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Endomorphisms of exceptional $\mathcal{D}$-elliptic sheaves

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Abstract We relate the endomorphism rings of certain $\mathcal{D}$-elliptic sheaves of finite characteristic to hereditary orders in central division algebras over function fields.

Keywords $\mathcal{D}$-elliptic sheaves · Hereditary orders

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1 Introduction

The endomorphism rings of abelian varieties have long been a subject of intensive investigation in number theory. One of the earliest results in this area was the determination by Deuring of the endomorphism rings of elliptic curves over finite fields. His results were later generalized to higher dimensional abelian varieties by Honda, Tate and Waterhouse, cf. [17]. These results have important applications, e.g., they play a key role in calculations of local zeta functions of Shimura varieties.

Drinfeld [4] introduced a certain function field analogue of abelian varieties; these objects are now called Drinfeld modules. Denote by $\mathbb{F}_q$ the finite field with q elements. Let X be a smooth, projective, geometrically connected curve defined over $\mathbb{F}_q$. Let $F = \mathbb{F}_q(X)$ be the function field of X. Fix a place ∞ of F (in Drinfeld's theory this plays the role of an archimedean place). Let $A = H^0(X - \infty, \mathcal{O}_X)$ be the subring of F consisting of functions on X which are regular away from ∞. Let $o \neq \infty$ be another place of F. Denote by $\mathbb{F}_o$ the residue field at o. Drinfeld [7] proved the analogue of Honda-Tate theorem for Drinfeld modules defined over extensions of $\mathbb{F}_o$. Gekeler [9] extended Drinfeld's results, in particular, he proved that for a rank-d supersingular Drinfeld module ϕ over $\overline{\mathbb{F}}_o$ the endomorphism ring

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$\mathrm{End}(\phi)$ is a maximal A-order in the central division algebra over F of dimension d^2, which is ramified exactly at o and ∞ with invariants $1/d$ and $-1/d$, respectively. Moreover, there is a bijection between the isomorphism classes of rank-d supersingular Drinfeld modules over $\overline{\mathbb{F}}_o$ and the left ideal classes of $\mathrm{End}(\phi)$ (see [9, Thm. 4.3]).

Laumon et al. [13] introduced the notion of $\mathcal{D}$-elliptic sheaves, which is a generalization of the notion of Drinfeld modules. (One can think of these objects as the function field analogues of abelian varieties equipped with an action of a maximal order in a simple algebra over $\mathbb{Q}$.) In Laumon-Rapoport-Stuhler theory one needs to fix a central simple algebra D over F of dimension d^2 which is split at ∞, and a maximal $\mathcal{O}_X$-order $\mathcal{D}$ in D. In [13, §9], the authors develop the analogue of Honda-Tate theory for $\mathcal{D}$-elliptic sheaves over $\overline{\mathbb{F}}_o$ with zero o and pole ∞, assuming D is split at o. The assumption that D is split at o is not superficial. When D is ramified at o, to obtain a reasonable theory of $\mathcal{D}$-elliptic sheaves with zero o and pole ∞, one has to assume at least that $D \otimes_F F_o$ is the division algebra with invariant $1/d$ over F_o, where F_o is the completion of F at o. Such $\mathcal{D}$-elliptic sheaves play a crucial role in the function field analogue of Čerednik-Drinfeld uniformization theory developed by Hausberger [11].

Assume $D_o := D \otimes_F F_o$ is the d^2-dimensional central division algebra with invariant $1/d$ over F_o. In this paper, we define a subclass of $\mathcal{D}$-elliptic sheaves over $\overline{\mathbb{F}}_o$, which we call *exceptional*, and which are distinguished by a particularly simple relationship between the actions of D_o and the Frobenius at o; see Definition 5.1. In general, exceptional $\mathcal{D}$-elliptic sheaves do not correspond to points on the moduli schemes constructed in [11] or [13]. Nevertheless, we show that the theory of endomorphism rings of these objects is similar to the theory of endomorphism rings of supersingular Drinfeld modules. The main result is the following (see Theorems 5.3 and 5.4):

Theorem 1.1 *Let $\mathbb{E}$ be an exceptional $\mathcal{D}$-elliptic sheaf over $\overline{\mathbb{F}}_o$ of type* $\mathbf{f}$. *Then* $\mathrm{End}(\mathbb{E})$ *is a hereditary A-order in the central division algebra $\bar{D}$ over F with invariants*

$$\mathrm{inv}_x(\bar{D}) = \begin{cases} 1/d, & x = \infty; \\ 0, & x = o; \\ \mathrm{inv}_x(D), & x \neq o, \infty. \end{cases}$$

This order is maximal at every place $x \neq o$, and at o it is isomorphic to a hereditary order determined by $\mathbf{f}$. *There is a bijection between the set of isomorphism classes of exceptional $\mathcal{D}$-elliptic sheaves over $\overline{\mathbb{F}}_o$ of type* $\mathbf{f}$ *modulo the action of $\mathbb{Z}$ and the isomorphism classes of locally free rank-*1 *right* $\mathrm{End}(\mathbb{E})$*-modules.*

The type of an exceptional $\mathcal{D}$-elliptic sheaf is determined by the action of the Frobenius at o. At the end of Sect. 5, we use Theorem 1.1 to prove a mass-formula for exceptional $\mathcal{D}$-elliptic sheaves, and discuss a geometric application of this formula. In Sect. 6, we explain how the argument in the proof of Theorem 1.1 can be used to prove a theorem about endomorphism rings of supersingular $\mathcal{D}$-elliptic sheaves over $\overline{\mathbb{F}}_o$, which implies Gekeler's result mentioned earlier as a special case (in Sect. 6 we assume that D is split at o).

Notation Unless specified otherwise, the following notation is fixed throughout the article.

- k is a fixed algebraic closure of $\mathbb{F}_q$ and $\mathrm{Fr}_q : k \to k$ is the automorphism $x \mapsto x^q$.
- $|X|$ denotes the set of closed points on X (equiv. the set of places of F).
- For $x \in |X|$, $\mathcal{O}_x$ is the completion of $\mathcal{O}_{X,x}$, and F_x (resp. $\mathbb{F}_x$) is the fraction field (resp. the residue field) of $\mathcal{O}_x$. The *degree* of x is $\deg(x) := [\mathbb{F}_x : \mathbb{F}_q]$, and $q_x := q^{\deg(x)} = \#\mathbb{F}_x$. We fix a uniformizer π_x of $\mathcal{O}_x$.

- $\mathbb{A}_F := \prod'_{x\in|X|} F_x$ is the adele ring of F, and for a set of places $S \subset |X|$, $\mathbb{A}_F^S := \prod'_{x\in|X|-S} F_x$ is the adele ring outside of S.
- The *zeta-function of X* is

$$\zeta_X(s) = \prod_{x\in|X|} (1 - q_x^{-s})^{-1}, \quad s \in \mathbb{C}.$$

- For a ring R with a unit element, we denote by $R^\times$ the group of all invertible elements in R.
- $\mathbb{M}_d$ denotes the ring of $d \times d$ matrices.

2 Orders

For the convenience of the reader, we recall some basic definitions and facts concerning orders over Dedekind domains. A standard reference for these topics is [15].

Let R be a Dedekind domain with quotient field K and let B be a central simple K-algebra. For any finite dimensional K-vector space V, a *full R-lattice* in V is a finitely generated R-submodule M in V such that $K \otimes_R M \cong V$. An *R-order* in the K-algebra B is a subring Λ of B, having the same unity element as B, and such that Λ is a full R-lattice in B. A *maximal R-order* in B is an R-order which is not contained in any other R-order in B. A *hereditary R-order* in B is an R-order Λ which is a hereditary ring, i.e., every left (equiv. right) ideal of Λ is a projective Λ-module. Maximal orders are hereditary. Being maximal or hereditary are local properties for orders: an R-order Λ in B is maximal (resp. hereditary) if and only if $\Lambda_\mathfrak{p} := \Lambda \otimes_R R_\mathfrak{p}$ is a maximal (resp. hereditary) $R_\mathfrak{p}$-order in $B_\mathfrak{p} := B \otimes_K K_\mathfrak{p}$ for all prime ideals $\mathfrak{p} \lhd R$, where $R_\mathfrak{p}$ and $K_\mathfrak{p}$ are the $\mathfrak{p}$-adic completions of R and K.

Let I be a full R-lattice in B. Define the *left order of I*

$$O_\ell(I) = \{b \in B \mid bI \subseteq I\}.$$

It is easy to see that $O_\ell(I)$ is an R-order in B. One similarly defines the right order $O_r(I)$ of I.

Assume R is a complete discrete valuation ring with a uniformizer π and fraction field K. Let $\mathbf{f} = (f_0, \dots, f_{d-1})$ be an ordered d-tuple of non-negative integers such that $\sum_{i=0}^{d-1} f_i = d$. Denote by $\mathbb{M}_d(\mathbf{f}, R)$ the subgroup of $\mathbb{M}_d(R)$ consisting of matrices of the form (m_{ij}), where m_{ij} ranges over all $f_i \times f_j$ matrices with entries in R if $i \geq j$, and over all $f_i \times f_j$ matrices with entries in πR if $i < j$ (a block of size 0 is assumed to be empty), e.g., if $\mathbf{f} = (d, 0, \dots, 0)$ then $\mathbb{M}_d(\mathbf{f}, R) = \mathbb{M}_d(R)$. Let $\mathbf{f}' = (n_1, \dots, n_r)$ be the ordered r-tuple obtained from $\mathbf{f}$ by deleting its zero entries but preserving the relative order of non-zero entries, e.g., if $\mathbf{f} = (0, 2, 0, 1)$ then $\mathbf{f}' = (2, 1)$. Note that $\mathbb{M}_d(\mathbf{f}_1, R) = \mathbb{M}_d(\mathbf{f}_2, R)$ if and only if $\mathbf{f}'_1 = \mathbf{f}'_2$.

Theorem 2.1 *Assume R is a complete discrete valuation ring. If B is a central division algebra over K, then the integral closure of R in B is the unique maximal R-order in B. Let Λ be an R-order in $\mathbb{M}_d(K)$. Λ is maximal if and only if there is an invertible element $u \in \mathbb{M}_d(K)$ such that $u\Lambda u^{-1} = \mathbb{M}_d(R)$. Λ is hereditary if and only if $u\Lambda u^{-1} = \mathbb{M}_d(\mathbf{f}, R)$ for some $\mathbf{f}$; Λ uniquely determines $\mathbf{f}'$, up to a cyclic permutation.*

Proof See (12.8), (17.3), (39.14) and (39.24) in [15]. □

Remark 2.2 Let $\Lambda \cong \mathbb{M}_d(\mathbf{f}, R)$ be a hereditary order as in Theorem 2.1, and let $\mathbf{f}' = (n_1, \dots, n_r)$. The number r is called the *type* of Λ and the ordered r-tuple $(n_1, \dots, n_r)$ the *invariants* of Λ; see [15, p. 360].

Let B be a central simple algebra over F. An $\mathcal{O}_X$*-order in* B is a coherent locally free sheaf $\mathcal{B}$ of $\mathcal{O}_X$-algebras with generic fibre B. The $\mathcal{O}_X$-order $\mathcal{B}$ is *maximal* if for every open affine $U = \mathrm{Spec}(R) \subset X$ the set of sections $H^0(U, \mathcal{B})$ is a maximal R-order in B. For $x \in |X|$ we write $B_x := B \otimes_F F_x$ and $\mathcal{B}_x := \mathcal{B} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_x$, so $\mathcal{B}_x$ is isomorphic to a subring of B_x. $\mathcal{B}$ is maximal if and only if $\mathcal{B}_x$ is a maximal $\mathcal{O}_x$-order in B_x.

3 Dieudonné modules

Let R be a complete discrete valuation ring of positive characteristic $p > 0$ and residue field $\mathbb{F}_q$. Fix a uniformizer π of R and identify $R = \mathbb{F}_q[\![\pi]\!]$. Let K be the fraction field of R. Let $\mathcal{R} = R\widehat{\otimes}_{\mathbb{F}_q} k \cong k[\![\pi]\!]$ be the completion of the maximal unramified extension of R, and $\mathcal{K} = K\widehat{\otimes}_{\mathbb{F}_q} k \cong k(\!(\pi)\!)$ be the field of fractions of $\mathcal{R}$. We will denote the canonical lifting of $\mathrm{Fr}_q \in \mathrm{Gal}(k/\mathbb{F}_q)$ to $\mathrm{Aut}(\mathcal{K})$ by the same symbol, so

$$\mathrm{Fr}_q\left(\sum_{i=n}^{\infty} a_i \pi^i\right) = \sum_{i=n}^{\infty} a_i^q \pi^i, \quad n \in \mathbb{Z}.$$

The following definition and Theorem 3.2 below are due to Drinfeld [8]; see also [12, 2.4].

Definition 3.1 A *Dieudonné R-module over k* is a free $\mathcal{R}$-module of finite rank M endowed with an injective Fr_q-linear map $\varphi : M \to M$ such that the cokernel of φ is finite dimensional as a k-vector space. The *rank* of (M, φ) is the rank of M as an $\mathcal{R}$-module. A *Dieudonné K-module over k* is a finite dimensional $\mathcal{K}$-vector space N endowed with a bijective Fr_q-linear map $\varphi : N \to N$. The *rank* of (N, φ) is the dimension of N as a $\mathcal{K}$-vector space. A *morphism* of Dieudonné R-modules (resp. K-modules) over k is a linear map between the underlying $\mathcal{R}$-modules (resp. $\mathcal{K}$-vector spaces) which commutes with the Fr_q-linear maps φ. If (M, φ) is a Dieudonné R-module over k, then $(K \otimes_R M, K \otimes_R \varphi)$ is a Dieudonné K-module over k.

Let $\mathcal{K}\{\tau\}$ be the non-commutative polynomial ring with commutation rule $\tau \cdot a = \mathrm{Fr}_q(a)\tau$, $a \in \mathcal{K}$. For each pair of integers (r, s) with $r \geq 1$ and $(r, s) = 1$, let

$$N_{r,s} = \mathcal{K}\{\tau\}/\mathcal{K}\{\tau\}(\tau^r - \pi^s).$$

Then $(N_{r,s}, \varphi_{r,s})$, where $\varphi_{r,s}$ is the left multiplication by τ, is a Dieudonné K-module over k of rank r.

Theorem 3.2 *The category of Dieudonné K-modules over k is K-linear and semi-simple. Its simple objects are $(N_{r,s}, \varphi_{r,s})$, $r, s \in \mathbb{Z}, r \geq 1, (r, s) = 1$. The K-algebra of endomorphisms $D_{r,s} = \mathrm{End}(N_{r,s}, \varphi_{r,s})$ of such an object is the central division algebra over K with invariant $-s/r$.*

This theorem implies that given a Dieudonné K-module (N, φ), its endomorphism algebra $\mathrm{End}(N, \varphi)$ is a finite dimensional semi-simple K-algebra such that the center of each simple component is K. It is clear that for a Dieudonné R-module (M, φ), the endomorphism ring $\mathrm{End}(M, \varphi)$ is an R-order in $\mathrm{End}(N, \varphi)$, where $(N, \varphi) = K \otimes (M, \varphi)$.

Proposition 3.3 *Let (M, φ) be a Dieudonné R-module over k of rank r such that $\varphi^n(M) \subset \pi M$ for some positive integer n and $\dim_k(M/\varphi(M)) = 1$. Then $K \otimes (M, \varphi) \cong N_{r,1}$, and $\mathrm{End}(M, \varphi)$ is the maximal R-order in $D_{r,1}$.*

Proof By [13, Lem. B.7], $K \otimes (M, \varphi) \cong N_{r,1}$. Consider the Dieudonné R-submodule

$$M_{r,1} = \mathcal{R}\{\tau\}/\mathcal{R}\{\tau\}(\tau^r - \pi)$$

of $N_{r,1}$. By [8, Prop. 2.7], $M = \varphi_{r,1}^m(M_{r,1})$ for some $m \in \mathbb{Z}$. Hence it is enough to prove that $\mathrm{End}(M_{r,1}, \varphi_{r,1})$ is the maximal R-order Λ in $D_{r,1}$. The proof of Theorem 3.2, as given in [12], implies that under the isomorphism $D_{r,1} \cong \mathrm{End}(N_{r,1}, \varphi_{r,1})$ one obtains an inclusion $\Lambda \subseteq \mathrm{End}(M_{r,1}, \varphi_{r,1})$. As the left hand-side of this inclusion is a maximal order, an equality must hold. □

Proposition 3.4 *Let (M, φ) be a Dieudonné R-module over k of rank n. Suppose $\varphi(M) = M$. Then*

$$(M, \varphi) \cong (R^n \widehat{\otimes}_{\mathbb{F}_q} k, \mathrm{Id} \widehat{\otimes}_{\mathbb{F}_q} \mathrm{Fr}_q).$$

Proof This is proven in [8, Prop. 2.5] using π-divisible groups. An alternative argument is as follows. Let (N, φ) be the associated Dieudonné K-module. By [12, Prop. 2.4.6], the assumption of the proposition is equivalent to

$$(N, \varphi) \cong (N_{1,0}, \varphi_{1,0})^n.$$

Hence

$$(N, \varphi) \cong (N^{\varphi} \widehat{\otimes}_{\mathbb{F}_q} k, \mathrm{Id} \widehat{\otimes}_{\mathbb{F}_q} \mathrm{Fr}_q),$$

where $N^{\varphi} = \{a \in N \mid \varphi(a) = a\}$ is an n-dimensional K-vector space. Since $N = M \otimes K$, $M^{\varphi} := M \cap N^{\varphi}$ is a full R-lattice in N^{φ} and $M = M^{\varphi} \widehat{\otimes}_{\mathbb{F}_q} k$. □

Let D be the d^2-dimensional central division algebra over K with invariant $1/d$. Let $\mathcal{D}$ be the maximal R-order in D. Denote by $R_d = \mathbb{F}_{q^d}[\![\pi]\!]$ the ring of integers of the degree d unramified extension of K. We can identify $\mathcal{D}$ with the R-algebra $R_d[\![\Pi]\!]$ of non-commutative formal power series in the indeterminate Π satisfying the relations

$$\Pi a = \mathrm{Fr}_q(a)\Pi \quad \text{for any } a \in R_d,$$
$$\Pi^d = \pi.$$

Definition 3.5 A Dieudonné R-module (M, φ) over k is *connected* if for all large enough positive integers m, $\varphi^m(M) \subset \pi M$. This is equivalent to saying that $(N_{1,0}, \varphi_{1,0})$ does not appear in the decomposition of the associated Dieudonné K-module into simple factors. A *Dieudonné $\mathcal{D}$-module over k* is a rank-d^2 connected Dieudonné R-module (M, φ) over k equipped with a right $\mathcal{D}$-action which commutes with φ and extends the natural action of R. A *morphism* of Dieudonné $\mathcal{D}$-modules is a morphism of the underlying Dieudonné R-modules which commutes with the action of $\mathcal{D}$. For each Dieudonné $\mathcal{D}$-module (M, φ) over k there is an associated Dieudonné D-module $(N, \varphi) = K \otimes (M, \varphi)$.

A Dieudonné $\mathcal{D}$-module (M, φ) over k is naturally a right $\mathcal{D} \widehat{\otimes}_{\mathbb{F}_q} k$-module. After fixing embeddings $\mathbb{F}_{q^d} \hookrightarrow k$ and $\mathbb{F}_{q^d} \hookrightarrow \mathcal{D}$, we obtain a grading

$$M = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} M_i,$$

where $M_i = \{m \in M \mid m(\lambda \widehat{\otimes} 1) = m(1 \widehat{\otimes} \lambda^{q^i}), \lambda \in \mathbb{F}_{q^d}\}$. Each M_i is a free finite rank $\mathcal{R}$-module. The action of $\Pi \widehat{\otimes} 1$ on M induces injective linear maps $\Pi_i : M_i \to M_{i+1}$, $i \in \mathbb{Z}/d\mathbb{Z}$. The composition

$$M_i \xrightarrow{\Pi_i} M_{i+1} \xrightarrow{\Pi_{i+1}} \cdots \xrightarrow{\Pi_{i+d-1}} M_{i+d} = M_i$$

is $M_i(\pi\widehat{\otimes}1) = \pi M_i$. In particular, all M_i have the same rank over $\mathcal{R}$, which must be d, since the rank of M is d^2. Since $\dim_k(\mathrm{coker}(\pi)) = d^2$, we have $\dim_k(\mathrm{coker}(\Pi)) = d$.

Similarly, φ induces injective Fr_q-linear maps

$$M_i \xrightarrow{\varphi_i} M_{i+1} \xrightarrow{\varphi_{i+1}} \cdots \xrightarrow{\varphi_{i+d-1}} M_{i+d} = M_i.$$

Let $f_i := \dim_k(\mathrm{coker}(\varphi_i))$, so $\sum_{i=0}^{d-1} f_i = \dim_k(\mathrm{coker}(\varphi))$. We call the ordered d-tuple $\mathbf{f} = (f_0, \ldots, f_{d-1})$ the *type* of M. Note that choosing another embedding $\mathbb{F}_{q^d} \hookrightarrow k$ induces a cyclic permutation of $\mathbf{f}$.

Definition 3.6 (cf. [10,16]) A Dieudonné $\mathcal{D}$-module (M, φ) over k is *exceptional* if $\mathrm{Im}(\varphi) = \mathrm{Im}(\Pi)$. (M, φ) is *special* if $f_i = 1$ for all i. (M, φ) is *superspecial* if it is special and exceptional.

Note that (M, φ) being exceptional is equivalent to $\mathrm{Im}(\varphi_i) = \mathrm{Im}(\Pi_i)$ for all $i \in \mathbb{Z}/d\mathbb{Z}$, i.e., every index of M is *critical* in the terminology of [10, Def. II.1.3]. In particular, if (M, φ) is exceptional of type $\mathbf{f}$ then $\sum_{i=0}^{d-1} f_i = d$.

Proposition 3.7 *Let (M, φ) be an exceptional Dieudonné $\mathcal{D}$-module over k of type $\mathbf{f}$. Then $\mathrm{End}_{\mathcal{D}}(M, \varphi) \cong \mathbb{M}_d(\mathbf{f}, R)$. In particular, $\mathrm{End}_{\mathcal{D}}(M, \varphi)$ is a hereditary R-order in $\mathrm{End}_D(N, \varphi) \cong \mathbb{M}_d(K)$.*

Proof Using the injections Π_i, we can identify all $M_i \otimes K$ with the same d-dimensional $\mathcal{K}$-vector space V. Then Π_i induces a bijective linear map $V \to V$, and φ_i induces a bijective Fr_q-linear map. Consider $\Pi_i^{-1} \circ \varphi_i : V \to V$ as a bijective Fr_q-linear map. Since (M, φ) is exceptional, $\mathrm{Im}(\Pi_i) = \mathrm{Im}(\varphi_i)$ for all $i \in \mathbb{Z}/d\mathbb{Z}$. Hence $\Pi_i^{-1} \circ \varphi_i$ is bijective on M_i. By Proposition 3.4, there are d full R-lattices Λ_i in K^d, $i \in \mathbb{Z}/d\mathbb{Z}$, such that $M_i = \Lambda_i \widehat{\otimes} k$. Since the action of $\mathcal{D}$ commutes with the action of φ, we have $\varphi_{i+1} \circ \Pi_i = \Pi_{i+1} \circ \varphi_i$. This implies that the identifications $M_i = \Lambda_i \widehat{\otimes} k$ can be made compatibly so that

$$\Lambda_0 \xrightarrow{\pi_0} \Lambda_1 \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{d-2}} \Lambda_{d-1} \xrightarrow{\pi_{d-1}} \Lambda_0, \tag{3.1}$$

where the π_i's are injections, $\Pi_i = \pi_i \widehat{\otimes} k$, $\varphi_i = \pi_i \widehat{\otimes} \mathrm{Fr}_q$. The inclusions π_i satisfy

$$\pi_{i+d-1} \circ \cdots \circ \pi_{i+1} \circ \pi_i = \pi,$$

so each $\mathrm{coker}(\pi_i)$ has no nilpotents and $\dim_{\mathbb{F}_q}(\mathrm{coker}(\pi_i)) = f_i$.

Now it is easy to see that giving an endomorphism of (M, φ) commuting with the action of $\mathcal{D}$ is equivalent to giving an endomorphism g of K^d which preserves the flag of lattices (3.1). Such endomorphisms form an R-algebra isomorphic to $\mathbb{M}_d(\mathbf{f}, R)$. That this is a hereditary order in $\mathbb{M}_d(K)$ follows from Theorem 2.1. □

Every Dieudonné $\mathcal{D}$-module over k satisfies the properties in [10, p. 20], hence corresponds to a formal $\mathcal{D}$-module of height d^2. It is instructive to give explicit examples of such formal modules. What follows below is motivated by [10, I.4.2].

The underlying formal group is isomorphic to $\widehat{\mathbb{G}}_{a,k}^d$, where $\widehat{\mathbb{G}}_{a,k} = \mathrm{Spf}(k[\![t]\!])$ is the formal additive group. Denote by τ the Frobenius isogeny of $\widehat{\mathbb{G}}_{a,k}$ corresponding to $t \mapsto t^q$. To give a formal $\mathcal{D}$-module essentially amounts to giving an embedding

$$\Phi : \mathbb{F}_{q^d}[\![\Pi]\!] = \mathcal{D} \hookrightarrow \mathrm{End}(\mathbb{G}_{a,k}^d) \cong \mathbb{M}_d(k\{\!\{\tau\}\!\}),$$

where $k\{\!\{\tau\}\!\}$ is the non-commutative ring of formal power series in τ satisfying $\tau a = a^q \tau$ for $a \in k$.

Define Φ by

$$\Phi(\Pi) = \tau \cdot \mathrm{Id}$$

and

$$\Phi(\lambda) = \mathrm{diag}(\chi_{ij}(\lambda))_{0 \le j \le d-1, 1 \le i \le f_j}, \quad \lambda \in \mathbb{F}_{q^d},$$

where $\chi_{ij}(\lambda) = \lambda^{q^j}$ if $f_j \neq 0$ and is omitted from $\mathrm{diag}(\cdot)$ otherwise. For example, if $d = 3$ and $\mathbf{f} = (2, 0, 1)$ then

$$\Phi(\Pi) = \begin{pmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{pmatrix} \quad \text{and} \quad \Phi(\lambda) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{q^2} \end{pmatrix}.$$

Then the Dieudonné $\mathcal{D}$-module (M, φ) corresponding to Φ is exceptional of type $\mathbf{f}$, and the endomorphism ring $\mathrm{End}_{\mathcal{D}}(M, \varphi)$ is isomorphic to the opposite algebra of the centralizer of $\Phi(\mathcal{D})$ in $\mathbb{M}_d(k\{\{\tau\}\})$. One can check as in [10, I.4.2] that this centralizer is isomorphic to $\mathbb{M}_d(\mathbf{f}, R)^{\mathrm{opp}}$.

4 $\mathcal{D}$-elliptic sheaves and their homomorphisms

In this section, we recall the definition of $\mathcal{D}$-elliptic sheaves of finite characteristic and their basic properties as given in [13].

Fix a closed point $\infty \in |X|$. Let D be a central simple algebra over F of dimension d^2. Assume D is split at ∞, i.e., $D \otimes_F F_\infty \cong \mathbb{M}_d(F_\infty)$. Fix a maximal $\mathcal{O}_X$-order $\mathcal{D}$ in D. Denote by $\mathrm{Ram} \subset |X|$ the set of places where D is ramified; hence for all $x \notin \mathrm{Ram}$ the couple $(D_x, \mathcal{D}_x)$ is isomorphic to $(\mathbb{M}_d(F_x), \mathbb{M}_d(\mathcal{O}_x))$. Fix another closed point $o \in |X| - \infty$, and an embedding $\mathbb{F}_o \hookrightarrow k$. Let z be the morphism determined by these choices

$$z : \mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{F}_o) \hookrightarrow X.$$

Definition 4.1 A *$\mathcal{D}$-elliptic sheaf of characteristic o over k* is a sequence $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i \in \mathbb{Z}}$, where $\mathcal{E}_i$ is a locally-free $\mathcal{O}_{X \otimes_{\mathbb{F}_q} k}$-module of rank d^2, equipped with a right action of $\mathcal{D}$ which extends the $\mathcal{O}_X$-action, and

$$\begin{aligned} j_i &: \mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1} \\ t_i &: {}^\tau\mathcal{E}_i := (\mathrm{Id}_X \otimes \mathrm{Fr}_q)^* \mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1} \end{aligned}$$

are injective $\mathcal{D}$-linear homomorphisms. Moreover, for each $i \in \mathbb{Z}$ the following conditions hold:

(1) The diagram

$$\begin{array}{ccc} \mathcal{E}_i & \xrightarrow{j_i} & \mathcal{E}_{i+1} \\ \uparrow{\scriptstyle t_{i-1}} & & \uparrow{\scriptstyle t_i} \\ {}^\tau\mathcal{E}_{i-1} & \xrightarrow{{}^\tau j_{i-1}} & {}^\tau\mathcal{E}_i \end{array}$$

commutes;

(2) $\mathcal{E}_{i+d\cdot\deg(\infty)} = \mathcal{E}_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(\infty)$, and the inclusion

$$\mathcal{E}_i \xrightarrow{j_i} \mathcal{E}_{i+1} \xrightarrow{j_{i+1}} \cdots \to \mathcal{E}_{i+d\cdot\deg(\infty)} = \mathcal{E}_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(\infty)$$

is induced by $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(\infty)$;

(3) $\dim_k H^0(X \otimes k, \operatorname{coker} j_i) = d$;

(4) $\mathcal{E}_i / t_{i-1}({}^\tau\mathcal{E}_{i-1}) = z_*\mathcal{H}_i$, where $\mathcal{H}_i$ is a d-dimensional k-vector space.

Definition 4.2 Let **DES** be the category whose objects are the $\mathcal{D}$-elliptic sheaves of characteristic o over k, and a morphism between two objects in this category

$$\psi = (\psi_i)_{i\in\mathbb{Z}} : \mathbb{E}' = (\mathcal{E}'_i, j'_i, t'_i)_{i\in\mathbb{Z}} \to \mathbb{E}'' = (\mathcal{E}''_i, j''_i, t''_i)_{i\in\mathbb{Z}}$$

is a sequence of sheaf morphisms $\psi_i : \mathcal{E}'_i \to \mathcal{E}''_{i+n}$ for some fixed $n \in \mathbb{Z}$ which are compatible with the action of $\mathcal{D}$ and commute with the morphisms j_i and t_i:

$$\psi_{i+1} \circ j'_i = j''_{i+n} \circ \psi_i \quad \text{and} \quad \psi_i \circ t'_{i-1} = t''_{i+n-1} \circ {}^\tau\psi_{i-1}.$$

Note that the group $\mathbb{Z}$ acts on **DES** by "shifting the indices":

$$n \cdot (\mathcal{E}_i, j_i, t_i) = (\mathcal{E}'_i, j'_i, t'_i)_{i\in\mathbb{Z}}$$

with $\mathcal{E}'_i = \mathcal{E}_{i+n}$, $j'_i = j_{i+n}$, $t'_i = t_{i+n}$, and there is an obvious morphism $\psi : (\mathcal{E}_i, j_i, t_i) \to n \cdot (\mathcal{E}_i, j_i, t_i)$ where $\psi_i : \mathcal{E}_i \xrightarrow{\sim} \mathcal{E}'_{i-n}$ for all $i \in \mathbb{Z}$.

Definition 4.3 A *φ-space* over k is a finite dimensional $F \otimes_{\mathbb{F}_q} k$-vector space N equipped with a bijective $F \otimes_{\mathbb{F}_q} \mathrm{Fr}_q$-linear map $\varphi : N \to N$. A *morphism* α between two φ-spaces (N', φ') and (N'', φ'') is an $F \otimes_{\mathbb{F}_q} k$-linear map $N' \xrightarrow{\alpha} N''$ such that $\varphi'' \circ \alpha = \alpha \circ \varphi'$.

Definition 4.4 Let $\mathbb{E} \in$ **DES**. Denote $N = H^0(\operatorname{Spec}(F \otimes_{\mathbb{F}_q} k), \mathcal{E}_0)$. This is a free $D \otimes_{\mathbb{F}_q} k$-module of rank 1. The t_i's induce a bijective $F \otimes_{\mathbb{F}_q} \mathrm{Fr}_q$-linear map $\varphi : N \to N$, compatible with the action of D on the right, so to $\mathbb{E}$ one can attach a φ-space over k equipped with an action of D, which commutes with φ. This action induces an F-algebra homomorphism

$$\iota : D^{\mathrm{opp}} \to \operatorname{End}(N, \varphi).$$

We denote by $\operatorname{End}_D(N, \varphi)$ the F-algebra of endomorphisms of (N, φ) which commute with the action of D. The triple (N, φ, ι) is called the *generic fibre of* $\mathbb{E}$. It is independent of the choice of $\mathcal{E}_0$ since the sheaves $\mathcal{E}_i$ are isomorphic over $(X - \infty) \otimes k$ via the j's.

Definition 4.5 For $x \in |X|$, denote $M_x = H^0(\operatorname{Spec}(\mathcal{O}_x \widehat{\otimes}_{\mathbb{F}_q} k), \mathcal{E}_0)$. This is a free $\mathcal{O}_x \widehat{\otimes}_{\mathbb{F}_q} k$-module of rank d^2 with a right action of $\mathcal{D}_x$. Let $N_x = F_x \otimes_{\mathcal{O}_x} M_x$. The t_i's induce a bijective $F_x \widehat{\otimes}_{\mathbb{F}_q} \mathrm{Fr}_q$-linear map $\varphi_x : N_x \to N_x$, compatible with the action of D_x. The pair (N_x, φ_x) will be called the *Dieudonné module* of $\mathbb{E}$ at x. Note that $(N_x, \varphi_x) = (F_x \widehat{\otimes}_F N, F_x \widehat{\otimes}_F \varphi)$.

Remark 4.6 Let $r := \deg(x)$. Since N_x is a free $F_x \widehat{\otimes}_{\mathbb{F}_q} k$-module, by fixing an embedding $\mathbb{F}_x \to k$, we obtain two actions of $\mathbb{F}_x$ on N_x. These actions induce a grading

$$N_x = \bigoplus_{i\in\mathbb{Z}/r\mathbb{Z}} N_{x,i},$$

where $N_{x,i} = \{a \in N_x \mid (\lambda^{q^i} \widehat{\otimes} 1)a = (1 \widehat{\otimes} \lambda)a,\ \lambda \in \mathbb{F}_x\}$. Now φ_x maps $N_{x,i}$ bijectively into $N_{x,i+1}$, and $N_{x,0}$ is an $F_x \widehat{\otimes}_{\mathbb{F}_x} k$-vector space. Hence $(N_{x,0}, \varphi_x^r)$ is a Dieudonné F_x-module over k in the sense of Sect. 3. We can recover (N_x, φ_x) uniquely from $(N_{x,0}, \varphi_x^r)$ since as $F_x \widehat{\otimes}_{\mathbb{F}_q} k$-vector space

$$N_x \cong \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} N_{x,0}$$

with the action of φ_x given by

$$(a_0, a_1, \ldots, a_{r-1}) \mapsto (\varphi_x^r(a_{r-1}), a_0, \ldots, a_{r-2}).$$

Finally, since the action of D_x commutes with φ_x, $(N_{x,0}, \varphi_x^r)$ is a Dieudonné D_x-module over k. A similar argument applies also to the lattices M_x $(x \neq \infty)$ and produces Dieudonné $\mathcal{D}_x$-modules over k.

As easily follows from definitions, the lattices M_x have the following properties (see [13, Lem. 9.3]):

(M1) If $x = \infty$, then

$$M_\infty \subset \varphi_\infty(M_\infty)$$
$$\dim_k(\varphi_\infty(M_\infty)/M_\infty) = d$$
$$\varphi_\infty^{d\cdot\deg(\infty)}(M_\infty) = \pi_\infty^{-1} M_\infty.$$

(M2) If $x = o$, then

$$\pi_o M_o \subset \varphi_o(M_o) \subset M_o$$

and the $\mathbb{F}_o \otimes_{\mathbb{F}_q} k$-module $M_o/\varphi_o(M_o)$ is of length d and is supported on the connected component of $\mathrm{Spec}(\mathbb{F}_o \otimes_{\mathbb{F}_q} k)$ which is the image of z.

(M3) If $x \neq o, \infty$, then

$$\varphi_x(M_x) = M_x.$$

(M4) Some basis of N generates M_x in N_x for all but finitely many $x \in |X|$.

Definition 4.7 Let **DMod** be the category whose objects are the pairs

$$((N, \varphi, \iota), (M_x)_{x\in|X|})$$

where (N, φ) is a φ-space of rank d^2 over $F \otimes k$, $\iota : D^{\mathrm{opp}} \to \mathrm{End}(N, \varphi)$ is an F-algebra homomorphism, and $(M_x)_{x\in|X|}$ is a collection of $\mathcal{D}_x$-lattices in $(N_x, \varphi_x) = (F_x \widehat{\otimes}_F N, F_x \widehat{\otimes}_F \varphi)$ which satisfy (M1)–(M4). A morphism α between two such objects

$$\alpha : ((N', \varphi', \iota'), (M'_x)_{x\in|X|}) \to ((N'', \varphi'', \iota''), (M''_x)_{x\in|X|})$$

is a morphism of the φ-spaces $\alpha : (N', \varphi') \to (N'', \varphi'')$ such that

$$\iota'' \circ \alpha = \alpha \circ \iota' \quad \text{and} \quad \alpha \widehat{\otimes} F_x(M'_x) \subset M''_x$$

for all $x \in |X| - \infty$.

Proposition 4.8 *The functor Υ :* **DES** $\to$ **DMod** *which associates to a $\mathcal{D}$-elliptic sheaf of characteristic o over k its generic fibre along with the lattices M_x in its Dieudonné modules is an equivalence of categories.*

Proof Since the generic fibre of $\mathbb{E} \in \mathbf{DES}$ and the lattices M_x, $x \in |X| - \infty$, do not depend on the choice of $\mathcal{E}_0$ in their definitions, it is easy to see that a morphism $\mathbb{E} \to \mathbb{E}'$ induces a morphism $\Upsilon(\mathbb{E}) \to \Upsilon(\mathbb{E}')$. Hence Υ is indeed a functor.

Next, we show that any object $\widetilde{\mathbb{E}} = ((N, \varphi, \iota), (M_x)_{x\in|X|}) \in \mathbf{DMod}$ is isomorphic to an object of the form $\Upsilon(\mathbb{E})$ for some $\mathbb{E} \in \mathbf{DES}$. For $i \in \mathbb{Z}$, define a sheaf $\mathcal{E}_i$ on $X \otimes_{\mathbb{F}_q} k$ as follows. Let $U \subset X$ be an open affine. If $\infty \notin |U|$, then let

$$\mathcal{E}_i(U \otimes_{\mathbb{F}_q} k) := \bigcap_{x\in|U|} (N \cap M_x),$$

where the inner intersections are taken in N_x, and the outer in N. If $\infty \in |U|$, let

$$\mathcal{E}_i(U \otimes_{\mathbb{F}_q} k) := \left(\bigcap_{x\in|U|-\infty} (N \cap M_x)\right) \bigcap \left(\varphi_\infty^i(M_\infty) \cap N\right).$$

Thanks to (M4), $\mathcal{E}_i$ is a locally-free $\mathcal{O}_{X\otimes_{\mathbb{F}_q} k}$-module of rank d^2. The inclusions $\varphi_\infty^i(M_\infty) \subset \varphi_\infty^{i+1}(M_\infty)$ induce inclusions $j_i : \mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$. The action of φ on N induces homomorphisms $t_i : {}^\tau\mathcal{E}_i \to \mathcal{E}_{i+1}$. The action of D on (N, φ) and $\mathcal{D}_x$ on M_x, defines an action of $\mathcal{D}$ on $\mathcal{E}_i$ compatible with j_i and t_i. Finally, the conditions (M1)–(M3) ensure that $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} \in \mathbf{DES}$, and clearly $\Upsilon(\mathbb{E}) \cong \widetilde{\mathbb{E}}$.

Now it is enough to show that Υ is fully faithful. Let $\widetilde{\mathbb{E}} = ((N, \varphi, \iota), (M_x)_{x\in|X|}) \in \mathbf{DMod}$. By (9.7) and (9.8) in [13], there is a canonical splitting

$$(N_\infty, \varphi_\infty) = (N_{d,-1}, \varphi_{d,-1})^d$$
$$M_\infty = {M_\infty^\dagger}^d,$$

where $M_\infty^\dagger$ is a lattice in $N_{d,-1}$ with $M_\infty^\dagger \subset \varphi_{d,-1}(M_\infty^\dagger)$, and the action of D_∞ on $(N_\infty, \varphi_\infty)$ is the natural right action of $\mathbb{M}_d(F_\infty)$ on $(N_{d,-1}, \varphi_{d,-1})^d$. Suppose

$$\alpha : \widetilde{\mathbb{E}} \to \widetilde{\mathbb{E}}' = ((N', \varphi', \iota'), (M'_x)_{x\in|X|})$$

is a non-trivial morphism. Then $\alpha\widehat{\otimes}F_\infty$ induces a non-zero endomorphism of the (irreducible) Dieudonné F_∞-module $(N_{d,-1}, \varphi_{d,-1})$. This implies that $M := \alpha\widehat{\otimes}F_\infty(M_\infty^\dagger)$ is a full lattice in $N_{d,-1}$ which satisfies $M \subset \varphi_{d,-1}(M)$. Now [13, Prop. B.10] can be used to conclude that

$$\alpha\widehat{\otimes}F_\infty(M_\infty) = (\varphi'_\infty)^n(M'_\infty)$$

for some $n \in \mathbb{Z}$. On the other hand, note that if $\Upsilon(\mathbb{E}) = ((N, \varphi, \iota), (M_x)_{x\in|X|})$ then

$$\Upsilon(n \cdot \mathbb{E}) = ((N, \varphi, \iota), (M_x)_{x\in|X|-\infty}, \varphi_\infty^n(M_\infty)).$$

Since a morphism between two locally free sheaves on a curve is uniquely determined by the induced morphisms on their stalks, using the previous discussion it is not hard to check that α uniquely lifts to a morphism between the preimages of $\widetilde{\mathbb{E}}$ and $\widetilde{\mathbb{E}}'$ in $\mathbf{DES}$. This finishes the proof of the proposition. □

Notation 4.9 Let $\mathbb{E}, \mathbb{E}' \in \mathbf{DES}$, and (N, φ, ι), (N', φ', ι') be their generic fibres, respectively. Let $(\widetilde{F}, \widetilde{\Pi})$ and $(\widetilde{F}', \widetilde{\Pi}')$ be the φ-pairs of (N, φ) and (N', φ'), respectively; see [13, App. A] for the definition. Let (W, ψ) be the irreducible φ-space corresponding to $(\widetilde{F}, \widetilde{\Pi})$; see [13, (A.6)]. By [13, Prop. 9.9], $\widetilde{F}$ is a separable field extension of F of degree dividing d and (N, φ) is isomorphic to $(W, \psi)^d$. Using [13, Cor. 9.10], we conclude that

$\mathrm{Hom}((N,\varphi,\iota),(N',\varphi',\iota'))=0$ if $(\widetilde{F},\widetilde{\Pi})\neq(\widetilde{F}',\widetilde{\Pi}')$, and is an F-vector space of dimension $(d/[\widetilde{F}:F])^2$, otherwise. Since a basis of N spans almost all the M_x, and similarly for N' and M'_x, we conclude that $\mathrm{Hom}(\Upsilon(\mathbb{E}),\Upsilon(\mathbb{E}'))$ has a natural structure of a projective finite rank A-module. From now on $\mathrm{Hom}(\mathbb{E},\mathbb{E}')$ denotes this module. (By Proposition 4.8, the elements of $\mathrm{Hom}(\mathbb{E},\mathbb{E}')$ are in bijection with the morphisms from $\mathbb{E}$ to $\mathbb{E}'$ as objects in **DES**.) We denote $\mathrm{End}(\mathbb{E})=\mathrm{Hom}(\mathbb{E},\mathbb{E})$.

Remark 4.10 There is another category closely related to **DES** which can be used to define the module $\mathrm{Hom}(\mathbb{E},\mathbb{E}')$. Since this is more in line with the definition of homomorphisms between Drinfeld modules [4], we also outline this alternative construction.

Let $O_D:=H^0(X-\infty,\mathcal{D})$, which is a maximal A-order in D. For $\mathbb{E}=(\mathcal{E}_i,j_i,t_i)_{i\in\mathbb{Z}}$, let

$$P:=H^0((X-\infty)\otimes_{\mathbb{F}_q}k,\mathcal{E}_0).$$

Since P is independent of the choice of $\mathcal{E}_0$, it is a left $k\{\tau\}$-module, where the operation of τ is induced from $t_i:{}^\tau\mathcal{E}_i\to\mathcal{E}_{i+1}$. (As in Sect. 3, $k\{\tau\}$ denotes the skew-polynomial ring with the commutation relation $\tau b=b^q\tau$.) In fact, P is a free $k\{\tau\}$-module of rank d; see [13, Lem. 3.5]. Since the action of $\mathcal{D}$ commutes with the t_i's, P also has a natural structure of right O_D-module which commutes with the left action of $k\{\tau\}$. The underlying $A\otimes_{\mathbb{F}_q}k$-module is locally free of rank d^2. We make P into a left $O_D^{\mathrm{opp}}\otimes_{\mathbb{F}_q}k\{\tau\}$-module ($\tau$ acts trivially on O_D). Let $z:A\to k$ be the composition of the quotient map $A\to\mathbb{F}_o$ with our fixed embedding $\mathbb{F}_o\hookrightarrow k$. Then $(a-z(a))$ acts as 0 on $P/\tau P$ for all $a\in A$, cf. [13, Thm. 3.17]. Overall, P is an *Anderson A-motive* of rank d^2 and τ-rank d equipped with an action of O_D, cf. [1,14]. Now, $\mathrm{Hom}_{O_D^{\mathrm{opp}}\otimes_{\mathbb{F}_q}k\{\tau\}}(P,P')$ has a natural structure of a projective A-module of rank not exceeding d^2, cf. [1,4]. It is easy to check that this module is isomorphic to $\mathrm{Hom}(\mathbb{E},\mathbb{E}')$.

In fact, the map $\mathbb{E}\mapsto(P,M_\infty)$ identifies **DES** with a full subcategory of so-called vector bundles of rank d over the non-commutative projective line over k; see [13, Thm. 3.17]. Combined with Anderson's theorem on the anti-equivalence of categories of abelian t-modules and t-motives [1, Thm. 1], this gives an analogue for $\mathcal{D}$-elliptic sheaves of Drinfeld's theorem of the anti-equivalence of categories of Drinfeld modules and elliptic sheaves [6].

5 Main theorems

Let D be as in Sect. 4. Assume D is a division algebra such that D_o is the d^2-dimensional central division algebra over F_o with invariant $1/d$. In this case $\mathcal{D}_o$ is the unique maximal order of D_o. After fixing an embedding $\mathbb{F}_o^{(d)}\hookrightarrow D_o$, we can identify $\mathcal{D}_o$ with $\mathbb{F}_o^{(d)}[\![\Pi_o]\!]$; here $\mathbb{F}_o^{(d)}$ is the degree d extension of $\mathbb{F}_o$ and

$$\Pi_o a=\mathrm{Fr}_q^{\deg(o)}(a)\Pi_o,$$
$$\Pi_o^d=\pi_o.$$

Definition 5.1 Let $\mathbb{E}\in\mathbf{DES}$. We say that $\mathbb{E}$ is *exceptional* if

$$\varphi_o^{\deg(o)}(M_o)=M_o\cdot\Pi_o.$$

Clearly $\mathbb{E}$ is exceptional if and only if the Dieudonné $\mathcal{D}_o$-module $(M_{o,0},\varphi_o^{\deg(o)})$ associated to (M_o,φ_o) (see Remark 4.6) is exceptional in the sense of Definition 3.6. The *type* of an exceptional $\mathbb{E}$ is the type of $(M_{o,0},\varphi_o^{\deg(o)})$. Similarly, we say that $\mathbb{E}$ is *special* (resp. *superspecial*) if $(M_{o,0},\varphi_o^{\deg(o)})$ is special (resp. superspecial). Denote by $\mathfrak{X}_\mathfrak{f}$ the set of isomorphism classes of exceptional $\mathcal{D}$-elliptic sheaves of characteristic o over k of type $\mathbf{f}$. Using Proposition 4.8, one can easily show that $\mathfrak{X}_{\mathbf{f}} \neq \emptyset$; cf. the proof of [13, Thm. 9.13]. The action of $\mathbb{Z}$ on **DES** preserves $\mathfrak{X}_{\mathbf{f}}$, and we denote the quotient set by $\mathfrak{X}_{\mathbf{f}}/\mathbb{Z}$.

Remark 5.2 Exceptional $\mathcal{D}$-elliptic sheaves do not correspond to points on the moduli schemes constructed in [11], unless they are superspecial.

Let $\bar{D}$ be the central division algebra over F with invariants

$$\mathrm{inv}_x(\bar{D}) = \begin{cases} 1/d, & x = \infty; \\ 0, & x = o; \\ \mathrm{inv}_x(D), & x \neq o, \infty. \end{cases} \tag{5.1}$$

Theorem 5.3 *If* $\mathbb{E} \in \mathfrak{X}_{\mathbf{f}}$, *then* $\mathrm{End}(\mathbb{E})$ *is a hereditary A-order in* $\bar{D}$. *This order is maximal at every* $x \in |X| - o - \infty$, *and at o it is isomorphic to* $\mathbb{M}_d(\mathbf{f}, \mathcal{O}_o)$.

Proof Let $((N, \varphi, \iota), (M_x)_{x\in|X|}) \in$ **DMod** be the object attached to $\mathbb{E}$ by Proposition 4.8. Let $(\widetilde{F}, \widetilde{\Pi})$ be the φ-pair of (N, φ). Since $\mathbb{E}$ is exceptional

$$\begin{aligned} \varphi_\infty^{d\cdot\deg(\infty)}(M_\infty) &= \pi_\infty^{-1} M_\infty, \\ \varphi_o^{d\cdot\deg(o)}(M_o) &= \pi_o M_o, \\ \varphi_x(M_x) &= M_x, \quad \text{if } x \neq o, \infty. \end{aligned}$$

Let h be the class number of X. The divisor $h(\deg(\infty)o - \deg(o)\infty)$ is principal, so from the previous equalities $\varphi^{dh} \in F$. By construction of $(\widetilde{F}, \widetilde{\Pi})$, this implies $\widetilde{F} = F$ and $\widetilde{\Pi} \in F$ has valuations

$$\mathrm{ord}_x(\widetilde{\Pi}) = \begin{cases} 1/d \deg(o), & x = o; \\ -1/d \deg(\infty), & x = \infty; \\ 0, & x \neq o, \infty. \end{cases} \tag{5.2}$$

Since (N, φ) is isotypical [13, Lem. 9.6], (5.2) and [13, Thm. A.6] imply that $B = \mathrm{End}(N, \varphi)$ is the central simple algebra over F of dimension d^4 with invariants $\mathrm{inv}_o(B) = -1/d$, $\mathrm{inv}_\infty(B) = 1/d$, $\mathrm{inv}_x(B) = 0$, $x \neq o, \infty$. $\mathrm{End}_D(N, \varphi)$ is exactly the centralizer of $\iota(D^{\mathrm{opp}})$ in $\mathrm{End}(N, \varphi)$. By the double centralizer theorem [15, Cor. 7.14]

$$\mathrm{End}_D(N, \varphi) \otimes_F D^{\mathrm{opp}} \cong B.$$

This implies that $\mathrm{End}_D(N, \varphi)$ is the central simple algebra over F of dimension d^2 with invariants

$$\mathrm{inv}_x(\mathrm{End}_D(N, \varphi)) = \mathrm{inv}_x(B) - \mathrm{inv}_x(D^{\mathrm{opp}}) = \mathrm{inv}_x(B) + \mathrm{inv}_x(D) \bmod \mathbb{Z}.$$

Comparing the invariants, we see that $\mathrm{End}_D(N, \varphi) \cong \bar{D}$.

Let $x \in |X| - \infty$. There is a natural homomorphism

$$\mathrm{End}(\mathbb{E}) \otimes_A \mathcal{O}_x \to \mathrm{End}_{\mathcal{D}_x}(M_x, \varphi_x), \tag{5.3}$$

which is injective with torsion-free cokernel, cf. [12, (2.5.6)]. On the other hand, according to [13, Lem. B.6, B.7], for $x \neq o$, we have $\mathrm{End}_{D_x}(N_x, \varphi_x) \cong \bar{D}_x$. The same isomorphism for $x = o$ follows from Proposition 3.7. Hence (5.3) becomes an isomorphism after tensoring with F_x, and therefore is an isomorphism itself. To finish the proof we need to show that $\mathrm{End}_{\mathcal{D}_x}(M_x, \varphi_x)$ is maximal for every $x \neq o$, and is hereditary for $x = o$.

If $x = o$, then by Proposition 3.7

$$\mathrm{End}_{\mathcal{D}_x}(M_x, \varphi_x) \cong \mathrm{End}_{\mathcal{D}_x}(M_{x,0}, \varphi_x^{\deg(x)}) \cong \mathbb{M}_d(\mathbf{f}, \mathcal{O}_x).$$

If $x \neq o, \infty$, then $\varphi_x(M_x) = M_x$. By Proposition 3.4,

$$(M_{x,0}, \varphi_x^{\deg(x)}) \cong (\Lambda_x \widehat{\otimes}_{\mathbb{F}_x} k, \mathrm{Id}\widehat{\otimes}_{\mathbb{F}_x} \mathrm{Fr}_q^{\deg(x)}),$$

where Λ_x is a free $\mathcal{O}_x$-module of rank d^2. The action of $\mathcal{D}_x$ commutes with φ_x, so $\mathcal{D}_x$ is in the right order of the full $\mathcal{O}_x$-lattice Λ_x in D_x. Since $\mathcal{D}_x$ is maximal, the left order $O_l(\Lambda_x)$ of Λ_x is also maximal in $D_x \cong \bar{D}_x$; see [15, (17.6)]. On the other hand, $O_l(\Lambda_x) \subset \mathrm{End}_{\mathcal{D}_x}(M_{x,0}, \varphi_x^{\deg(x)})$, which forces $\mathrm{End}_{\mathcal{D}_x}(M_x, \varphi_x)$ to be maximal. □

Theorem 5.4 *Let $\mathbb{E} \in \mathfrak{X}_{\mathbf{f}}$ be fixed. The map*

$$\mathbb{E}' \mapsto I = \mathrm{Hom}(\mathbb{E}, \mathbb{E}')$$

*establishes a bijection between $\mathfrak{X}_{\mathbf{f}}/\mathbb{Z}$ and the isomorphism classes of locally free rank-*1 *right* End$(\mathbb{E})$*-modules. We have*

$$\mathrm{End}(\mathbb{E}') \cong O_\ell(\mathrm{Hom}(\mathbb{E}, \mathbb{E}')),$$

which thus corresponds to $O_\ell(I)$ under this bijection.

Proof Let $\mathbb{E}' \in \mathfrak{X}_{\mathbf{f}}$. Let $((N, \varphi, \iota), (M_x)_{x\in|X|})$ and $((N', \varphi', \iota'), (M'_x)_{x\in|X|})$ be the objects in **DMod** corresponding to $\mathbb{E}$ and $\mathbb{E}'$, respectively, under the equivalence of Proposition 4.8. From the proof of Theorem 5.3 we know that the φ-pairs $(\widetilde{F}, \widetilde{\Pi})$ associated to the generic fibres of $\mathbb{E}$ and $\mathbb{E}'$ are the same. By [13, (9.12)], this implies that the generic fibres (N, φ, ι) and (N', φ', ι') are isomorphic. (In the terminology of [13] this is equivalent to saying that $\mathbb{E}$ and $\mathbb{E}'$ are isogenous.) Hence the Dieudonné modules (N_x, φ_x) and (N'_x, φ'_x) of $\mathbb{E}$ and $\mathbb{E}'$ are also isomorphic for all $x \in |X| - \infty$. Consider the $\mathcal{D}_x$-lattices $M_x \subset N_x$ and $M'_x \subset N'_x$. We claim that there is an isomorphism $\alpha_x : (N_x, \varphi_x) \cong (N'_x, \varphi'_x)$ which commutes with D_x and $\alpha(M_x) = M'_x$. When $x \neq o, \infty$, this follows from Proposition 3.4 and the fact that any one-sided ideal of $\mathcal{D}_x$ is principal [15, Thm. 18.10]. When $x = o$, the claim follows from the proof of Proposition 3.7, using the assumption that $\mathbb{E}$ and $\mathbb{E}'$ have the same type. Moreover, thanks to (M4), if we fix an isomorphism $\alpha : (N, \varphi, \iota) \cong (N', \varphi', \iota')$, then for almost all x we can take $\alpha_x = \alpha \widehat{\otimes} F_x$. The argument which shows that (5.3) is an isomorphism also implies that for $x \in |X| - \infty$

$$\mathrm{Hom}(\mathbb{E}, \mathbb{E}') \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_x \cong \mathrm{Hom}_{\mathcal{D}_x}((M_x, \varphi_x), (M'_x, \varphi'_x)).$$

From what was said above we conclude that $\mathrm{Hom}(\mathbb{E}, \mathbb{E}')$ is a locally free rank-1 right Λ-module, where $\Lambda := \mathrm{End}(\mathbb{E})$.

From the definition of $\mathrm{Hom}(\mathbb{E}, \mathbb{E}')$ and the action of $\mathbb{Z}$ on **DES**, it is clear that $\mathrm{Hom}(\mathbb{E}, \mathbb{E}') \cong \mathrm{Hom}(\mathbb{E}, n \cdot \mathbb{E}')$. Hence the map in the statement of the theorem factors through $\mathfrak{X}_{\mathbf{f}}/\mathbb{Z}$.

Let I be a locally free rank-1 right Λ-module. Define $(N', \varphi', \iota') = (N, \varphi, \iota)$, $M'_x = I_x \otimes_{\Lambda_x} M_x$, for $x \in |X| - \infty$, and $M'_\infty = M_\infty$. It is easy to check that $((N', \varphi', \iota'), (M'_x)_{x\in|X|}) \in$ **DMod**, hence defines a $\mathcal{D}$-elliptic sheaf of characteristic o over k. We denote this $\mathcal{D}$-elliptic sheaf by $I \otimes_\Lambda \mathbb{E}$. Since $(M_o, \varphi_o) \cong (M'_o, \varphi'_o)$, $I \otimes_\Lambda \mathbb{E}$ is exceptional of the same type as $\mathbb{E}$.

There is a natural homomorphism of right Λ-modules

$$I \to \mathrm{Hom}(\mathbb{E}, I \otimes_\Lambda \mathbb{E}),$$

which is an isomorphism, as one checks locally. Next, $\mathrm{Hom}(\mathbb{E}, \mathbb{E}') \otimes_\Lambda \mathbb{E}$ is isomorphic to $n \cdot \mathbb{E}'$ for some n. Hence these two constructions are inverses of each other, and the bijection of the theorem follows.

Finally, it is clear that $O_\ell(I) \subset \mathrm{End}(I \otimes_\Lambda \mathbb{E})$, and since both sides are locally isomorphic hereditary orders, an equality must hold. □

Fix some $\mathbb{E} \in \mathfrak{X}_{\mathbf{f}}$. Denote $\Lambda = \mathrm{End}(\mathbb{E})$, $\bar{D}(\mathbb{A}_F^\infty) = \bar{D} \otimes_F \mathbb{A}_F^\infty$ and

$$\Lambda(\mathbb{A}_F^\infty) = \prod_{x \in |X| - \infty} \Lambda_x \hookrightarrow \bar{D}(\mathbb{A}_F^\infty).$$

The ring $\bar{D}(F)$ embeds diagonally into $\bar{D}(\mathbb{A}_F^\infty)$.

Corollary 5.5 *There is a bijection between $\mathfrak{X}_{\mathbf{f}}/\mathbb{Z}$ and the double coset space*

$$\bar{D}(F)^\times \backslash \bar{D}(\mathbb{A}_F^\infty)^\times / \Lambda(\mathbb{A}_F^\infty)^\times.$$

Proof This is a consequence of the bijection in Theorem 5.4. □

By the strong approximation theorem for $\bar{D}^\times$, the double coset space in Corollary 5.5 has finite cardinality. Unfortunately, in general, an explicit expression for class numbers of hereditary orders over Dedekind domains is not known (e.g. the order of the double coset space above). But one can at least give an estimate on this number using an analogue of Eichler's mass-formula.

Definition 5.6 For $\mathbb{E} \in \mathfrak{X}_{\mathbf{f}}$, let $\mathrm{Aut}(\mathbb{E}) := \mathrm{End}(\mathbb{E})^\times$.

Lemma 5.7 $\mathrm{Aut}(\mathbb{E}) \cong \mathbb{F}_{q^s}^\times$ *for some s dividing d.*

Proof Let $\Lambda := \mathrm{End}(\mathbb{E})$. Since $\bar{D}_\infty$ is a division algebra, $\bar{D}(F_\infty)^\times / F_\infty^\times$ is compact in the ∞-adic topology, and contains $\Lambda^\times / \mathbb{F}_q^\times$ as a discrete subgroup. Hence $\Lambda^\times$ is finite. Let $\lambda \in \Lambda^\times$. Since $\lambda^n = 1$ for some n, λ is algebraic over $\mathbb{F}_q$. Conversely, it is clear that if $\lambda \in \Lambda$ is algebraic over $\mathbb{F}_q$ and $\lambda \neq 0$, then $\lambda \in \Lambda^\times$. Let Λ^{alg} be the subset of Λ consisting of elements which are algebraic over $\mathbb{F}_q$. It is not hard to show that Λ^{alg} is a field extension of $\mathbb{F}_q$; see [3, p. 383]. Let $\Lambda^{\mathrm{alg}} \cong \mathbb{F}_{q^s}$. Then $\mathbb{F}_{q^s} F$ is a field extension of F of degree s contained in $\bar{D}$. This implies that s divides d (see [12, Prop. A.1.4]), and $\Lambda^\times = \Lambda^{\mathrm{alg}} - 0 = \mathbb{F}_{q^s}^\times$. □

Let $\mathbb{E}_1, \ldots, \mathbb{E}_h$ be representatives of $\mathfrak{X}_{\mathbf{f}}/\mathbb{Z}$, and let $w_i = \#\mathrm{Aut}(\mathbb{E}_i)$, $1 \leq i \leq h$. Each w_i is finite by Lemma 5.7, so we can consider the sum

$$\mathrm{Mass}(\mathbf{f}) := (q-1) \sum_{i=1}^{h} \frac{1}{w_i}.$$

Let $I_1, \ldots, I_h$ represent the isomorphism classes of locally free rank-1 right $\mathrm{End}(\mathbb{E}_1)$-modules. By Theorem 5.4,

$$\mathrm{Mass}(\mathbf{f}) = \sum_{i=1}^{h} (O_\ell(I_i)^\times : \mathbb{F}_q^\times)^{-1}.$$

According to [3], it is possible to give a formula for this last sum in terms of the invariants of F, D and $\mathbf{f} = (f_0, \ldots, f_{d-1})$. For $x \in |X|$, $\bar{D}_x \cong \mathbb{M}_{\kappa_x}(\Delta_x)$, where Δ_x is a central division algebra over F_x of index e_x. We always have $\kappa_x e_x = d$, and $e_x = 1$ if $x \notin \mathrm{Ram} \cup \infty$.

Let

$$\mathcal{T}^o = \prod_{\substack{x \in \mathrm{Ram} \cup \infty \\ x \neq o}} \prod_{\substack{1 \le j \le d-1 \\ j \not\equiv 0 \bmod e_x}} (q_x^j - 1)$$

and

$$\mathcal{T}_o = \frac{\prod_{1 \le j \le d} (q_o^j - 1)}{\prod_{0 \le i \le d-1} \prod_{1 \le j \le f_i} (q_o^j - 1)}.$$

If we denote by $h(A)$ the class number of A, then [3, (1)] specializes to

$$\mathrm{Mass}(\mathbf{f}) = h(A) \cdot q^{(d^2-1)(g_X-1)} \cdot \mathcal{T}^o \cdot \mathcal{T}_o \cdot \prod_{i=2}^{d} \zeta_X(i), \tag{5.4}$$

where g_X is the genus of X. From this we get our desired explicit estimate:

$$\mathrm{Mass}(\mathbf{f}) \le \#(\mathfrak{X}_{\mathbf{f}}/\mathbb{Z}) \le \frac{q^d - 1}{q - 1} \cdot \mathrm{Mass}(\mathbf{f}).$$

We end this section with a geometric application of previous results. Let $d = 2$, and fix a closed finite subscheme $\mathfrak{n} \neq \emptyset$ of $X - \infty - o$. Denote by $\mathcal{E}\ell\ell_{\mathcal{D},\mathfrak{n}}$ the modular curve of $\mathcal{D}$-elliptic sheaves which are special at o in the sense of [11], equipped with level-$\mathfrak{n}$ structures, modulo the action of $\mathbb{Z}$.

Remark 5.8 The definition of $\mathcal{D}$-elliptic sheaves in [11] includes a "normalization" condition which requires the Euler-Poincaré characteristic of $\mathcal{E}_0$ to be in the interval $[0, d)$. The resulting category is equivalent to the quotient of the category of $\mathcal{D}$-elliptic sheaves by the action of $\mathbb{Z}$ as is done in [13].

According to Theorems 6.4 and 8.1 in [11], $\mathcal{E}\ell\ell_{\mathcal{D},\mathfrak{n}}$ is a fine moduli scheme which is projective of relative dimension 1 over $X' = (X - \mathfrak{n} - \infty - \mathrm{Ram}) \cup \{o\}$. It is smooth over X' except at o. The fibre of $\mathcal{E}\ell\ell_{\mathcal{D},\mathfrak{n}}$ over o is a reduced singular curve whose only singular points are ordinary double points, and whose normalization is a disjoint union of finitely many rational curves.

Proposition 5.9 *The singular points of $\mathcal{E}\ell\ell_{\mathcal{D},\mathfrak{n}} \times_{X'} \mathrm{Spec}(\overline{\mathbb{F}}_o)$ are represented by the isomorphisms classes of pairs $(\mathbb{E}, \theta_\mathfrak{n})$, where $\mathbb{E}$ is a superspecial $\mathcal{D}$-elliptic sheaf of characteristic o over k, and $\theta_\mathfrak{n}$ is a level-$\mathfrak{n}$ structure on $\mathbb{E}$.*

Proof Denote $Y = \mathcal{E}\ell\ell_{\mathcal{D},\mathfrak{n}} \times_{X'} \mathrm{Spec}(\overline{\mathbb{F}}_o)$. As follows from [10, Prop. 4.1.2] and [11, Prop. 2.16], a $\mathcal{D}$-elliptic sheaf of characteristic o over k is special in the sense of Definition 5.1 if and only if it is special in the sense of [11, Def. 3.5]. Hence Y classifies the pairs $(\mathbb{E}, \theta_\mathfrak{n})$, where $\mathbb{E}$ is a special $\mathcal{D}$-elliptic sheaf over k in the sense of Definition 5.1, and $\theta_\mathfrak{n}$ is a level-$\mathfrak{n}$ structure on $\mathbb{E}$.

Let $\mathcal{T} := \widehat{\Omega}^2 \otimes_{\mathbb{F}_o} k$, where $\widehat{\Omega}^2$ is the formal scheme over $\mathrm{Spf}(\mathcal{O}_o)$ corresponding to Drinfeld's upper-half plane. By a theorem of Drinfeld [5], $\mathcal{T}$ parametrizes special Dieudonné $\mathcal{D}$-modules over k ("special" in the sense of Definition 3.6) equipped with some extra data. Proposition II.2.7.1 and the main result of Chapter III in [10] imply that the singular points of $\mathcal{T}$ correspond exactly to the superspecial Dieudonné $\mathcal{D}$-modules.

Finally, Hausberger's uniformization theorem [11, Thm. 8.1] relates $\mathcal{T}$ and Y as functors. This theorem, combined with the previous two paragraphs, implies that a closed point on Y corresponding to $(\mathbb{E}, \theta_\mathfrak{n})$ is singular if and only if $\mathbb{E}$ is superspecial. □

Let $\mathcal{D}_\mathfrak{n} = \mathcal{D} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{N})$, where $\mathcal{N}$ is the ideal sheaf of $\mathfrak{n}$. $\mathbb{F}_q^\times$ embeds diagonally into $\mathcal{D}_n^\times$. Denote $d(\mathfrak{n}) = \#(\mathcal{D}_\mathfrak{n}^\times/\mathbb{F}_q^\times)$.

Corollary 5.10 *The number of singular points on $\mathcal{E}\ell\ell_{\mathcal{D},\mathfrak{n}} \times_{X'} \mathrm{Spec}(\overline{\mathbb{F}}_o)$ is equal to*

$$d(\mathfrak{n}) \cdot h(A) \cdot q^{3(g_X-1)} \cdot \zeta_X(2) \cdot (q_o+1) \cdot \prod_{\substack{x \in \mathrm{Ram}\cup\infty \\ x \neq o}} (q_x - 1).$$

Proof A superspecial $\mathcal{D}$-elliptic sheaf over k is the same thing as an exceptional $\mathcal{D}$-elliptic sheaf of type $\mathbf{f} = (1,1)$ (we assume $d = 2$). Fix such a $\mathcal{D}$-elliptic sheaf $\mathbb{E}$ and let $w = \#\mathrm{Aut}(\mathbb{E})$. The number of all level-$\mathfrak{n}$ structures on $\mathbb{E}$, up to an isomorphism, is equal to $d(\mathfrak{n})\frac{q-1}{w}$. This combined with Proposition 5.9 implies that the number of singular points on $\mathcal{E}\ell\ell_{\mathcal{D},\mathfrak{n}} \times_{X'} \mathrm{Spec}(\overline{\mathbb{F}}_o)$ is equal to $d(\mathfrak{n}) \cdot \mathrm{Mass}(1,1)$. Now the corollary follows from (5.4). □

6 Supersingular $\mathcal{D}$-elliptic sheaves

We keep the notation and assumptions of Sect. 4. In this section, we assume $o \notin \mathrm{Ram}$, i.e., $D_o \cong \mathbb{M}_d(F_o)$.

Definition 6.1 $\mathbb{E} \in \mathbf{DES}$ is *supersingular* if for all large enough integers n

$$\varphi_o^n(M_o) \subset \pi_o M_o.$$

Let $\tilde{D}$ be the central division algebra over F with invariants

$$\mathrm{inv}_x(\tilde{D}) = \begin{cases} 1/d, & x = \infty; \\ -1/d, & x = o; \\ \mathrm{inv}_x(D), & x \neq o, \infty. \end{cases} \tag{6.1}$$

Theorem 6.2 *If $\mathbb{E}$ is a supersingular $\mathcal{D}$-elliptic sheaf of characteristic o over k, then* $\mathrm{End}(\mathbb{E})$ *is a maximal A-order in $\tilde{D}$. There is a bijection between the set of isomorphism classes of supersingular $\mathcal{D}$-elliptic sheaves over k modulo the action of $\mathbb{Z}$ and the isomorphism classes of locally free rank-*1 *right* $\mathrm{End}(\mathbb{E})$*-modules.*

Proof There is a canonical splitting $M_o = M_o'^{\,d}$, where (M_o', φ_o') is a Dieudonné $\mathcal{O}_o$-module over k satisfying $\dim_k(M_o'/\varphi_o'(M_o')) = 1$; see [13, Lem. 9.8]. Moreover, the action of $\mathcal{D}_o$ becomes the natural right action of $\mathbb{M}_d(\mathcal{O}_o)$ on $M_o'^{\,d}$. The supersingularity assumption implies that $(\varphi_o')^n(M_o') \subset \pi_o M_o'$ for all large enough integers n. Using Proposition 3.3, we conclude that $\mathrm{End}_{D_o}(N_o, \varphi_o) \cong \tilde{D}_o$. This implies $\mathrm{End}(\mathbb{E}) \otimes_A F \cong \tilde{D}$; see Proposition 9.9 and Corollary 9.10 in [13]. Now, as in the proof of Theorem 5.3, to show that $\mathrm{End}(\mathbb{E})$ is a maximal A-order in $\tilde{D}$, it is enough to show that $\mathrm{End}_{\mathcal{D}_x}(M_x, \varphi_x)$ is a maximal order in $\mathrm{End}_{D_x}(N_x, \varphi_x)$ for all $x \in |X| - \infty$. For $x \neq o$ the proof is the same as in Theorem 5.3, and for $x = o$ this follows from Proposition 3.3.

The second statement of the theorem follows from the same argument as in the proof of Theorem 5.4. □

Suppose $D = \mathbb{M}_d(F)$ and $\mathcal{D} = \mathbb{M}_d(\mathcal{O}_X)$. Morita equivalence establishes an equivalence between $\mathbf{DES}/\mathbb{Z}$ and the category of normalized rank-d elliptic sheaves over k with pole at ∞; cf. [2, 3.1.4]. On the other hand, by a theorem of Drinfeld [6] this latter category is anti-equivalent to the category of rank-d Drinfeld A-modules over k. Hence $\mathbf{DES}/\mathbb{Z}$ is anti-equivalent to the category of rank-d Drinfeld A-modules over k. Under this anti-equivalence, supersingular $\mathcal{D}$-elliptic sheaves correspond to supersingular Drinfeld modules (see [9] for the definition of supersingular Drinfeld modules). Indeed, the supersingular $\mathcal{D}$-elliptic sheaves and Drinfeld modules are uniquely characterized by the fact that the center of their endomorphism algebra is F. One concludes that in this case Theorem 6.2 specializes to [9, Thm. 4.3] (note that due to the anti-equivalence the endomorphism algebra of a supersingular Drinfeld module is $\tilde{D}^{\mathrm{opp}}$, not $\tilde{D}$).

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