K^\times)$. In this subsection following [Wei82] we relate this volume to the residue at 0 of a certain zeta function. The calculation itself is very similar to the calculation of $\mathrm{Vol}(\mathbb{A}_1^\times/K^\times)$ in Tate's thesis, where $\mathbb{A}_1^\times$ are the ideles of norm 1. Recall that this is nothing else but the celebrated relation between the residues of ζ-functions of global fields, and the invariants of the field (units, class numbers and etc.).

Let as before D_K be a central division algebra over a function field K, and let Φ be a Schwartz function on $D_{\mathbb{A}}^{\times}$. Consider the following integral

$$\zeta_D(s,\Phi) = \int_{D_{\mathbb{A}}^{\times}} \Phi(x)\,\|x\|^s\, d^{\times}x. \tag{2.1}$$

It is standard (and logically more correct) to consider the local decomposition of this integral first. The local calculations enable one to establish the regions of absolute convergence of this global integral which are crucial for the argument we are about to present. Nevertheless, for expository purposes we will postpone the local calculations till the next subsection. The fact we need is that (2.1) converges absolutely for $\mathrm{Re}(s) > 1$; c.f. [Wei82, §3.1]. As in *loc. cit.* we introduce a function λ on $\mathbb{R}^+$ as follows: $\lambda(t) = 1$ for $0 < t < 1$, $\lambda(1) = 1/2$, and $\lambda(t) = 0$ for $t > 1$. For $x \in D_{\mathbb{A}}^{\times}$, put

$$f_+(x) = \lambda(\|x\|^{-1}), \quad f_-(x) = \lambda(\|x\|),$$

so that we have $f_+ + f_- = 1$. Write

$$\zeta_D^+(s,\Phi) = \int_{D_{\mathbb{A}}^{\times}} f_+(x)\Phi(x)\,\|x\|^s\, d^{\times}x,$$
$$\zeta_D^-(s,\Phi) = \int_{D_{\mathbb{A}}^{\times}} f_-(x)\Phi(x)\,\|x\|^s\, d^{\times}x.$$

Clearly $\zeta_D = \zeta_D^+ + \zeta_D^-$. Moreover, $\zeta_D^+(s,\Phi)$ is an entire function. Indeed, this integral converges absolutely for $\mathrm{Re}(s) > 1$ (as $\zeta_D(s,\Phi)$ converges absolutely in that region), and when $\|x\| \geq 1$ decreasing $\mathrm{Re}(s)$ only improves the convergence. We consider more carefully the second part of the integral ζ_D. We may write any element of $D_{\mathbb{A}}^{\times}$ as $x\alpha$, where $\alpha \in D_K^{\times}$ and x is an element of some fixed set of representatives of the space of right cosets $D_{\mathbb{A}}^{\times}/D_K^{\times}$. Thus

$$\begin{aligned}
\zeta_D^-(s,\Phi) &= \sum_{\alpha\in D_K^{\times}} \int_{D_{\mathbb{A}}^{\times}/D_K^{\times}} f_-(x\alpha)\Phi(x\alpha)\,\|x\|^s\, d^{\times}x \\
&= \int_{D_{\mathbb{A}}^{\times}/D_K^{\times}} \left[\sum_{\alpha\in D_K^{\times}} \Phi(x\alpha)\right] f_-(x)\,\|x\|^s\, d^{\times}x \\
&= \int_{D_{\mathbb{A}}^{\times}/D_K^{\times}} \left[\sum_{\alpha\in D_K} \Phi(x\alpha)\right] f_-(x)\,\|x\|^s\, d^{\times}x - \Phi(0)\int_{D_{\mathbb{A}}^{\times}/D_K^{\times}} f_-(x)\,\|x\|^s\, d^{\times}x,
\end{aligned}$$

where the second equality follows from the fact that $\mathcal{N}(x) \in K^{\times}$ for $x \in D_K^{\times}$ and the idelic norm of elements in $K^{\times}$ is 1, hence $\|x\| = 1$. By Poisson summation, we have

$$\sum_{\alpha\in D_K} \Phi(x\alpha) = \|x\|^{-1} \sum_{\beta\in D_K} \Psi(\beta x^{-1}),$$

where Ψ is the Fourier transform of Φ; see [Wei82, §3.1]. Hence

$$\begin{aligned}\int_{D_{\mathbb{A}}^\times/D_K^\times}&\left[\sum_{\alpha\in D_K}\Phi(x\alpha)\right] f_-(x)\,\|x\|^s\,d^\times x\\ &=\int_{D_{\mathbb{A}}^\times/D_K^\times}\left[\sum_{\beta\in D_K}\Psi(\beta x^{-1})\right] f_-(x)\,\|x\|^{s-1}\,d^\times x\\ &=\int_{D_{\mathbb{A}}^\times/D_K^\times}\left[\sum_{\beta\in D_K}\Psi(x\beta)\right] f_+(x)\,\|x\|^{1-s}\,d^\times x\\ &=\zeta_D^+(1-s,\Psi)+\Psi(0)\int_{D_{\mathbb{A}}^\times/D_K^\times} f_+(x)\,\|x\|^{1-s}\,d^\times x,\end{aligned}$$

since the substitution $x\mapsto x^{-1}$ does not affect the Haar measure and $f_-(x^{-1})=f_+(x)$.

Lemma 2.4.

$$\int_{D_{\mathbb{A}}^\times/D_K^\times} f_-(x)\,\|x\|^s\,d^\times x=\mathrm{Vol}(D_{\mathbb{A}}^{(1)}/D_K^\times)\left(\frac{1}{1-q^{-s}}-\frac{1}{2}\right).$$

Proof. Indeed, it is known that the idelic norm gives a surjection $\mathbb{A}^\times\to q^{\mathbb{Z}}$, and also that the reduced norm $\mathcal{N}: D_{\mathbb{A}}^\times\to\mathbb{A}^\times$ is surjective. Hence we have a surjective homomorphism $D_{\mathbb{A}}^\times\to q^{\mathbb{Z}}$ given by $x\mapsto\|x\|$. The volume is translation invariant, so

$$\int_{D_{\mathbb{A}}^\times/D_K^\times} f_-(x)\,\|x\|^s\,d^\times x=\mathrm{Vol}(D_{\mathbb{A}}^{(1)}/D_K^\times)(\frac{1}{2}+q^{-s}+q^{-2s}+\cdots).$$

□

Note that in the lemma we did not specify how we normalized the Haar measure for computing $\mathrm{Vol}(D_{\mathbb{A}}^{(1)}/D_K^\times)$. There is no discrepancy here as the measure is implicitly fixed in the integral $\zeta_D(s,\Phi)$.

Finally we have the following expression:

$$\begin{aligned}\zeta_D(s,\Phi)=\ &\zeta_D^+(s,\Phi)+\zeta_D^+(1-s,\Psi)\\ &-\Phi(0)\mathrm{Vol}(D_{\mathbb{A}}^{(1)}/D_K^\times)\left(\frac{1}{1-q^{-s}}-\frac{1}{2}\right)\\ &+\Psi(0)\mathrm{Vol}(D_{\mathbb{A}}^{(1)}/D_K^\times)\left(\frac{1}{1-q^{-(s-1)}}-\frac{1}{2}\right).\end{aligned}$$

Proposition 2.5.

$$\operatorname*{Res}_{s=0}\zeta_D(s,\Phi)=-\Phi(0)\mathrm{Vol}(D_{\mathbb{A}}^{(1)}/D_K^\times)\frac{1}{\log q}.$$

2.4. **Third expression for mass formula.** We need to compute $\mathrm{Vol}(D_{\mathbb{A}}^{(1)}/D_K^\times)$, and Proposition 2.5 has reduced the problem to computing a residue of $\zeta_D(s,\Phi)$. We will compute this latter quantity for a particular choice of Φ and Haar measure normalized as in §2.2.

Fix a standard maximal $\mathcal{O}_v$-order Γ_v in each D_v; for places where D is ramified Γ_v is uniquely determined and at the places where $D_v\cong M_n(K_v)$ we take $\Gamma_v=$

$M_n(\mathcal{O}_v)$. Let Φ_v be the characteristic function of Γ_v, and let $\Phi = \prod_v \Phi_v$. We have the decomposition

$$\zeta_D(s,\Phi) = \int_{D_{\mathbb{A}}^\times} \Phi(x)\,\|x\|^s\, d^\times x = \prod_v \zeta_{D_v^\times}(s,\Phi_v),$$

where

$$\int_{D_v^\times} \Phi_v(x_v)|\mathcal{N}(x_v)|_v^s d^\times x_v.$$

Lemma 2.6. *Suppose D splits at v. Then we have*

$$\zeta_{D_v^\times}(s,\Phi) = \zeta_v(s)\cdot\zeta_v(s-1)\cdots\zeta_v(s-(n-1)),$$

where $\zeta_v(s) = (1-q_v^{-s})^{-1}$ is the zeta function of the local field K_v. (Here q_v is the number of elements in the residue field k_v of K_v.)

Proof. Consider the decomposition of $M_n(\mathcal{O}_v)$ into left $\mathrm{GL}_n(\mathcal{O}_v)$-cosets:

$$M_n(\mathcal{O}_v) = \coprod \mathrm{GL}_n(\mathcal{O}_v)T.$$

Fix a uniformizer π_v of $\mathcal{O}_v$. Under the action of $\mathrm{GL}_n(\mathcal{O}_v)$ on the left we can transform any non-zero matrix $T \in M_n(\mathcal{O}_v)$ into a matrix $(\alpha_{i,j})$ with $\alpha_{i,j} = 0$ for $i > j$, $\alpha_{i,i} = \pi_v^{d_i}$, $d_i \in \mathbb{Z}_{\geq 0}$, and $\alpha_{i,j}$ is a uniquely determined element modulo (π^{d_i}) for $i < j$. Indeed, it is easy to check that any matrix T can be transformed into an upper-triangular form using elementary row operations, such that $\mathrm{ord}_v(\alpha_{i,i}) > \mathrm{ord}_v(\alpha_{i,j})$ for $j > i$. Next, no two such matrices are in the same $\mathrm{GL}_n(\mathcal{O}_v)$ coset. This can be shown using induction. Indeed, if $MT_1 = T_2$ with $M \in \mathrm{GL}_n(\mathcal{O}_v)$ and T_1, T_2 upper-triangular as above, then starting with $m_{n,1}$, and working up the diagonals $(m_{s,1}, m_{s-1,2}, \ldots, m_{1,s})$, one shows that M must be a diagonal matrix with entries in $\mathcal{O}_v^\times$. Since we have fixed a uniformizer π_v, M in fact must be the identity matrix.

For an element $M \in M_n(\mathcal{O}_v)$, we have $|\mathcal{N}(M)|_v = |\det(M)|_v$. Hence

$$\begin{aligned}
\int_{D_v^\times} \Phi_v(x_v)|\mathcal{N}(x_v)|_v^s d^\times x_v &= \int\limits_{M_n(\mathcal{O}_v)-\{0\}} |\mathcal{N}(x_v)|_v^s d^\times x_v \\
&= \int\limits_{\substack{\coprod \mathrm{GL}_n(\mathcal{O}_v)T \\ T\neq 0}} |\mathcal{N}(x_v)|_v^s d^\times x_v \\
&= \mathrm{Vol}(\mathrm{GL}_n(\mathcal{O}_v)) \sum_{(d_i)\in\mathbb{Z}_{\geq 0}^n} q_v^{(-sd_1+(-s+1)d_2+\cdots+(-s+n-1)d_n)} \\
&= \zeta_v(s)\cdot\zeta_v(s-1)\cdots\zeta_v(s-(n-1)),
\end{aligned}$$

since we have normalized the Haar measure on $D_v^\times$ by $\mathrm{Vol}(\mathrm{GL}_n(\mathcal{O}_v)) = 1$. □

Lemma 2.7. *Suppose D is ramified at v. Then we have*

$$\zeta_{D_v^\times}(s,\Phi) = \zeta_v(s).$$

Proof. Recall that we assume that at the places where our division algebra D_K is ramified, D_v is still a division algebra. A division algebra over a local ring has a unique maximal $\mathcal{O}_v$-order $\mathcal{O}_{D_v}$ and a unique maximal ideal $\mathfrak{P}$; see [Rei75, (12.8),(13.2)]. Moreover, by [Rei75, (24.13)], $\mathcal{N}(\mathfrak{P}) = \pi_v$, where π_v is a uniformizer of $\mathcal{O}_v$. Hence

$$\begin{aligned}\int_{D_v^\times} \Phi_v(x_v)|\mathcal{N}(x_v)|_v^s d^\times x_v &= \int_{\mathcal{O}_{D_v}-\{0\}} |\mathcal{N}(x_v)|_v^s d^\times x_v \\ &= \mathrm{Vol}(\mathcal{O}_{D_v}^\times) \sum_{d\in\mathbb{Z}_{\geq 0}} q_v^{-sd} \\ &= \zeta_v(s),\end{aligned}$$

since we normalized the Haar measure by $\mathrm{Vol}(\mathcal{O}_{D_v}^\times) = 1$. □

Combining the two previous lemmas we have for Φ being the characteristic function of $\prod_v \Gamma_v$

$$\zeta_D(s,\Phi) = \prod_{i=0}^{n-1} \zeta_K(s-i) \prod_{\substack{v \\ D_v \text{ div.alg.}}} (1-q_v^{1-s})(1-q_v^{2-s})\cdots(1-q_v^{n-1-s}).$$

Proposition 2.8. *With previous notations the residue of $\zeta_D(s,\Phi)$ at $s=0$ is equal to*

$$-\frac{P(1)}{(q-1)\log q}\prod_{i=1}^{n-1}\zeta_{K,S}(-i).$$

Finally, Theorem 1.1 is a trivial consequence of Propositions 2.3, 2.5, 2.8, and the well-known fact $h(A) = d_\infty \cdot P(1)$.

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