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RESEARCH

Drinfeld–Stuhler modules

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Abstract

We study $\mathscr{D}$-elliptic sheaves in terms of their associated modules, which we call Drinfeld–Stuhler modules. First, we prove some basic results about Drinfeld–Stuhler modules and give explicit examples. Then we examine the existence and properties of Drinfeld–Stuhler modules with large endomorphism rings, which are analogous to CM and supersingular Drinfeld modules. Finally, we examine the fields of moduli of Drinfeld–Stuhler modules.

Keywords: Drinfeld modules, $\mathscr{D}$-elliptic sheaves, Central simple algebras, Fields of moduli

Mathematics Subject Classification: 11G09, 11R52

1 Introduction

The idea of $\mathscr{D}$-elliptic sheaves was proposed by Ulrich Stuhler, as a natural generalization of Drinfeld's elliptic sheaves [4,10]. The moduli varieties of $\mathscr{D}$-elliptic sheaves were studied by Laumon, Rapoport, and Stuhler in [22], with the aim of proving the local Langlands correspondence for GL_d in positive characteristic. In this paper, we study some of the arithmetic properties of $\mathscr{D}$-elliptic sheaves, and in particular their endomorphism rings and fields of moduli.

Let C be a smooth, projective, geometrically connected curve over the finite field $\mathbb{F}_q$. Let F be the function field of C. Let $\infty \in C$ be a fixed closed point, and let $A \subset F$ be the ring of functions regular outside ∞. Denote by F_∞ the completion of F at ∞ and by $\mathbb{C}_\infty$ the completion of an algebraic closure of F_∞. Let D be a central simple algebra over F of dimension d^2, which is split at ∞, i.e., $D \otimes_F F_\infty$ is isomorphic to the matrix algebra $M_d(F_\infty)$. Fix a maximal A-order O_D in D. An *A-field* is a field L equipped with an A-algebra structure, i.e., with a homomorphism $\gamma : A \to L$. A $\mathscr{D}$-elliptic sheaf over an A-field L is essentially a vector bundle of rank d^2 on $C \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(L)$ equipped with an action of O_D and with a meromorphic O_D-linear Frobenius satisfying certain conditions (see Sect. 3). One can think of these objects as being analogous to abelian varieties equipped with an action of an order in a central simple algebra over $\mathbb{Q}$.

In this paper, we study $\mathscr{D}$-elliptic sheaves in terms of their associated modules, which we call *Drinfeld–Stuhler modules*. The relationship between $\mathscr{D}$-elliptic sheaves and Drinfeld–Stuhler modules is similar to the relationship between elliptic sheaves and Drinfeld modules; cf. [4,10]. Let L be an A-field and $L[\tau]$ be the skew polynomial ring with the commutation relation $\tau b = b^q \tau$, $b \in L$. A Drinfeld–Stuhler O_D-module over L is an embedding

$$\phi : O_D \longrightarrow M_d(L[\tau])$$

satisfying certain conditions; see Sect. 2.2. This concept implicitly appears in [22, Sect. 3], although it does not play an important role in that paper since its "shtuka" incarnation (the $\mathscr{D}$-elliptic sheaf) seems better suited for the study of moduli spaces. The advantage of the concept of Drinfeld–Stuhler module is that it is relatively elementary and one can easily write down explicit examples of these objects. We expect that the reader familiar with the theory of Drinfeld modules, but not necessarily with [22], will find it easier to understand the results of this paper in terms of Drinfeld–Stuhler modules, rather than $\mathscr{D}$-elliptic sheaves.

The theory of Drinfeld modules can be seen as a special case of the theory of Drinfeld–Stuhler modules for $D \cong M_d(F)$; cf. Sect. 2.4. In the other direction, some of the properties of general Drinfeld–Stuhler modules are similar to, and in fact can be deduced from, the properties of Drinfeld modules, e.g., uniformizability and CM theory. There are also some notable differences. The most significant is probably the fact that, when D is a division algebra, the modular varieties of Drinfeld–Stuhler modules are projective [22], unlike the Drinfeld modular varieties, which are affine [8]. Another difference is that Drinfeld–Stuhler modules can be defined only over fields which split D (cf. Lemma 2.5), so for $D \ncong M_d(F)$ there are no Drinfeld–Stuhler modules over F itself, even in the simplest case when $A = \mathbb{F}_q[T]$.

The main results of this paper concern the endomorphism rings of Drinfeld–Stuhler modules and their fields of moduli. The outline of the paper is the following:

In Sect. 2, we introduce the concept of Drinfeld–Stuhler O_D-modules and prove some of its basic properties. Moreover, we give several explicit examples, which will be revisited throughout the paper. Finally, we explain the so-called Morita equivalence in the context of Drinfeld–Stuhler modules, which gives an equivalence between the categories of Drinfeld A-modules of rank d and Drinfeld–Stuhler $M_d(A)$-modules.

In Sect. 3, we recall three categories equivalent to the category of Drinfeld–Stuhler modules: O_D-motives, $\mathscr{D}$-elliptic sheaves, and O_D-lattices. The tools provided by these alternative points of view on Drinfeld–Stuhler modules are crucial for the proofs of the main results of the paper.

In Sect. 4, given a Drinfeld–Stuhler O_D-module ϕ over L, we prove that the endomorphism ring $\mathrm{End}_L(\phi)$(= the centralizer of $\phi(O_D)$ in $M_d(L[\tau])$) is a projective A-module of rank $\leq d^2$ such that $\mathrm{End}_L(\phi) \otimes_A F_\infty$ is isomorphic to a subalgebra of the central division algebra over F_∞ with invariant $-1/d$. Moreover, if $\gamma : A \to L$ is injective, then $\mathrm{End}_L(\phi)$ is an A-order in an *imaginary* field extension K of F which embeds into D, so, in particular, $\mathrm{End}_L(\phi)$ is commutative and its rank over A divides d. ("Imaginary" in this context means that there is a unique place ∞' of K over ∞.) Next, we study Drinfeld–Stuhler modules over $\mathbb{C}_\infty$ with large endomorphism rings, namely the analogue of "complex multiplication". The results here are similar to those for Drinfeld modules; cf. [11,17]. We prove that if K is an imaginary field extension of F of degree d which embeds into D and O_K is the integral closure of A in K, then, up to isomorphism, the number of Drinfeld–Stuhler O_D-modules over $\overline{F}$ with $\mathrm{End}_{\overline{F}}(\phi) = O_K$ is finite and nonzero, and any such module can be defined over the Hilbert class field of K(= the maximal unramified abelian extension of K in which ∞' totally splits). We also compute the number of isomorphism classes of Drinfeld–Stuhler modules over $\mathbb{C}_\infty$ with the largest possible automorphism group $\mathbb{F}_{q^d}^\times$, assuming $A = \mathbb{F}_q[T]$.

In Sect. 5, we fix a maximal ideal $\mathfrak{p} \lhd A$ and study Drinfeld–Stuhler O_D-modules over the algebraic closure of $A/\mathfrak{p}$ with large endomorphism rings, namely the analogue of "supersingularity". It turns out that the cases when D is ramified/unramified at $\mathfrak{p}$ have to be studied separately, but in either case the endomorphism ring of a supersingular Drinfeld–Stuhler module is essentially a maximal A-order in the central division algebra over F of dimension d^2 whose invariants are closely related to the invariants of D. These results are again similar to those for Drinfeld modules; cf. [12]. This potentially opens up the way to use Drinfeld–Stuhler modules in the arithmetic of division algebras over function fields.

In Sect. 6, we prove a Hilbert's 90th-type theorem for $M_d(L^{\mathrm{sep}}[\tau])$ and use this theorem to give conditions under which a field of moduli for a Drinfeld–Stuhler module is a field of definition. In particular, we prove that a field of moduli is a field of definition if and only if it splits D. We also prove that if d and $q^d - 1$ are coprime, then a field of moduli is always a field of definition. These results have applications to the existence/nonexistence of rational points on the coarse moduli scheme of Drinfeld–Stuhler modules. An interesting application of this is the construction of concrete examples of varieties over function fields violating the Hasse principle; we will discuss this application in a future publication.

2 Basic properties and examples

In this section, after introducing the notation and terminology that will be used throughout the paper, we define the key concept of Drinfeld–Stuhler module. We then examine the basic properties of these objects and give explicit examples.

2.1 Notation and terminology

Let F be the field of rational functions on a smooth and geometrically irreducible projective curve C defined over the finite field $\mathbb{F}_q$ of q elements, where q is a power of a prime number. Fix a place ∞ of F (equiv. a closed point of C), and let A be the subring of F consisting of functions which are regular away from ∞. A is a Dedekind domain.

An *imaginary* field extension of F is an extension K/F in which ∞ does not split, i.e., there is a unique place ∞' of K over ∞. For a field L we denote by L^{alg} (resp. L^{sep}) its algebraic (resp. separable) closure.

For a place v of F, we denote by F_v, O_v, and $\mathbb{F}_v$ the completion of F at v, the ring of integers in F_v, and the residue field at v, respectively. If $v \neq \infty$, so corresponds to a nonzero prime ideal $\mathfrak{p}$ of A, we sometimes write $A_\mathfrak{p}$ or A_v instead of O_v, and $\mathbb{F}_\mathfrak{p}$ instead of $\mathbb{F}_v$. The maximal ideal of O_v will be denoted $\mathfrak{p}$ if $v = \mathfrak{p}$ is finite, and $\mathfrak{p}_\infty$ if $v = \infty$.

Given a unitary ring R, we denote by $R^\times$ the group of multiplicative units in R. Let $M_d(R)$ be the ring of $d \times d$ matrices with entries in R; the group of units in $M_d(R)$ is denoted by $\mathrm{GL}_d(R)$. Given $r_1, \dots, r_d \in R$, we denote by $\mathrm{diag}(r_1, \dots, r_d) \in M_d(R)$ the matrix which has r_i as the (i, i)th entry, $1 \leq i \leq d$, and zeros everywhere else.

Let D be a central simple algebra over F of dimension d^2. Let $\mathrm{Ram}(D)$ be the set of places of F which ramify in D, i.e., $v \in \mathrm{Ram}(D)$ if and only if $D_v := D \otimes_F F_v$ is not isomorphic to $M_d(F_v)$. From now on we assume that $\infty \notin \mathrm{Ram}(D)$, so that the places in $\mathrm{Ram}(D)$ correspond to prime ideals of A. We denote

$$\mathfrak{r}(D) = \prod_{\mathfrak{p} \in \mathrm{Ram}(D)} \mathfrak{p}.$$

An empty product is assumed to be 1, so $\mathfrak{r}(M_d(F)) = A$. We fix a maximal A-order O_D in D; see [27] for the definitions. Note that A is the center of O_D.

Let L be an A-field, i.e., a field equipped with an A-algebra structure $\gamma : A \to L$. The *A-characteristic of* L is the prime ideal $\mathrm{char}_A(L) := \ker(\gamma) \lhd A$; we say that L has *generic A-characteristic* if $\ker(\gamma) = 0$. Note that $\mathbb{F}_q$ is a subfield of L. Let τ be the $\mathbb{F}_q$-linear Frobenius endomorphism of the additive group scheme $\mathbb{G}_{a,L} = \mathrm{Spec}(L[x])$ over L; the morphism τ is given on the underlying ring by $x \mapsto x^q$. The ring of $\mathbb{F}_q$-linear endomorphisms $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L})$ is canonically isomorphic to the skew polynomial ring $L[\tau]$ with the commutation relation $\tau b = b^q \tau$, $b \in L$. There is also a canonical isomorphism (cf. [33, Prop. 1.1])

$$\mathrm{End}_{\mathbb{F}_q}\left(\mathbb{G}_{a,L}^d\right) \cong M_d\left(L[\tau]\right).$$

One can write the elements of $M_d(L[\tau])$ as finite sums $\sum_{i\geq 0} B_i\tau^i$, where $B_i \in M_d(L)$ and, by slight abuse of notation, τ^i denotes $\mathrm{diag}(\tau^i, \dots, \tau^i)$. An element $S = \sum_{i\geq 0} B_i\tau^i \in M_d(L[\tau])$ acts on the tangent space $\mathrm{Lie}(\mathbb{G}_{a,L}^d) \cong L^d$ via $\partial(S) := B_0$. The map

$$\partial : M_d(L[\tau]) \longrightarrow M_d(L), \quad S \longmapsto \partial(S)$$

is a surjective homomorphism.

2.2 Definitions and basic properties

Definition 2.1 A *Drinfeld–Stuhler O_D-module* defined over L is an embedding

$$\begin{aligned} \phi : O_D &\longrightarrow M_d(L[\tau]) \\ b &\longmapsto \phi_b \end{aligned}$$

satisfying the following conditions:

(i) For any $b \in O_D \cap D^\times$ the endomorphism ϕ_b of $\mathbb{G}_{a,L}^d$ is surjective with kernel $\phi[b] := \ker \phi_b$ a finite group scheme over L of order $\#(O_D/O_D \cdot b)$.

(ii) The composition

$$A \longrightarrow O_D \xrightarrow{\phi} M_d(L[\tau]) \xrightarrow{\partial} M_d(L)$$

maps $a \in A$ to $\mathrm{diag}(\gamma(a), \dots, \gamma(a))$, where the map $A \to O_D$ identifies A with the center of O_D.

Remark 2.2 A homomorphism $f : \mathbb{G}_{a,L}^d \to \mathbb{G}_{a,L}^d$ is surjective if and only if $\ker f$ is finite; cf. [15, Prop. 5.2]. In particular, (i) in Definition 2.1 can be simplified to the assumption that $\#\phi[b] = \#(O_D/O_D \cdot b)$ for $b \in O_D \cap D^\times$.

Remark 2.3 Drinfeld–Stuhler modules are a special case of abelian Anderson A-modules, as follows from [1, Sect. 1] and the discussion in Sect. 3. If $d = 1$, so that $D = F$, then the definition of Drinfeld–Stuhler O_D-modules becomes the definition of Drinfeld A-modules of rank 1. One can introduce a notion of rank for Drinfeld–Stuhler modules so that Definition 2.1 corresponds to the case of rank 1. Lafforgue studied the $\mathscr{D}$-shtukas of arbitrary rank and their modular varieties in [20].

Remark 2.4 Let $b \in O_D \cap D^\times$. If we consider D as a vector space over F, then the left multiplication by b induces a linear transformation. Let $\det(b)$ denote the determinant of this linear transformation. Note that O_D is an A-lattice in D in the sense of [29, Ch. III, Sect. 1]. By Proposition 3 in [29, Ch. III, Sect. 1], we have $\#(O_D/O_Db) = \#(A/\det(b)A)$. Finally, recall that $(-1)^d\det(b) =: \mathrm{Nr}(b)$ is the *non-reduced norm* of b; cf. [27, Sect. 9a]. Hence, condition (i) is equivalent to saying that $\phi[b]$ is a finite group scheme of order $\#(A/\mathrm{Nr}(b)A)$.

The action of $\phi(O_D)$ on the tangent space $\mathrm{Lie}(\mathbb{G}_{a,L}^d)$ gives a homomorphism

$$\partial_\phi : O_D \longrightarrow M_d(L),$$

which extends linearly to a homomorphism

$$\partial_{\phi,L} : O_D \otimes_A L \longrightarrow M_d(L).$$

Lemma 2.5 *If* $\mathrm{char}_A(L)$ *does not divide* $\mathfrak{r}(D)$*, then* $\partial_{\phi,L}$ *is an isomorphism.*

Proof Both sides are rings with 1, so $\partial_{\phi,L}$ is nonzero, as it maps 1 to 1. If L has generic A-characteristic, then L is an extension of F; hence, $O_D \otimes_A L$ is a central simple algebra over L. Therefore, $\partial_{\phi,L}$ is injective, and comparing the dimensions, we see that it is in fact an isomorphism. Now assume that $\mathrm{char}_A(L) = \mathfrak{p} \neq 0$. Then $O_D \otimes_A L$ is obtained by extension of scalars from $O_D \otimes_A A_\mathfrak{p} \to O_D \otimes_A \mathbb{F}_\mathfrak{p}$. On the other hand, $O_D \otimes_A A_\mathfrak{p} \cong M_d(A_\mathfrak{p})$ since $\mathfrak{p} \nmid \mathfrak{r}(D)$. Now it is clear that $O_D \otimes_A \mathbb{F}_\mathfrak{p} \cong M_d(\mathbb{F}_\mathfrak{p})$; hence, $O_D \otimes_A L \cong M_d(\mathbb{F}_\mathfrak{p}) \otimes_{\mathbb{F}_\mathfrak{p}} L \cong M_d(L)$. Since $M_d(L)$ is a central simple algebra over L, the previous argument again implies that $\partial_{\phi,L}$ is an isomorphism. □

Remark 2.6 If $\mathrm{char}_A(L)$ divides $\mathfrak{r}(D)$, then $\partial_{\phi,L}$ is not an isomorphism, since $O_D \otimes_A L$ is not isomorphic to $M_d(L)$; cf. Sect. 5.2.

Remark 2.7 We recall some necessary and sufficient conditions for a finite field extension L of F to *split* D, i.e., $D \otimes_F L \cong M_d(L)$, since, by Lemma 2.5, if there is a Drinfeld–Stuhler O_D-module defined over L, then L necessarily splits D. (In particular, if $D \not\cong M_d(F)$, then a Drinfeld–Stuhler module cannot be defined over F itself.) Let $\mathfrak{p} \lhd A$. The Wedderburn structure theorem says that $D \otimes_F F_\mathfrak{p} \cong M_{\kappa_\mathfrak{p}}(D'_\mathfrak{p})$, where $D'_\mathfrak{p}$ is a central division algebra of dimension $d_\mathfrak{p}^2 = (d/\kappa_\mathfrak{p})^2$. The integer $d_\mathfrak{p}$ is called the *local index* of D at $\mathfrak{p}$; cf. [27, p. 272]. By [27, (32.15)], L splits D if and only if for each prime $\mathfrak{p} \lhd A$ and for all primes $\mathfrak{P}$ of L lying above $\mathfrak{p}$, $d_\mathfrak{p}$ divides $[L_\mathfrak{P} : K_\mathfrak{p}]$. If D is a division algebra and $[L : F] = d$, then L splits D if and only if L embeds into D; moreover, every maximal subfield L of D contains F and $[L : F] = d$; see [27, (7.15)].

Definition 2.8 Let ϕ, ψ be Drinfeld–Stuhler O_D-modules over L. A *morphism* $u : \phi \to \psi$ over L is $u \in M_d(L[\tau])$ such that $u\phi_b = \psi_b u$ for all $b \in O_D$. We say that u is an *isomorphism* if u is invertible in the ring $M_d(L[\tau])$. We say that u is an *isogeny* if $\ker(u)$ is a finite group scheme over L; an isogeny u is *separable* if $\ker(u)$ is étale. Note that ϕ_b, $b \in O_D \cap D^\times$, defines an isogeny $\phi \to \phi$. The set of morphisms $\phi \to \psi$ over L is an A-module $\mathrm{Hom}_L(\phi, \psi)$, where A acts by $a \circ u := u\phi_a$. (Using the fact that $a \in A$ is in the center of O_D, it is easy to check that $u\phi_a \in \mathrm{Hom}_L(\phi, \psi)$.) We denote $\mathrm{End}_L(\phi) = \mathrm{Hom}_L(\phi, \phi)$; this is a subring of $M_d(L[\tau])$. For an arbitrary field extension $\mathcal{L}$ of L we can consider ϕ, ψ as Drinfeld–Stuhler O_D-modules over $\mathcal{L}$, so we have the corresponding module $\mathrm{Hom}_\mathcal{L}(\phi, \psi)$ of morphisms over $\mathcal{L}$. We will denote $\mathrm{Hom}(\phi, \psi) = \mathrm{Hom}_{L^{\mathrm{alg}}}(\phi, \psi)$ and $\mathrm{End}(\phi) = \mathrm{End}_{L^{\mathrm{alg}}}(\phi)$.

A Drinfeld–Stuhler O_D-module ϕ over L *can be defined* over a subfield K of L (equiv. K is a *field of definition for* ϕ) if there is a Drinfeld–Stuhler O_D-module ψ over K which is isomorphic to ϕ over L.

Lemma 2.9 *Let ϕ be a Drinfeld–Stuhler O_D-module over L and $b \in O_D \cap D^\times$. Then ϕ_b is separable if and only if* $\mathrm{char}_A(L)$ *does not divide* $\mathrm{Nr}(b)$.

Proof This follows from Proposition 3.10 in [22] (see also Corollary 5.11 in [15]). □

Lemma 2.10 *For a nonzero ideal $\mathfrak{n} \lhd A$ and a Drinfeld–Stuhler O_D-module ϕ over L we define*

$$\phi[\mathfrak{n}] = \bigcap_{a\in\mathfrak{n}} \phi[a],$$

where the intersection is the scheme-theoretic intersection of subgroup schemes of $\mathbb{G}_{a,L}^d$. Then $\phi[\mathfrak{n}]$ is invariant under $\phi(O_D)$. Moreover, if $\mathrm{char}_A(L)$ *does not divide* $\mathfrak{n}$, *then*

$$\phi[\mathfrak{n}](L^{\mathrm{sep}}) \cong O_D/O_D\mathfrak{n}$$

as left O_D-modules.

Proof Since $a \in A$ is in the center of O_D, it is clear that each $\phi[a]$, and thus also $\phi[\mathfrak{n}]$, is $\phi(O_D)$-invariant. The second claim essentially follows from theorem 6.4 in [15]. More precisely, in the terminology of Sect. 3, let $M(\phi)$ be the O_D-motive associated with ϕ. By [15, Thm. 6.4], $\phi[\mathfrak{n}]$ is dual to $M(\phi)/\mathfrak{n}M(\phi)$. This isomorphism is compatible with the action of O_D. On the other hand, $M(\phi)$ is a locally free left $O_D^{\mathrm{opp}} \otimes_{\mathbb{F}_q} L$-module of rank 1. Hence, $M(\phi)/\mathfrak{n}M(\phi) \cong O_D^{\mathrm{opp}}/\mathfrak{n}O_D^{\mathrm{opp}}$ as left O_D^{opp}-modules. □

Remark 2.11 In general, $\phi[b]$ is not necessarily O_D-invariant for $b \in O_D \cap D^\times$, so condition (i) in Definition 2.1 cannot be stated in the stronger form of isomorphism of left O_D-modules $\phi[b] \cong O_D/O_D \cdot b$; see Remark 2.17.

Lemma 2.12 *Let ϕ and ψ be Drinfeld–Stuhler O_D-modules over L. Assume L has generic A-characteristic. Then:*

1. *The map* $\partial : \mathrm{Hom}_L(\phi, \psi) \to M_d(L)$ *is injective.*
2. $\mathrm{End}_L(\phi)$ *is a commutative ring.*

Proof Suppose $u \in \mathrm{Hom}_L(\phi, \psi)$ is nonzero, but $\partial(u) = 0$. Then $u = B_m\tau^m + B_{m+1}\tau^{m+1} + \cdots$, where $m \geq 1$ is the smallest index such that $B_m \neq 0$. For $a \in A$, the equality $u\phi_a = \psi_a u$ leads to $B_m\gamma(a)^{q^m} = \gamma(a)B_m$. Since $B_m \in M_d(L)$ has at least one nonzero entry, we must have $\gamma(a)^{q^m} = \gamma(a)$. Since a was arbitrary, this implies $\gamma(A) \subseteq \mathbb{F}_{q^m}$. On the other hand, since L has generic A-characteristic, $\gamma(A)$ is infinite, which leads to a contradiction.

By the first claim, ∂ maps $\mathrm{End}_L(\phi)$ isomorphically to its image in $M_d(L)$. On the other hand, $\partial(\mathrm{End}_L(\phi))$ is in the centralizer of $\partial_\phi(O_D)$. By Lemma 2.5, $\partial_\phi(O_D)$ contains a basis of $M_d(L)$, so $\partial(\mathrm{End}_L(\phi))$ is in the center of $M_d(L)$, which consists of scalar matrices. Hence, ∂ identifies $\mathrm{End}_L(\phi)$ with an A-subalgebra of L. □

Lemma 2.13 *Let ϕ and ψ be Drinfeld–Stuhler O_D-modules over L. If $u \in \mathrm{Hom}_L(\phi, \psi)$ is nonzero, then u is an isogeny.*

Proof We will prove the lemma assuming $\mathrm{char}_A(L) \nmid \mathfrak{r}(D)$. At the end of Sect. 2.4 we give a different proof, which avoids this assumption.

Without loss of generality, we can assume that L is algebraically closed. Suppose $u \in \mathrm{Hom}(\phi, \psi)$ is nonzero and has infinite kernel. Since $\ker(u) \subset \mathbb{G}_{a,L}^d$ is an algebraic subgroup with infinitely many geometric points, the connected component $\ker(u)^0$ of the identity has positive dimension. We can decompose $u = u_0\tau^s$ for some $s \geq 0$, so that $\partial(u_0) \neq 0$. Note that ∂_{u_0} is not invertible since it acts as 0 on the tangent space of $\ker(u)^0$. Thus, $0 \subsetneq \ker(\partial_{u_0}) \subsetneq L^d$. Denote by $\partial_{\phi,L}^{q^s}$ the composition of $\partial_{\phi,L}$ and $\tau^s : M_d(L) \to M_d(L)$, which raises the entries of a matrix to q^sth powers. By Lemma 2.5, since L is algebraically closed, we have $\partial_{\phi,L}^{q^s}(O_D \otimes L) = M_d(L)$. On the other hand, $\partial_{u_0}\partial_{\phi,L}^{q^s}(b) = \partial_{\psi,L}(b)\partial_{u_0}$ for all $b \in O_D$, which comes from $u\phi_b = \psi_b u$. This implies that the subspace $\ker(\partial_{u_0})$ of L^d is invariant under $M_d(L)$, which leads to a contradiction. □

Lemma 2.14 *Let ϕ and ψ be Drinfeld–Stuhler O_D-modules over L. If $u : \phi \to \psi$ is an isogeny, then there is an element $0 \neq a \in A$ and an isogeny $w : \psi \to \phi$ such that $wu = \phi_a$ and $uw = \psi_a$.*

Proof This follows from Corollary 5.15 in [15]. We remark that an isogeny $u : \phi \to \psi$ between abelian Anderson A-modules is defined in [15] with an extra assumption that u, as an endomorphism of $\mathbb{G}_{a,L}^d$, is surjective. On the other hand, the surjectivity of u follows from the finiteness of $\ker(u)$ (see [15, Prop. 5.2]), so Hartl's definition is equivalent to Definition 2.8. □

Lemmas 2.9 and 2.14 imply that any isogeny $u : \phi \to \psi$ between Drinfeld–Stuhler O_D-modules over a field L of generic A-characteristic is separable. In fact, this is true more generally for isogenies between abelian Anderson A-modules over L; see [15, Cor. 5.17].

Lemmas 2.13 and 2.14 imply that $\mathrm{End}_L(\phi) \otimes_A F$ is a division algebra over F. In particular, if L has generic A-characteristic, then $\mathrm{End}_L(\phi) \otimes_A F$ is a field, since it is commutative by Lemma 2.12 (see Theorem 4.1 for a more precise statement).

2.3 Examples

As a consequence of the Grunwald–Wang theorem, every central simple F-algebra is cyclic; see [27, (32.20)]. This means that there is a Galois extension K/F with $\mathrm{Gal}(K/F) \cong \mathbb{Z}/d\mathbb{Z}$, a generator σ of $\mathrm{Gal}(K/F)$, and $f \in F^\times$ such that

$$D \cong (K/F, \sigma, f) = \bigoplus_{i=0}^{d-1} Kz^i, \qquad z^d = f, \qquad z \cdot y = \sigma(y)z \quad \text{for } y \in K, \tag{2.1}$$

where we identify z^0 with the identity element of D. Moreover, one can choose f to be in A; cf. [27, (30.4)].

Assume K/F is imaginary, and let O_K be the integral closure of A in K. Consider the A-order

$$O_D = \bigoplus_{i=0}^{d-1} O_K z^i \tag{2.2}$$

in D. This order is not necessarily maximal. It is not hard to compute that its discriminant is equal to $f^{d(d-1)}\mathrm{disc}(K/F)^d$; see [5, Cor. 7]. For an A-order in D to be maximal, it is necessary and sufficient for its discriminant to be equal to the discriminant of a maximal order. The discriminant of a maximal order in D can be computed from the invariants of D; see [27, (32.1)] and [5, Prop. 25]. For $\mathfrak{p} \in \mathrm{Ram}(D)$, let the reduced fraction $s_\mathfrak{p}/r_\mathfrak{p} \in \mathbb{Q}/\mathbb{Z}$ be the invariant of D at $\mathfrak{p}$. Set $r = \mathrm{lcm}(r_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ram}(D))$. Then a maximal order in D has discriminant $\left(\prod_{\mathfrak{p}\in\mathrm{Ram}(D)} \mathfrak{p}^{r-\frac{r}{r_\mathfrak{p}}}\right)^r$. For example, if d is prime, then the discriminant of a maximal order is equal to $\mathfrak{r}(D)^{d(d-1)}$. Comparing the discriminant of O_D with the discriminant of a maximal order gives an explicit criterion for the order O_D to be maximal; see [5, Cor. 26].

Example 2.15 Assume the order O_D in (2.2) is maximal (For the definition of Drinfeld modules we refer to [14, p. 69]). Let $\Phi : O_K \to L[\tau], b \mapsto \Phi_b$, be a Drinfeld O_K-module of rank 1 defined over some field L. Observe that the restriction of Φ to A defines a Drinfeld A-module of rank d over L. Let

$$\phi : O_D \longrightarrow M_d(L[\tau])$$

be defined as follows:

$$\phi_\alpha = \mathrm{diag}(\Phi_\alpha, \Phi_{\sigma\alpha}, \dots, \Phi_{\sigma^{d-1}\alpha}), \quad \alpha \in O_K,$$

$$\phi_z = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ \Phi_f & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Using the fact that $\Phi_\alpha \Phi_f = \Phi_f \Phi_\alpha$, it is easy to check that $\phi_z\phi_\alpha = \phi_{\sigma\alpha}\phi_z$ and $\phi_z^d = \phi_f$. Thus, ϕ is an embedding. Moreover, for $a \in A$, we have $\phi_a = \mathrm{diag}(\Phi_a, \dots, \Phi_a)$, which maps under ∂ to $\mathrm{diag}(\gamma(a), \dots, \gamma(a))$ by the definition of Drinfeld modules. Finally,

$$\#\phi[z] = \#\ker \Phi_f = \#(A/fA)^d = \#(A/f^d A) = \#(A/\mathrm{Nr}(z)A),$$

and

$$\#\phi[\alpha] = \#(O_K/O_K\alpha)^d = \#(A/\mathrm{Nr}(\alpha)A).$$

Thus, ϕ is a Drinfeld–Stuhler O_D-module.

Example 2.16 As a more explicit version of Example 2.15, let $A = \mathbb{F}_q[T]$ and $F = \mathbb{F}_q(T)$. Let $\mathbb{F}_{q^d}$ denote the degree d extension of $\mathbb{F}_q$. Let $K = \mathbb{F}_{q^d}(T)$, which is a cyclic imaginary extension as ∞ is inert in K. In this case, $O_K = \mathbb{F}_{q^d}[T]$ and the Galois group $\mathrm{Gal}(K/F) \cong \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ has a canonical generator σ given by the Frobenius automorphism (i.e., σ induces the qth power morphism on $\mathbb{F}_{q^d}$). Let $\mathfrak{r} \in A$ be a monic square-free polynomial with prime decomposition $\mathfrak{r} = \mathfrak{p}_1 \cdots \mathfrak{p}_m$. Assume the degree of each prime $\mathfrak{p}_i$ is coprime to d. Let D be the cyclic algebra $D = (K/F, \sigma, \mathfrak{r})$. Then, by [14, Thm. 4.12.4], for any prime $\mathfrak{p} \lhd A$ one has

$$\mathrm{inv}_\mathfrak{p}(D) = \frac{\mathrm{ord}_\mathfrak{p}(\mathfrak{r})\deg(\mathfrak{p})}{d} \in \mathbb{Q}/\mathbb{Z}. \tag{2.3}$$

Since the sum of the invariants of D over all places of F is 0, if we assume that $\sum_{i=1}^{m} \deg(\mathfrak{p}_i)$ is divisible by d, then D will be split at ∞ and will ramify only at the primes of A dividing $\mathfrak{r}$.

The order $O_D = \bigoplus_{i=0}^{d-1} O_K z^i$ is maximal in D, since its discriminant is equal to $\mathfrak{r}^{d(d-1)}$. Let L be an O_K-field and $\gamma : A \to O_K \to L$ be the composition homomorphism. Let $\Phi : O_K \to L[\tau]$ be defined by $\Phi_T = \gamma(T) + \tau^d$; this is a rank-1 Drinfeld O_K-module and a rank-d Drinfeld A-module. Then

$$\phi : O_D \longrightarrow M_d(L[\tau])$$

given by

$$\begin{aligned}
\phi_T &= \operatorname{diag}(\Phi_T, \dots, \Phi_T),\\
\phi_h &= \operatorname{diag}(h, h^q, \dots, h^{q^{d-1}}), \quad h \in \mathbb{F}_{q^d},\\
\phi_z &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1\\ \Phi_{\mathfrak{r}} & 0 & 0 & \cdots & 0 \end{pmatrix},
\end{aligned}$$

is a Drinfeld–Stuhler module.

Remark 2.17 It is easy to see from the previous example that for general $b \in O_D$ the kernel $\phi[b]$ is not necessarily O_D-invariant. Indeed, take $d = 2$ and $b = h + z$ with $h \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$. A nonzero element $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{G}_{a,K}^2(K^{\mathrm{alg}})$ is in $\phi[b]$ only if $h\alpha + \beta = 0$. On the other hand, $\phi_h \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} h\alpha \\ h^q\beta \end{pmatrix}$, so $\phi_h \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \phi[b]$ only if $h^2\alpha + h^q\beta = 0$. This implies $h^2\alpha = h^{q+1}\alpha$. Since $h^{q-1} \neq 1$, we must have $\alpha = 0$, but then $\beta = 0$.

Example 2.18 Let $D = M_d(F)$ and $O_D = M_d(A)$. Let $\Phi : A \to L[\tau]$ be a Drinfeld A-module over L of rank d. Define

$$\begin{aligned}
\phi : O_D &\longrightarrow M_d(L[\tau])\\
(a_{ij}) &\longmapsto \left(\Phi_{a_{ij}}\right).
\end{aligned}$$

It is easy to check that ϕ is an injective homomorphism using the fact that $\Phi : A \to L[\tau]$ is an injective homomorphism. That (ii) is satisfied follows from the definition of Drinfeld modules. The non-reduced norm on O_D in this case is simply the dth power of the determinant map, up to a sign. Condition (i) is easy to check for diagonal and unipotent matrices in $M_d(A)$. Since these matrices generate the semigroup of matrices in $M_d(A)$ with nonzero determinants, it follows that condition (i) holds. Hence, ϕ is a Drinfeld–Stuhler module.

2.4 Morita equivalence for Drinfeld–Stuhler modules

The main result of this subsection is the fact that any Drinfeld–Stuhler $M_d(A)$-module arises from some Drinfeld module of rank d via the construction of Example 2.18. This fact for $\mathscr{D}$-elliptic sheaves is mentioned in [22, p. 224].

Let R be an arbitrary unitary ring (not necessarily commutative). We denote by $e_{ij} \in M_d(R)$ the matrix which has 1 at the (i,j)th entry, and 0 everywhere else. We have the relations

$$e_{ij}e_{ks} = \begin{cases} e_{is} & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.19 *Let R be a unitary ring for which every left ideal is principal. Let $\phi : M_d(A) \to M_d(R)$ be an injective homomorphism. Then, up to conjugation by an element of $\mathrm{GL}_d(R)$, we have $\phi(e_{ij}) = e_{ij}$ for $1 \le i,j \le d$.*

Proof Let M be a free left R-module of rank d. Let $e_1, \ldots, e_d \in \mathrm{End}_R(M)$ be nonzero elements which satisfy the following conditions:

$$e_i \cdot e_i = e_i, \quad e_i \cdot e_j = 0 \quad \text{if } i \neq j, \quad e_1 + \cdots e_d = \mathrm{id}.$$

Let $M_i = e_i(M) \subset M$ be the image of e_i. Then $M_i \subset M$ is a nonzero R-submodule. We observe the following:

- e_i acts as id on M_i. (If $m \in M_i$ then $m = e_i(m')$ so $e_i m = e_i^2(m') = e_i(m') = m$.)
- e_i acts as 0 on M_j (the same argument as above).
- $M_i \cap M_j = 0$. (If $m \in M_i \cap M_j$ then $e_i m = m$ since $m \in M_i$; on the other hand, $e_i m = 0$, since $m \in M_j$.)
- $M = M_1 + \cdots + M_d$ (since $m = \mathrm{id} m = \sum e_i m$).

Hence, M is an internal direct sum of the submodules M_i. We see that M_i is a projective left R-module, and since every left R-ideal is principal, M_i is free. Since $M_i \neq 0$, $\mathrm{rank}_R M_i \geq 1$. Comparing the ranks of $\sum M_i$ and M, we see that $\mathrm{rank}_R M_i = 1$. If we choose the generators of $M_1, \ldots, M_d$ as an R-basis of M, then we get an isomorphism $\mathrm{End}_R(M) \cong M_d(R)$ such that $e_i = e_{ii}$.

Now let $e'_{ij} = \phi(e_{ij})$. Since e'_{ii} satisfy the conditions listed above, after a conjugation corresponding to mapping a given basis to the basis of the previous paragraph, we can assume $e'_{ii} = e_{ii}$. Next, $e'_{ij}e'_{jj} = e'_{ij}$ and $e'_{ii}e'_{ij} = e'_{ij}$ shows that e'_{ij} has zero entries except possibly at (i,j)th entry, which we denote a_{ij}. Since $e'_{ij}e'_{ji} = e'_{ii}$, we see that $a_{ij}a_{ji} = 1$. Hence, all $a_{ij} \in R^\times$. After conjugating $\phi(M_d(A))$ by $\mathrm{diag}(a_{11}, a_{12}, \ldots, a_{1d})$, we get $a_{1i} = a_{i1} = 1$ for all i. On the other hand, $a_{ij} = a_{i1}a_{1j}$, so e'_{ij} become e_{ij}. □

Theorem 2.20 *The category of Drinfeld–Stuhler $M_d(A)$-modules over L is equivalent to the category of Drinfeld A-modules of rank d over L.*

Proof Suppose ϕ is a Drinfeld–Stuhler $M_d(A)$-module. By the previous lemma, we can assume that $\phi(e_{ij}) = e_{ij}$. (Note that every left ideal of $L[\tau]$ is principal; cf. [14, Cor. 1.6.3].) The map $\Phi : A \to L[\tau]$ which sends a to the nonzero entry of $\phi(a \cdot e_{11})$ is a Drinfeld A-module of rank d, as easily follows from considering the kernel of $\phi(\mathrm{diag}(a, 1, 1, \ldots, 1))$. Next, $ae_{ij} = e_{i1}(ae_{11})e_{1j}$, which implies that $\phi(ae_{ij})$ is the matrix $\Phi_a e_{ij}$. Hence, ϕ arises from a unique Drinfeld A-module Φ of rank d by the construction of Example 2.18.

Now suppose $u : \Phi \to \Phi'$ is a morphism of Drinfeld modules, i.e., $u \in L[\tau]$ is such that $u\Phi_a = \Phi'_a u$ for all $a \in A$. Mapping u to $U := \mathrm{diag}(u, \dots, u)$, we obtain a morphism $U\phi_b = \phi'_b U$, $b \in M_d(A)$, of the corresponding Drinfeld–Stuhler modules. By an argument similar to the argument of the previous paragraph, it is not hard to check that any morphism $\phi \to \phi'$ arises in this manner. This proves the theorem. □

Remark 2.21 Given a unitary ring R and a left R-module M, the direct sum $M^{\oplus d}$ is a left $M_d(R)$-module with $M_d(R)$ acting on elements of $M^{\oplus d}$ as column vectors with entries in M. The functor $M \mapsto M^{\oplus d}$ from the category of left R-modules to the category of left $M_d(R)$-modules is an equivalence of categories, known as *Morita equivalence*; cf. [27]. The inverse functor is $M' \mapsto e_{11}M'$.

The Morita equivalence can be modified so that certain problems concerning Drinfeld–Stuhler O_D-modules reduce to the case of Drinfeld modules, even when $D \ncong M_d(F)$. This idea is due to Taelman [32], who used it in the context of O_D-motives to prove a fact equivalent to the existence of analytic uniformization of Drinfeld–Stuhler modules. To end this section, we sketch Taelman's construction in the setting of Drinfeld–Stuhler modules and indicate one application.

First, recall the following fact. Let F'/F be a finite extension. Then $D \otimes_F F'$ is a central simple algebra over F', and for a place w of F' over a place v of F we have (cf. [21, Lem. A.3.2])

$$\mathrm{inv}_w(D \otimes_F F') = [F'_w : F_v] \cdot \mathrm{inv}_v(D) \in \mathbb{Q}/\mathbb{Z}.$$

Now suppose $F' = \mathbb{F}_{q^n}F$ is obtained by extending the constants. In this case F'_w/F_v is unramified of degree $n/\gcd(n, \deg(v))$. Hence, using the above formula for the invariants of $D \otimes_F F'$, we see that there is n, e.g., $n = d \prod_{\mathrm{inv}_v(D) \notin \mathbb{Z}} \deg(v)$, such that the invariants of $D \otimes_F F'$ at all places of F' are 0, which is equivalent to $D \otimes_F F' \cong M_d(F')$. This fact is known as *Tsen's theorem*.

Now let ϕ be a Drinfeld–Stuhler O_D-module over L and let n be such that $F' = \mathbb{F}_{q^n}F$ splits D. Let A' be the integral closure of A in F'. Assume $\mathbb{F}_{q^n} \subset L$. Denote $\sigma = \tau^n$, and consider the composition

$$\phi' : O_D \xrightarrow{\phi} M_d(L[\tau]) \xrightarrow{\tau \mapsto \sigma} M_d(L[\sigma]).$$

(The second map is a formal substitution $\tau \mapsto \sigma$; it is not a homomorphism.) Note that ϕ' is not a Drinfeld–Stuhler module according to our definition, but the definition can be easily generalized so that ϕ' is a Drinfeld–Stuhler module of "rank n". Denote $O_{D'} := O_D \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, and extend ϕ' to an embedding

$$\widetilde{\phi}' : O_{D'} \longrightarrow M_d(L[\sigma])$$

by mapping $1 \otimes \alpha \mapsto \mathrm{diag}(\alpha, \dots, \alpha)$. It is easy to check that $O_{D'}$ is a maximal A'-order in $D' := D \otimes_F F' \cong M_d(F')$, for example, by calculating its discriminant. Finally, using the Morita equivalence, one associates with $\widetilde{\phi}'$ a Drinfeld A'-module Φ' over L. (One technical complication that should be pointed out is that $O_{D'}$ might not be conjugate to $M_d(A')$ in $M_d(F')$ if A' is not a P.I.D., but for the Morita equivalence to work one only needs an idempotent e in $O_{D'}$ which commutes with σ; cf. [32, p. 68].)

As for the promised application of the above construction, we prove Lemma 2.13.

Proof of Lemma 2.13 Let ϕ and ψ be Drinfeld–Stuhler O_D-modules over L and $u : \phi \to \psi$ be a nonzero morphism. We want to show that u is an isogeny. Without loss of generality, we assume that L is algebraically closed. Explicitly, u is a matrix in $M_d(L[\tau])$. Substituting σ for τ in the entries of u, we get a matrix $u(\sigma) \in M_d(L[\sigma])$. It is clear that $u(\sigma)$ gives a morphism $\widetilde{\phi}' \to \widetilde{\psi}'$ and hence also a nonzero morphism $w : \Phi' \to \Psi'$. But now w is a nonzero polynomial in $L[\sigma]$, so $\ker(w)$ is obviously finite, i.e., w is an isogeny. Finally, it is easy to check that this implies that $u(\sigma)$, and thus also u itself, is an isogeny. □

3 O_D-motives, $\mathscr{D}$-elliptic sheaves, and O_D-lattices

In this section, we introduce three categories closely related to the category of Drinfeld–Stuhler modules. These alternative points of view on Drinfeld–Stuhler modules will be important for the proofs of the main results of this paper. None of the results of this section are original—they are due to Anderson [1], Laumon, Rapoport, Stuhler [22], and Taelman [32]. We keep the notation and assumptions of Sect. 2. In particular, L is an A-field. Let O_D^{opp} denote the opposite ring of O_D (see [27, p. 91]), i.e., O_D^{opp} is O_D with the same addition, but multiplication defined by $\alpha * \beta = \beta \cdot \alpha$, where $\beta \cdot \alpha$ is the multiplication in O_D.

The first category is a variant of Anderson's motives.

Definition 3.1 An O_D-*motive* over L is a left $O_D^{\mathrm{opp}} \otimes_{\mathbb{F}_q} L[\tau]$-module M with the following properties (cf. [32, p. 68], [22, p. 228]):

(i) M is a locally free $O_D^{\mathrm{opp}} \otimes_{\mathbb{F}_q} L$-module of rank 1.
(ii) M is a free $L[\tau]$-module of rank d.
(iii) For all $a \in A$,

$$(a \otimes 1 - 1 \otimes \gamma(a))\overline{M} \subset \tau\overline{M},$$

where $\overline{M} := M \otimes_L L^{\mathrm{alg}}$ is considered as a left $A \otimes_{\mathbb{F}_q} L^{\mathrm{alg}}[\tau]$-module.

The morphisms between O_D-motives are the homomorphisms of $O_D^{\mathrm{opp}} \otimes_{\mathbb{F}_q} L[\tau]$-modules. We denote the corresponding category by **DMot**. (An O_D-motive is a pure abelian Anderson A-motive, in the sense of [33] or [6], of rank d^2, dimension d, and weight $1/d$; see [32, Sect. 9.2].)

Given a Drinfeld–Stuhler O_D-module ϕ over L, let $M(\phi)$ be the group

$$\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{G}_{a,L}^d, \mathbb{G}_{a,L}) \cong L[\tau]^d$$

equipped with the unique $O_D^{\mathrm{opp}} \otimes_{\mathbb{F}_q} L[\tau]$-module structure such that

$$(\ell m)(e) = \ell(m(e)), \quad (\tau m)(e) = m(e)^q, \quad (bm)(e) = m(\phi(b)e),$$

for all $e \in \mathbb{G}_{a,L}^d$, $\ell \in L$, $b \in O_D$, and morphisms $m : \mathbb{G}_{a,L}^d \to \mathbb{G}_{a,L}$. It is easy to see that $M(\phi)$ is an O_D-motive.

Theorem 3.2 *The functor $\phi \mapsto M(\phi)$ gives an anti-equivalence of categories between the category of Drinfeld–Stuhler O_D-modules and* **DMot***.*

Proof This can be proven by a slight modification of Anderson's method; see [33, Thm. 2.3] or [15, Thm. 3.5]. □

The second category arises from $\mathscr{D}$-elliptic sheaves mentioned in Introduction.

Definition 3.3 Fix a maximal $\mathcal{O}_C$-order $\mathscr{D}$ in D such that $H^0(C-\infty, \mathscr{D}) = O_D$. A $\mathscr{D}$-*elliptic sheaf over* L is a sequence $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$, where $\mathcal{E}_i$ is a locally free $\mathcal{O}_{C\otimes_{\mathbb{F}_q}L}$-module of rank d^2 equipped with a right action of $\mathscr{D}$ which extends the $\mathcal{O}_C$-action, and

$$j_i : \mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$$
$$t_i : {}^\tau\mathcal{E}_i := (\mathrm{Id}_C \otimes \mathrm{Frob}_q)^*\mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$$

are injective $\mathscr{D}$-linear homomorphisms. Moreover, for each $i \in \mathbb{Z}$ the following conditions hold:

(i) The diagram

$$\begin{array}{ccc} \mathcal{E}_i & \xrightarrow{j_i} & \mathcal{E}_{i+1} \\ \uparrow{\scriptstyle t_{i-1}} & & \uparrow{\scriptstyle t_i} \\ {}^\tau\mathcal{E}_{i-1} & \xrightarrow{{}^\tau j_{i-1}} & {}^\tau\mathcal{E}_i \end{array}$$

commutes;

(ii) $\mathcal{E}_{i+d\cdot\deg(\infty)} = \mathcal{E}_i \otimes_{\mathcal{O}_C} \mathcal{O}_C(\infty)$, and the inclusion

$$\mathcal{E}_i \xrightarrow{j_i} \mathcal{E}_{i+1} \xrightarrow{j_{i+1}} \cdots \to \mathcal{E}_{i+d\cdot\deg(\infty)} = \mathcal{E}_i \otimes_{\mathcal{O}_C} \mathcal{O}_C(\infty)$$

is induced by $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(\infty)$;

(iii) $\dim_L H^0(C\otimes L, \mathrm{coker} j_i) = d$;

(iv) $\mathcal{E}_i/t_{i-1}({}^\tau\mathcal{E}_{i-1}) = z_*\mathcal{V}_i$, where $\mathcal{V}_i$ is a d-dimensional L-vector space, and z is the morphism induced by γ:

$$z : \mathrm{Spec}(L) \to \mathrm{Spec}(A) \to C.$$

A *morphism* between two $\mathscr{D}$-elliptic sheaves over L

$$\psi = (\psi_i)_{i\in\mathbb{Z}} : \mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} \to \mathbb{E}' = (\mathcal{E}'_i, j'_i, t'_i)_{i\in\mathbb{Z}}$$

is a sequence of sheaf morphisms $\psi_i : \mathcal{E}_i \to \mathcal{E}'_{i+n}$ for some fixed $n \in \mathbb{Z}$ which are compatible with the action of $\mathscr{D}$ and commute with the morphisms j_i and t_i:

$$\psi_{i+1} \circ j_i = j'_{i+n} \circ \psi_i \quad \text{and} \quad \psi_i \circ t_{i-1} = t'_{i+n-1} \circ {}^\tau\psi_{i-1}.$$

Note that the group $\mathbb{Z}$ acts freely on the objects of the category of $\mathscr{D}$-elliptic sheaves by "shifting the indices":

$$n \cdot (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} = (\mathcal{E}'_i, j'_i, t'_i)_{i\in\mathbb{Z}}$$

with $\mathcal{E}'_i = \mathcal{E}_{i+n}, j'_i = j_{i+n}, t'_i = t_{i+n}$. Let $\mathbf{DES}/\mathbb{Z}$ be the quotient of the category of $\mathscr{D}$-elliptic sheaves by this action of $\mathbb{Z}$.

Let $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ be a $\mathscr{D}$-elliptic sheaf over L. Consider

$$M(\mathbb{E}) := H^0((C - \{\infty\}) \otimes L, \mathcal{E}_i).$$

This is independent of i since $\mathrm{supp}(\mathcal{E}_i/\mathcal{E}_{i-1}) \subset \{\infty\} \times \mathrm{Spec}(L)$. It is an $L[\tau]$-module, where the operation of τ is induced from $t_i : {}^\tau\mathcal{E}_i \to \mathcal{E}_{i+1}$. In fact, $M(\mathbb{E})$ is an O_D-motive; see [22, (3.17)].

Theorem 3.4 *The functor* $\mathbb{E} \mapsto M(\mathbb{E})$ *gives an equivalence of* $\mathbf{DES}/\mathbb{Z}$ *with* $\mathbf{DMot}$.

Proof This is implicitly proven in [22, (3.17)] and explicitly in [32, 10.3.5]. We outline the main steps of the proof since part of this argument will be used later in the paper.

First note that since $M(\mathbb{E})$ does not depend on the choice of $\mathcal{E}_i$, the map is indeed a functor from $\mathbf{DES}/\mathbb{Z}$ to $\mathbf{DMot}$. Next, let $W_\infty := H^0(\mathrm{Spec}(O_\infty \hat{\otimes} L), \mathcal{E}_0)$. From the definition of $\mathscr{D}$-elliptic sheaf one deduces that W_∞ has a natural structure of a free $L[\![\tau^{-1}]\!]$-module of rank d; see [22, p. 231]. In addition, W_∞ is a right $\mathscr{D}_\infty$-module so that we get an injective $\mathbb{F}_q$-algebra homomorphism

$$\varphi_\infty : \mathscr{D}_\infty^{\mathrm{opp}} \to \mathrm{End}_{L[\![\tau^{-1}]\!]}(W_\infty),$$

and if we denote by π_∞ a uniformizer of O_∞ and $\tau_\infty = \tau^{\deg(\infty)}$, then W_∞ has the property that $\tau_\infty^{-d} W_\infty = \pi_\infty W_\infty$.

The pair $(M(\mathbb{E}), W_\infty)$ is a vector bundle of rank d over the non-commutative projective line over L in the sense of [22, (3.13)]. Hence, by [22, (3.16)],

$$(M(\mathbb{E}), W_\infty) \cong \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_d),$$

where $\mathcal{O}(n) = (L[\tau], \tau^n L[\![\tau^{-1}]\!])$. Since $(M(\mathbb{E}), W_\infty)$ is equipped with a coherent right $\mathscr{D}$-action (cf. [22, (3.14)]), we have $n_1 = \cdots = n_d$. Hence, $(M(\mathbb{E}), W_\infty) \cong \mathcal{O}(n)^{\oplus d}$ for some $n \in \mathbb{Z}$. If we define $W'_\infty = H^0(\mathrm{Spec}(O_\infty \hat{\otimes} L), \mathcal{E}_i)$, then $(M(\mathbb{E}), W'_\infty)$ is again a vector bundle of rank d over the non-commutative projective line. Moreover, $(M(\mathbb{E}), W'_\infty) = (M(\mathbb{E}), \tau^i W_\infty)$; see [22, p. 235]. Hence, up to the action of $\mathbb{Z}$, $M(\mathbb{E})$ uniquely determines the vector bundle $(M(\mathbb{E}), W_\infty)$. On the other hand, by [22, (3.17)], the vector bundle $(M(\mathbb{E}), W_\infty)$ with its coherent $\mathscr{D}$-action uniquely determines $\mathbb{E}$ and any O_D-motive is isomorphic to $M(\mathbb{E})$ for some $\mathbb{E}$. This proves that the functor in question is fully faithful and essentially surjective. □

The third category arises in the theory of analytic uniformization of Drinfeld–Stuhler modules. Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of F_∞. Let ϕ be a Drinfeld–Stuhler O_D-module over $\mathbb{C}_\infty$. By fixing an isomorphism $\mathrm{Lie}(\mathbb{G}_{a,\mathbb{C}_\infty}^d) \cong \mathbb{C}_\infty^d$, we get an action of O_D on $\mathbb{C}_\infty^d$ via ∂_ϕ.

Theorem 3.5 *There is a discrete O_D-submodule Λ_ϕ of $\mathbb{C}_\infty^d$, which is locally free of rank* 1*, and an entire $\mathbb{F}_q$-linear function $\exp_\phi : \mathbb{C}_\infty^d \to \mathbb{C}_\infty^d$, which is surjective with kernel Λ_ϕ, such that for any $b \in O_D$ the following diagram is commutative:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda_\phi & \longrightarrow & \mathbb{C}_\infty^d & \xrightarrow{\exp_\phi} & \mathbb{C}_\infty^d & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_\phi(b)} & & \downarrow{\scriptstyle \partial_\phi(b)} & & \downarrow{\scriptstyle \phi_b} & & \\
0 & \longrightarrow & \Lambda_\phi & \longrightarrow & \mathbb{C}_\infty^d & \xrightarrow{\exp_\phi} & \mathbb{C}_\infty^d & \longrightarrow & 0.
\end{array}$$

Proof The exponential function $\exp_\phi$ is the function constructed by Anderson in [1, Sect. 2]. The existence of Λ_ϕ (which is equivalent to the surjectivity of $\exp_\phi$ by [1, Thm. 4]) was proved by Taelman [32, Sect. 9–10] in the terminology of O_D-motives. A crucial point in Taelman's proof is the use of Morita equivalence (see Sect. 2.4), which reduces the proof to the analytic uniformization of Drinfeld modules (already known by the work of Drinfeld [8] and Anderson [1]). □

Corollary 3.6 *The ring* $\mathrm{End}(\phi)$ *is canonically isomorphic to the ring*

$$\mathrm{End}(\Lambda_\phi) := \{c \in \mathbb{C}_\infty \mid c\Lambda_\phi \subseteq \Lambda_\phi\}.$$

Proof The functorial properties of $\exp_\phi$ (cf. [1, p. 473]) imply that ∂ maps $\mathrm{End}(\phi)$ isomorphically to the ring

$$\left\{P \in M_d(\mathbb{C}_\infty) \mid P\Lambda \subseteq \Lambda, P\partial_\phi(b) = \partial_\phi(b)P \text{ for all } b \in O_D\right\}.$$

Since any matrix which commutes with $\partial_\phi(O_D)$ must be a scalar, we get the desired isomorphism. □

Now suppose $\mathbb{C}_\infty^d$ is equipped with an action of O_D via some embedding $\iota : O_D \to M_d(\mathbb{C}_\infty)$. Suppose there is a discrete $\iota(O_D)$-submodule $\Lambda \subset \mathbb{C}_\infty^d$ which is locally free of rank one. Then there is a unique Drinfeld–Stuhler O_D-module such that $\iota = \partial_\phi$ and $\Lambda = \Lambda_\phi$; this follows from [32, Sect. 10.1.3], which itself crucially relies on [1, Thm. 6]. Hence, the category of Drinfeld–Stuhler modules over $\mathbb{C}_\infty$ is equivalent to the category of O_D-lattices as above. One can use this equivalence to give an analytic description of the set of isomorphism classes of Drinfeld–Stuhler modules over $\mathbb{C}_\infty$ as follows: Let

$$\Omega^d = \mathbb{P}^{d-1}(\mathbb{C}_\infty) - \bigcup_H H(\mathbb{C}_\infty)$$

be the Drinfeld symmetric space, where H runs through the set of F_∞-rational hyperplanes in $\mathbb{P}^{d-1}(\mathbb{C}_\infty)$. Similar to the ring of finite adèles

$$\mathbb{A}_f = \left\{(a_v) \in \prod_{v\neq\infty} F_v \mid a_v \in A_v \text{ for almost all } v\right\},$$

define

$$D(\mathbb{A}_f) = \left\{(a_v) \in \prod_{v\neq\infty} D_v \mid a_v \in O_D \otimes_A A_v \text{ for almost all } v\right\}.$$

Let $\hat{A} := \prod_{v\neq\infty} A_v$ and $\widehat{O}_D := \prod_{v\neq\infty} O_D \otimes_A A_v$. We embed D in $D(\mathbb{A}_f)$ diagonally. Fixing an isomorphism $D_\infty \cong M_d(F_\infty)$ identifies $D^\times$ with a subgroup of $\mathrm{GL}_d(F_\infty)$ and therefore induces an action of $D^\times$ on Ω.

Proposition 3.7 *There is a one-to-one correspondence between the set of isomorphism classes of Drinfeld–Stuhler O_D-modules over $\mathbb{C}_\infty$ and the double coset space*

$$D^\times \backslash \Omega^d \times D(\mathbb{A}_f)^\times / \widehat{O}_D^\times,$$

where $D^\times$ acts on both Ω^d and $D(\mathbb{A}_f)^\times$ on the left, and $\widehat{O}_D^\times$ acts on $D(\mathbb{A}_f)^\times$ on the right:

$$\gamma \cdot (z, \alpha) \cdot k = (\gamma z, \gamma\alpha k), \quad \gamma \in D^\times, \quad z \in \Omega^d, \quad \alpha \in D(\mathbb{A}_f)^\times, \quad k \in \widehat{O}_D^\times.$$

Proof This can be proved by a standard argument [32, p. 74] (see also [4, Thm. 4.4.11]). We recall this argument, since we will use it later on. Let $\Lambda \subset \mathbb{C}_\infty^d$ be an O_D-lattice, where D acts on $\mathbb{C}_\infty^d$ via the fixed isomorphism $D_\infty \cong M_d(F_\infty)$. The F-span $F\Lambda$ is a free module over D of rank 1. A choice of generator of this module defines a point in $\mathbb{P}^{d-1}(\mathbb{C}_\infty)$. One checks that this point lies in Ω^d if and only if Λ is discrete. The embedding $\Lambda \subset F\Lambda = D$ can be tensored to an embedding $\hat{A}\Lambda \subset D(\mathbb{A}_f)$, and the former can be recovered from the latter as $\Lambda = \hat{A}\Lambda \cap D$. Now $\hat{A}\Lambda$ is a locally free module over $\widehat{O}_D$. Since all such modules are free, we conclude that the locally free O_D-submodules $\Lambda \subset D$ of rank one are in bijection with the free rank one $\widehat{O}_D$-submodules of $D(\mathbb{A}_f)$, and the latter are in bijection with $D(\mathbb{A}_f)^\times / \widehat{O}_D^\times$. Finally, factoring out the choice of the generator of $F\Lambda$, that is, by factoring $D^\times$, we get the desired one-to-one correspondence. □

4 Complex multiplication

This section contains our main results about the endomorphism rings of Drinfeld–Stuhler modules. The proofs rely on the concepts introduced in Sect. 3.

Theorem 4.1 *Let ϕ be a Drinfeld–Stuhler O_D-module over an A-field L. Then:*

1. $\mathrm{End}_L(\phi)$ *is a projective A-module of rank* $\leq d^2$.
2. $\mathrm{End}_L(\phi) \otimes_A F_\infty$ *is isomorphic to a subalgebra of the central division algebra over* F_∞ *with invariant* $-1/d$.
3. *If L has generic A-characteristic, then* $\mathrm{End}_L(\phi)$ *is an A-order in an imaginary field extension of F which embeds into D. In particular,* $\mathrm{End}_L(\phi)$ *is commutative and its rank over A divides d.*
4. *The automorphism group* $\mathrm{Aut}_L(\phi) := \mathrm{End}_L(\phi)^\times$ *is isomorphic to* $\mathbb{F}_{q^s}^\times$ *for some s dividing d.*

Proof It is enough to prove (1), (2), and (3) after extending L to its algebraic closure, so we will assume that L is algebraically closed.

Since the O_D-motive $M(\phi)$ associated with ϕ is an Anderson A-motive of dimension d and rank d^2, the argument in [1, Sect. 1.7] implies that $\mathrm{End}_{A\otimes L[\tau]}(M(\phi))$ is a projective A-module of rank $\leq d^4$ (see also [6, Thm. 9.5] and [15, Cor. 2.6]). Hence, thanks to Theorem 3.2, $\mathrm{End}(\phi)$ is a projective A-module of rank $\leq d^4$.

Let W_∞ be the $\mathscr{D}_\infty \otimes L[\![\tau^{-1}]\!]$-module attached to ϕ in the proof of Theorem 3.4. As we discussed, W_∞ is well defined up to the shifts $W_\infty \mapsto \tau W_\infty$. Since $\mathscr{D}_\infty \cong \mathbb{M}_d(O_\infty)$, using the Morita equivalence, cf. [22, p. 262], one concludes that W_∞ is equivalent to an $O_\infty \otimes L[\![\tau^{-1}]\!]$-module W'_∞ which is free of rank 1 over O_∞, free of rank 1 over $L[\![\tau^{-1}]\!]$, and $\tau_\infty^{-d} W'_\infty = \pi_\infty W'_\infty$. From W'_∞ we get an $\mathbb{F}_q$-algebra homomorphism

$$\phi_\infty : O_\infty \to \mathrm{End}_{L[\![\tau^{-1}]\!]}\left(W'_\infty\right) = L[\![\tau^{-1}]\!], \quad \phi_\infty(\pi_\infty) = \tau_\infty^{-d}.$$

Thus,

$$\begin{aligned}\mathrm{End}_{O_\infty\otimes L[\![\tau^{-1}]\!]}(W'_\infty)^{\mathrm{opp}} &= \mathrm{End}(\phi_\infty)\\ &= \{f \in L[\![\tau^{-1}]\!] \mid f\phi_\infty(b) = \phi_\infty(b) f \text{ for all } b \in O_\infty\}.\end{aligned}$$

Since $O_\infty = \mathbb{F}_{q^{\deg(\infty)}}[\![\pi_\infty]\!]$, the image of O_∞ under ϕ_∞ is the subring $\mathbb{F}_{q^{\deg(\infty)}}[\![\tau_\infty^{-d}]\!]$ of $L[\![\tau^{-1}]\!]$. Now it is easy to see that

$$\mathrm{End}(\phi_\infty) \cong \mathbb{F}_{q^{d\deg(\infty)}}[\![\tau_\infty^{-1}]\!],$$

which is the maximal order in the central division algebra over F_∞ with invariant $-1/d$; cf. [22, Appendix B]. Definition 3.14 and theorem 3.17 in [22] imply that $\mathrm{End}(\phi)$ acts faithfully on W'_∞, and this action gives an embedding $\mathrm{End}(\phi)\otimes_A F_\infty \hookrightarrow \mathrm{End}(\phi_\infty)\otimes_{O_\infty} F_\infty$ (see also [6, Thm. 8.6]). Since $\mathrm{rank}_{O_\infty}\mathrm{End}(\phi_\infty) = d^2$, we get $\mathrm{rank}_A\mathrm{End}(\phi) \leq d^2$. This proves (1) and (2).

To prove (3), note that ϕ is defined over some finitely generated subfield of L which can be embedded into $\mathbb{C}_\infty$. So, without loss of generality, we assume $L = \mathbb{C}_\infty$. Combining (1) and (2) with Lemma 2.12 already implies that $\mathrm{End}(\phi)$ is an A-order in an imaginary field extension of F. We need to show that $\mathrm{End}(\phi)$ embeds into D. Let Λ_ϕ be the O_D-lattice associated with ϕ by Theorem 3.5. By Corollary 3.6, $\alpha \in \mathrm{End}(\phi)$ corresponds to $c \in \mathbb{C}_\infty$ such that $c\Lambda_\phi \subseteq \Lambda_\phi$. On the other hand, the F-span $F\Lambda_\phi$ is a free module over D of rank 1, so c corresponds to a unique element of D. Mapping α to that element gives an embedding $\mathrm{End}(\phi) \hookrightarrow D$. Finally, the rank of $\mathrm{End}(\phi)$ over A is equal to the degree of $\mathrm{End}(\phi) \otimes_F F$ over F, and it is well-known that a subfield of D containing F has degree over F dividing d; cf. [27, Sect. 7].

To prove (4), note that we have established that $E := \mathrm{End}_L(\phi)$ is an A-order in the division algebra $H := E \otimes_A F$ over F. In this situation, $\alpha \in E$ is a unit if and only if $\mathrm{Nr}_{H/F}(\alpha) \in A^\times \cong \mathbb{F}_q^\times$, where $\mathrm{Nr}_{H/F}$ is the norm on H; cf. [27, p. 224]. We also proved that $H_\infty := H \otimes F_\infty$ is a subalgebra of the central division algebra over F_∞ with invariant $-1/d$. It is known that $\mathcal{H} = \{\alpha \in H_\infty \mid \mathrm{Nr}_{H_\infty/F_\infty}(\alpha) \in O_\infty\}$ is the unique maximal order of H_∞, $\mathcal{M} = \{\alpha \in H_\infty \mid \mathrm{Nr}_{H_\infty/F_\infty}(\alpha) \in \mathfrak{p}_\infty\}$ is the unique maximal two-sided ideal of $\mathcal{H}$, and $\mathcal{H}/\mathcal{M}$ is a subfield of $\mathbb{F}_{q^d}$; see (12.8), (13.2), (14.3) in [27]. Since the norm of any element of E is in A, the subring $k := E \cap \mathcal{H}$ maps injectively into $\mathcal{H}/\mathcal{M} \hookrightarrow \mathbb{F}_{q^d}$. Thus, k is isomorphic to a subfield of $\mathbb{F}_{q^d}$. Finally, it is clear that $\mathrm{Aut}_L(\phi) = k^\times$. □

Remark 4.2 When $D = M_d(F)$, via the equivalence of Sect. 2.4, the statements of Theorem 4.1 are equivalent to some well-known facts about the endomorphism rings of Drinfeld A-modules of rank d; cf. [13].

Example 4.3 Let ϕ be a Drinfeld–Stuhler O_D-module over an algebraically closed field L of generic A-characteristic. From Theorem 4.1 we know that $\mathrm{End}(\phi)$ is an A-order in an imaginary field extension of F of degree dividing d. We show that this bound is the best possible. Consider ϕ from Example 2.15. Let Φ be the rank 1 Drinfeld O_K-module over L from the same example. Let

$$E := \left\{\mathrm{diag}(\Phi_\alpha, \ldots, \Phi_\alpha) \mid \alpha \in O_K\right\} \subset M_d(L[\tau]).$$

It is clear that $E \cong O_K$. One easily checks that the elements of E commute with $\phi_\alpha, \alpha \in O_K$, and ϕ_z. Therefore, $E \subseteq \mathrm{End}(\phi)$. Since O_K is a maximal A-order in K, Theorem 4.1 implies that $\mathrm{End}(\phi) \cong O_K$.

Definition 4.4 For the rest of this section, unless indicated otherwise, K denotes an imaginary field extension of F of degree d and O_K denotes the integral closure of A in K. An *A-order* E in K is an A-subalgebra of O_K, which has the same unity element as O_K and such that O_K/E has finite cardinality. We say that a Drinfeld–Stuhler O_D-module ϕ over a field of generic A-characteristic has *complex multiplication* by E (or has *CM*, for short) if $E \cong \mathrm{End}(\phi)$. Note that in that case K necessarily embeds into D by Theorem 4.1 (3). A *CM subfield* of D is a commutative subfield of D which is an imaginary extension of F of degree d.

Lemma 4.5 *Let ϕ be a Drinfeld–Stuhler O_D-module over a field L of generic A-characteristic. Assume $E \subset \mathrm{End}_L(\phi)$. There is a Drinfeld–Stuhler O_D-module ψ which is isogenous to ϕ over L and $\mathrm{End}_L(\psi) \cong O_K$.*

Proof Let $\mathfrak{c}$ be the conductor of E, i.e., the largest ideal of O_K which is also an ideal of E. Let $H := \bigcap_{c\in\mathfrak{c}} \ker(c)$, where the intersection is taken in $\mathbb{G}_{a,L}^d$. Since the action of O_D on $\mathbb{G}_{a,L}^d$ commutes with E, the finite étale subgroup scheme H of $\mathbb{G}_{a,L}^d$ is invariant under $\phi(O_D)$. From the discussion on page 155 in [14] it follows that there is $u \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L}^d)$ with $\ker(u) = H$. Let $b \in O_D$. Consider the endomorphism $u\phi_b$ of $\mathbb{G}_{a,L}^d$. Since H is invariant under $\phi(O_D)$, we have $H \subseteq \ker(u\phi_b)$. Then we can factor $u\phi_b$ as $\psi_b u$ for some $\psi_b \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L}^d)$. (See Proposition 5.2 and Corollary 5.3 in [15] for some relevant scheme-theoretic facts about morphisms $\mathbb{G}_{a,L}^d \to \mathbb{G}_{a,L}^d$; in particular, note that $\mathbb{G}_{a,L}^d/H \cong \mathbb{G}_{a,L}^d$.) It is easy to see that $b \mapsto \psi_b$ gives an embedding $O_D \to M_d(L[\tau])$ and $\#\psi[b] = \#\phi[b]$. Since $\partial(u) \in M_d(L)$ is an invertible matrix, and $\partial_\phi(a)$ ($a \in A$) is the scalar matrix $\mathrm{diag}(\gamma(a), \dots, \gamma(a))$, we also get that $\partial_\psi(a) = \mathrm{diag}(\gamma(a), \dots, \gamma(a))$. Thus, there is a Drinfeld–Stuhler O_D-module ψ over L and an isogeny $u : \phi \to \psi$ whose kernel is H. Now one can apply the argument in the proof of [14, Prop. 4.7.19] to deduce that $\mathrm{End}(\psi) \cong O_K$. □

We further investigate the properties of Drinfeld–Stuhler modules with CM using analytic uniformization. We fix an embedding $D^\times \to \mathrm{GL}_d(F_\infty)$ through which $D^\times$ acts on Ω. For $(z, \alpha) \in \Omega^d \times D(\mathbb{A}_f)^\times$, let $\Lambda_{(z,\alpha)}$ be the O_D-lattice corresponding to (z, α); see Proposition 3.7. Let $K_z^\times := \{\gamma \in D^\times \mid \gamma z = z\}$. From now on we will implicitly assume that D is a division algebra.

Lemma 4.6 *$K_z := K_z^\times \cup \{0\}$ is a subfield of D and* $\mathrm{End}(\Lambda_{(z,\alpha)}) = K_z \cap \alpha\widehat{O}_D\alpha^{-1}$.

Proof Let $\tilde{z} \in \mathbb{C}_\infty^d$ be an element mapping to z; such $\tilde{z}$ is well defined up to a scalar multiple. Denote $O = D \cap \alpha\widehat{O}_D\alpha^{-1}$. The lattice $\Lambda = O\tilde{z}$ is in the isomorphism class of $\Lambda_{(z,\alpha)}$. We have

$$c \in \mathrm{End}(\Lambda) \iff c\Lambda \subset \Lambda \iff cO\tilde{z} \subset O\tilde{z} \iff Oc\tilde{z} \subset O\tilde{z},$$

where $c \in \mathbb{C}_\infty$ acts on $\tilde{z}$ as a scalar matrix. The inclusion $Oc\tilde{z} \subset O\tilde{z}$ is equivalent to the existence of $\gamma \in O$ such that $\gamma\tilde{z} = c\tilde{z}$. This γ obviously fixes z, and since $\gamma \in \alpha\widehat{O}_D\alpha^{-1}$, we get $\gamma \in K_z \cap \alpha\widehat{O}_D\alpha^{-1} =: E_{(z,\alpha)}$. Conversely, suppose $\gamma \in E_{(z,\alpha)}$, so $\gamma \in O$ and $\gamma\tilde{z} = c\tilde{z}$ for some nonzero $c \in \mathbb{C}_\infty$ (because $\gamma \in K_z$). Reversing the previous argument we see that $c \in \mathrm{End}(\Lambda)$.

Observe that $E_{(z,\alpha)}$ is a subring of D since for $\gamma, \gamma' \in E_{(z,\alpha)}$ with $\gamma\tilde{z} = c\tilde{z}$, $\gamma'\tilde{z} = c'\tilde{z}$, we have $(\gamma + \gamma')\tilde{z} = (c + c')\tilde{z}$. Hence, $K_z = E_{(z,\alpha)} \otimes_A F$ is a commutative subalgebra of D, i.e., K_z is a subfield of D. Since the map $E_{(z,\alpha)} \to \mathrm{End}(\Lambda)$, $\gamma \mapsto c$, is a homomorphism which extends to $K_z \to \mathbb{C}_\infty$, it must be injective. But we have seen that $E_{(z,\alpha)} \to \mathrm{End}(\Lambda)$ is also surjective; thus, it is an isomorphism. □

Remark 4.7 For any $\alpha \in D(\mathbb{A}_f)^\times$ and a CM field $K \subset D$, the intersection $K \cap \alpha\widehat{O}_D\alpha^{-1}$ is an A-order in K. To prove this, first observe that $D \cap \alpha\widehat{O}_D\alpha^{-1}$ is a maximal order in D. Hence, it is enough to prove that for any maximal order $\mathcal{M}$ in D the intersection $E := K \cap \mathcal{M}$ is an A-order. It is clear that $A \subset E$. By exercise 4, p. 131, [27], there is a maximal order $\mathcal{M}'$ in D such that $K \cap \mathcal{M}' = O_K$. It is easy to see that $\mathcal{M}'' := \mathcal{M} \cap \mathcal{M}'$ is an A-order in D. (In fact, it is a hereditary order; cf. [27, Sect. 40].) Hence, $\mathcal{M}''$ has finite index in $\mathcal{M}'$. On the other hand, since $E = O_K \cap \mathcal{M}''$, under the natural homomorphism $\mathcal{M}' \to \mathcal{M}'/\mathcal{M}''$ the module O_K maps onto O_K/E. Thus, E has finite index in O_K, i.e., it is an order.

Lemma 4.8 *Let K be a CM subfield of D. The number of fixed points of $K^\times$ in Ω^d is nonzero and is at most d.*

Proof Since F has transcendence degree 1 over $\mathbb{F}_q$, we can find a primitive element $\gamma \in K$ such that $K = F(\gamma)$; cf. [2]. It is enough to prove that γ has at least one and at most d fixed points in Ω^d. Note that the minimal polynomial of γ over F has degree d and divides the characteristic polynomial of γ considered as an element of $\mathrm{GL}_d(F_\infty)$. Thus, the minimal polynomial is equal to the characteristic polynomial. The claim then follows from the fact that a matrix in $\mathrm{GL}_d(\mathbb{C}_\infty)$, whose characteristic and minimal polynomials are equal, has at least one and at most d eigenvectors, up to scaling. $\square$

Notation 4.9 Let K be a CM subfield of D and E be an A-order in K. Let

$$\mathbb{T}_E := \left\{\alpha \in D(\mathbb{A}_f)^\times \mid K \cap \alpha \widehat{O}_D \alpha^{-1} = E\right\}.$$

It is easy to check that $K^\times$ acts on $\mathbb{T}_E$ from the left by multiplication and $\widehat{O}_D^\times$ acts from the right. It is known that $\mathbb{T}_E$ is non-empty, and the double coset space $K^\times\backslash\mathbb{T}_E/\widehat{O}_D^\times$ has finite cardinality divisible by the class number of E; cf. [34, pp. 92–93]. The elements of $\mathbb{T}_E$ correspond to optimal embeddings of K into the maximal orders of D with respect to E.

Theorem 4.10 *Let S_K be the set of fixed points of $K^\times$ in Ω^d. We have:*

1. *Up to isomorphism, the number of Drinfeld–Stuhler O_D-modules over $\mathbb{C}_\infty$ having CM by E is equal to $\#(K^\times\backslash S_K \times \mathbb{T}_E/\widehat{O}_D^\times)$. In particular, that number is finite and nonzero.*
2. *A Drinfeld–Stuhler O_D-module having CM by O_K can be defined over the Hilbert class field of K.*

Proof (1) In our setup, we have fixed an embedding of K into D. For each $(z, \alpha) \in S_K \times \mathbb{T}_E$ we have $\mathrm{End}(\Lambda_{(z,\alpha)}) = K_z \cap \alpha\widehat{O}_D\alpha^{-1} = K \cap \alpha\widehat{O}_D\alpha^{-1} = E$; cf. Lemma 4.6. Note that for any $\gamma \in D^\times$, we have $K_{\gamma z} = \gamma K_z \gamma^{-1}$, and so

$$K_{\gamma z} \cap \gamma\alpha\widehat{O}_D\alpha^{-1}\gamma^{-1} = \gamma(K \cap \alpha\widehat{O}_D\alpha^{-1})\gamma^{-1} = \gamma E\gamma^{-1} \cong E,$$

which implies $\mathrm{End}(\Lambda_{\gamma(z,\alpha)}) \cong \mathrm{End}(\Lambda_{(z,\alpha)})$.

Now suppose $(z, \alpha) \in \Omega^d \times D(\mathbb{A}_f)^\times$ is such that $\mathrm{End}(\Lambda_{(z,\alpha)}) \cong E$. Then K_z must be isomorphic to K, so K_z is another embedding of K into D. By the Skolem–Noether theorem [27, (7.21)], two embeddings $K \rightrightarrows D$ differ by an inner automorphism of D. Thus, there is $\gamma \in D^\times$ such that $K_z = \gamma K\gamma^{-1}$ and $\mathrm{End}(\Lambda_{(z,\alpha)}) = \gamma E\gamma^{-1}$. This implies that we can find $z' \in S_K$ such that $\gamma z' = z$. We also have $\gamma K\gamma^{-1} \cap \alpha\widehat{O}_D\alpha^{-1} = \gamma E\gamma^{-1}$, which implies $\gamma^{-1}\alpha \in \mathbb{T}_E$. Hence, we can find $\alpha' \in \mathbb{T}_E$ such that $\alpha = \gamma\alpha'$. Overall, we conclude that $(z, \alpha) = \gamma(z', \alpha')$ for some $(z', \alpha') \in S_K \times \mathbb{T}_E$. The stabilizer in $D^\times$ of any $z \in S_K$ is $K^\times$. Hence, the set of images in $D^\times\backslash\Omega^d \times D(\mathbb{A}_f)^\times/\widehat{O}_D^\times$ of $(z, \alpha) \in \Omega^d \times D(\mathbb{A}_f)^\times$ with CM by E is the double coset space $K^\times\backslash S_K \times \mathbb{T}_E/\widehat{O}_D^\times$.

(2) Let ϕ be a Drinfeld–Stuhler O_D-module with $\mathrm{End}(\phi) \cong O_K$. Let $M(\phi)$ be the O_D-motive associated with ϕ. By definition, the action of O_K on $\mathbb{G}^d_{a,\mathbb{C}_\infty}$ commutes with $\phi(O_D)$; hence, $M(\phi)$ is an $O_D^{\mathrm{opp}} \otimes_A O_K$-module. On the other hand, $O_D^{\mathrm{opp}} \otimes_A O_K$ is an A-order in $D^{\mathrm{opp}} \otimes_F K \cong M_d(K)$; cf. exercise 6 on page 131 of [27]. Computing the discriminants, one checks that $O_D^{\mathrm{opp}} \otimes_A O_K$ is a maximal order in $M_d(K)$. By the Morita equivalence (cf. [22, p. 262] and [32, p. 68]), $M(\phi)$ is equivalent to an O_K-motive M' of rank 1 and dimension 1 (as defined in [33]). Through a generalization of Anderson's result (cf. [33, Thm. 2.9]), M' corresponds to a Drinfeld O_K-module Φ of rank 1. Since Φ can be defined over H (see [17, Sect. 8]), ϕ also can be defined over H. $\square$

Remark 4.11 If $d = 2$, K is a separable quadratic extension of F, and $E = O_K$, then (cf. [34, p. 94])

$$\#(K^\times\backslash S_K \times \mathbb{T}_{O_K}/\widehat{O}_D^\times) = \#\mathrm{Pic}(O_K) \prod_{\mathfrak{p}|\mathfrak{r}(D)} \left(1 - \left(\frac{K}{\mathfrak{p}}\right)\right),$$

where

$$\left(\frac{K}{\mathfrak{p}}\right) = \begin{cases} -1, & \text{if } \mathfrak{p} \text{ remains inert in } K; \\ 1, & \text{if } \mathfrak{p} \text{ splits in } K; \\ 0, & \text{if } \mathfrak{p} \text{ ramifies in } K. \end{cases}$$

From Theorem 4.1 (4) we know that $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^s}^\times$ for some $s \mid d$. Moreover, Examples 2.16 and 4.3 show that there do exist Drinfeld–Stuhler modules with maximal possible automorphism group $\mathbb{F}_{q^d}^\times$. We can now give necessary and sufficient conditions for the existence of Drinfeld–Stuhler modules with given automorphism group.

Corollary 4.12 *Let L be an algebraically closed field of generic A-characteristic. Given $s \mid d$, there is a Drinfeld–Stuhler module ϕ over L with* $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^s}^\times$ *if and only if*

(i) *s is coprime to* $\deg(\infty)$ *and*
(ii) $\frac{d}{\gcd(s,\deg(v))}\mathrm{inv}_v(D) \equiv 0 \pmod{\mathbb{Z}}$ *for all* $v \in \mathrm{Ram}(D)$.

Proof Let $K := \mathbb{F}_{q^s}F$. Note that K/F is a field extension of F of degree s. If w is a place of K over a place v of F, then the local extension K_w/F_v is unramified of degree $s/\gcd(s, \deg(v))$. This implies that K is imaginary if and only if (i) holds. On the other hand, by [21, Prop. A.3.4], K embeds into D if and only if $[K : F]$ divides d and

$$\frac{d}{[K:F]}[K_w : F_v]\mathrm{inv}_v(D) \equiv 0 \pmod{\mathbb{Z}} \quad \text{for all } v \in \mathrm{Ram}(D).$$

Thus, $K \hookrightarrow D$ if and only if (ii) holds.

Suppose there is ϕ over L with $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^s}^\times$. Then $K \subset \mathrm{End}(\phi) \otimes_A F$, and Theorem 4.1 (3) implies that K is isomorphic to an imaginary subfield of D. Conversely, suppose K is an imaginary subfield of D. We can extend K to a maximal subfield K' of D so that $\mathbb{F}_{q^s}$ is algebraically closed in K' and the place of K over ∞ does not split in K'. Then K' is a CM subfield of D and $O_{K'}^\times \cong \mathbb{F}_{q^s}^\times$. By a "Lefschetz principle"-type argument we can assume $L = \mathbb{C}_\infty$. The existence of ϕ with $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^s}^\times$ then follows from Theorem 4.10 (1) by taking $E = O_{K'}$. □

Next, in a special case, we compute the number of isomorphism classes of Drinfeld–Stuhler modules over $\mathbb{C}_\infty$ with maximal possible automorphism group $\mathbb{F}_{q^d}^\times$. In principle, this amounts to an explicit computation of the order of the double coset space in Theorem 4.10 (1), but we will take a somewhat different approach which also provides a group-theoretic interpretation of this number.

Proposition 4.13 *Assume $A = \mathbb{F}_q[T]$ and d is coprime to* $\deg(v)$ *for all* $v \in \mathrm{Ram}(D)$*. Up to isomorphisms, there are $d^{\#\mathrm{Ram}(D)}$ Drinfeld–Stuhler modules ϕ over $\mathbb{C}_\infty$ with* $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^d}^\times$.

Proof Note that in this case $\deg(\infty) = 1$, so assumption (i) in Corollary 4.12 is automatically satisfied. Also note that the local index of D at any $v \in \mathrm{Ram}(D)$ divides d, so the assumption that d is coprime to $\deg(v)$ for all $v \in \mathrm{Ram}(D)$ is a stronger assumption than (ii) in Corollary 4.12; it is equivalent to assuming that D_v, $v \in \mathrm{Ram}(D)$, is a division algebra.

Via our fixed embedding $D^\times \to \mathrm{GL}_d(F_\infty)$, we get an action of $\Gamma := O_D^\times$ on Ω^d. The stabilizer Γ_z in Γ of a point $z \in \Omega^d$ is a finite subgroup. In fact, Lemma 4.6 implies that Γ_z is the automorphism group of the Drinfeld–Stuhler module corresponding to the lattice $\Lambda_{(z,1)}$. Hence, $\Gamma_z \cong \mathbb{F}_{q^s}^\times$ for some $s \mid d$. We say that z is *special* if $\Gamma_z \cong \mathbb{F}_{q^d}^\times$. It is clear that for a special point $z \in \Omega$ and any $\gamma \in \Gamma$, the point $\gamma(z)$ is also special since $\Gamma_{\gamma(z)} = \gamma\Gamma_z\gamma^{-1}$. Denote the set of Γ-orbits of special points by $\mathcal{S}(\Gamma)$. Denote the set of conjugacy classes of subgroups of Γ isomorphic to $\mathbb{F}_{q^d}^\times$ by $\mathcal{G}(\Gamma)$.

For a subgroup $G \cong \mathbb{F}_{q^d}^\times$ of Γ, let $\mathrm{Fix}(G) := \{z \in \Omega \mid gz = z \text{ for all } g \in G\}$. If we fix a generator g of G, then $\mathrm{Fix}(G)$ coincides with the set of fixed points of g. The characteristic polynomial of g is separable and irreducible over F_∞ (in fact it has coefficients in $\mathbb{F}_q$), so $\#\mathrm{Fix}(G) = d$; cf. Lemma 4.8. Let G_1 and G_2 be two subgroups of Γ isomorphic to $\mathbb{F}_{q^d}^\times$. If $G_2 = \gamma G_1\gamma^{-1}$ for some $\gamma \in \Gamma$, then clearly $\mathrm{Fix}(G_2) = \gamma\mathrm{Fix}(G_1)$. Conversely, suppose $\mathrm{Fix}(G_2) = \gamma\mathrm{Fix}(G_1)$. Since $G_i = \Gamma_z$ for any $z \in \mathrm{Fix}(G_i)$ $(i = 1, 2)$ and $\Gamma_{\gamma(z)} = \gamma\Gamma_z\gamma^{-1}$, we see that G_1 and G_2 are conjugate. We claim that two distinct points in $\mathrm{Fix}(G)$ are not in the same Γ-orbit. If this is not the case, then there is $\gamma \notin G$ such that $\gamma G\gamma^{-1} = G$. This contradicts the statement at the end of the proof of [25, Prop. 3.11]. (This uses the assumption that there is $v \in \mathrm{Ram}(D)$ for which D_v is a division algebra.) Overall, we see that the map $\mathcal{S}(\Gamma) \to \mathcal{G}(\Gamma)$, $z \mapsto \Gamma_z$, is well defined and d-to-1.

When $A = \mathbb{F}_q[T]$, the number of isomorphism classes of Drinfeld–Stuhler modules over $\mathbb{C}_\infty$ with $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^d}^\times$ is equal to $\#\mathcal{S}(\Gamma)$, since in this case the double coset set of Proposition 3.7 is in natural bijection with $\Gamma\backslash\Omega^d$. Thus, we need to show that $\#\mathcal{G}(\Gamma) = d^{\#\mathrm{Ram}(D)-1}$. This last equality is proved in [25, Prop. 3.11] under the assumptions of the proposition. □

Remark 4.14 Let $\mathcal{B}^d$ be the geometric realization of the Bruhat–Tits building of $\mathrm{PGL}_d(F_\infty)$; see [7, Ch. 3]. Let $\lambda : \Omega^d \to \mathcal{B}^d$ be the $\mathrm{GL}_d(F_\infty)$-equivariant map defined in [7, pp. 64–65]. One can show that the special points in $\mathrm{Fix}(G)$ are conjugate under a natural action of $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ and map to the same vertex of $\mathcal{B}^d$ under λ. The action of Γ on $\mathcal{B}^d$ is studied in [25].

5 Supersingularity

The results on this section are not new. We essentially rephrase some of the results in [22] and [24] for $\mathscr{D}$-elliptic sheaves in terms of Drinfeld–Stuhler modules. This reformulation might be useful for the purposes of future reference. Moreover, Examples 5.2 and 5.10 are quite instructive, since in special cases they deduce the main result about the endomorphism rings of supersingular Drinfeld–Stuhler modules by a direct calculation.

In this section we fix a maximal ideal $\mathfrak{p} \lhd A$. Let L be a field extension of $\mathbb{F}_\mathfrak{p}$ of degree m, so L is a finite field of order q^n, where $n = m \cdot \deg(\mathfrak{p})$. Let $\pi = \tau^n$ be the associated Frobenius morphism. With abuse of notation, denote by π also the diagonal matrix $\mathrm{diag}(\pi, \ldots, \pi) \in M_d(L[\tau])$. Note that π is in the center of $M_d(L[\tau])$ since $\tau^n\ell = \ell\tau^n$ for all $\ell \in L$. We assume that the A-field structure $\gamma : A \to L$ factors through the quotient morphism $A \to A/\mathfrak{p}$; in particular, $\mathrm{char}_A(L) = \mathfrak{p}$.

As we will see, the theory of Drinfeld–Stuhler modules over L differs considerably depending on whether D ramifies at $\mathfrak{p}$ or not. (Note that this difference already appeared in Lemma 2.5.)

5.1 Case 1: $\mathfrak{p} \notin \mathrm{Ram}(D)$

Theorem 5.1 *Let ϕ be a Drinfeld–Stuhler O_D-module defined over L. Since π commutes with $\phi(O_D)$, we have $\pi \in \mathrm{End}_L(\phi)$. Let $\widetilde{F} := F(\pi)$ be the subfield of $D' := \mathrm{End}_L(\phi) \otimes_A F$ generated over F by π. Then:*

1. *$[\widetilde{F} : F]$ divides d, and ∞ does not split in $\widetilde{F}/F$.*
2. *Let $\widetilde{\infty}$ be the unique place of $\widetilde{F}$ over ∞. There is a unique prime $\tilde{\mathfrak{p}} \neq \widetilde{\infty}$ of $\widetilde{F}$ that divides π. Moreover, $\tilde{\mathfrak{p}}$ lies above $\mathfrak{p}$.*
3. *D' is a central division algebra over $\widetilde{F}$ of dimension $(d/[\widetilde{F} : F])^2$ and with invariants*

$$\mathrm{inv}_{\tilde{v}}(D') = \begin{cases} -[\widetilde{F} : F]/d & \text{if } \tilde{v} = \widetilde{\infty}, \\ [\widetilde{F} : F]/d & \text{if } \tilde{v} = \tilde{\mathfrak{p}}, \\ -[\widetilde{F}_{\tilde{v}} : F_v] \cdot \mathrm{inv}_v(D) & \text{otherwise,} \end{cases}$$

for each place v of F and each place $\tilde{v}$ of $\widetilde{F}$ dividing v.

Proof Observe that $D' \cong \mathrm{End}(M(\phi) \otimes_A F)^{\mathrm{opp}}$. The theorem then follows from [22, (9.10)] and the equivalences of Sect. 3. (We should mention that in Section 9 of [22] the $\mathscr{D}$-elliptic sheaves are considered over the algebraic closure of $\mathbb{F}_\mathfrak{p}$. On the other hand, the arguments in that section apply also over L with our choice of $(\widetilde{F}, \pi)$ in place of a "φ-pair" in [22], since theorem A.6 in [22] can be proved for $(\widetilde{F}, \pi)$ as in [21, Sect. 2.2].) □

Example 5.2 Let $A = \mathbb{F}_q[T]$. Let L be the degree d extension of $A/TA \cong \mathbb{F}_q$. Let ϕ be the Drinfeld–Stuhler O_D-module from Example 2.16. Assume (T) does not divide $\mathfrak{r} := \mathfrak{r}(D)$. Fix a generator h of $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$. Our Drinfeld–Stuhler module ϕ is generated over $\mathbb{F}_q$ by $\phi_T = \mathrm{diag}(\tau^d, \dots, \tau^d) = \pi$, ϕ_h, and ϕ_z, which satisfy the relations

$$\phi_T\phi_h = \phi_h\phi_T, \quad \phi_T\phi_z = \phi_z\phi_T, \quad \phi_z\phi_h = \phi_{h^q}\phi_z, \quad \phi_z^d = \phi_\mathfrak{r} = \mathrm{diag}(\Phi_\mathfrak{r}, \dots, \Phi_\mathfrak{r}).$$

With abuse of notation, for $i \geq 1$ let

$$\tau^i := \mathrm{diag}(\tau^i, \dots, \tau^i) \quad \text{and} \quad h := \mathrm{diag}(h, \dots, h).$$

Define

$$\kappa_i = \phi_z^i \tau^{d-i}, \quad 1 \leq i \leq d-1.$$

Note that, since the image of Φ is in $\mathbb{F}_q[\tau]$, we have

$$\phi_z^i \tau^{d-i} = \tau^{d-i}\phi_z^i. \tag{5.1}$$

In particular, h and κ_i commute with ϕ_z. It is clear that these elements also commute with $\phi_T = \tau^d$. Finally, h obviously commutes with ϕ_h, and so does κ_i:

$$\kappa_i\phi_h = \phi_z^i\tau^{d-i}\phi_h = \phi_z^i\phi_{h^{q^{d-i}}}\tau^{d-i} = \phi_{h^{q^d}}\phi_z^i\tau^{d-i} = \phi_h\kappa_i.$$

We conclude that $E := \mathbb{F}_q[\phi_T, h, \kappa_1, \dots, \kappa_{d-1}] \subseteq \mathrm{End}_L(\phi)$.

Note that h and κ_i do not commute,

$$\kappa_i h = h^{q^{d-i}}\kappa_i = \sigma^{-i}(h)\kappa_i, \tag{5.2}$$

where σ is the Frobenius automorphism in $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$. Let $E_i := \mathbb{F}_q[\phi_T, h, \kappa_i] \subset E$. Since $\mathbb{F}_q[\phi_T, h] \cong \mathbb{F}_{q^d}[T]$, we have $E_i = O_K[\kappa_i]$, where $K = \mathbb{F}_{q^d}(T)$. Denote $D_i = E_i \otimes_A F$.

Combining relation (5.2) with

$$\kappa_i^d = \left(\phi_z^i \tau^{d-i}\right)^d \overset{(5.1)}{=} \phi_z^{di} \tau^{d(d-i)} = \phi_{\mathfrak{r}}^i \phi_T^{d-1} = \phi_{\mathfrak{r}^i T^{d-i}},$$

we see that for i coprime to d we have

$$D_i \cong (K/F, \sigma^{-i}, \mathfrak{r}^i T^{d-i})$$

(see (2.1) for the notation). By [27, (30.4)], for i coprime to d, we have

$$(K/F, \sigma^{-i}, \mathfrak{r}^i T^{d-i}) \cong (K/F, \sigma, \mathfrak{r}^{-1} T).$$

Hence, for $1 \leq i, i' \leq d-1$ coprime to d we have $D_i \cong D_{i'}$, and we denote this cyclic algebra by $\overline{D}$. The invariants of $\overline{D}$ are easy to compute using (2.3):

$$\mathrm{inv}_v(\overline{D}) = \begin{cases} 1/d & \text{if } v = (T), \\ -1/d & \text{if } v = \infty, \\ -\mathrm{inv}_v(D) & \text{otherwise.} \end{cases}$$

Let $D' := \mathrm{End}_L(\phi) \otimes_A F$. By Theorem 4.1, we have $\dim_F(D') \leq d^2$. Since $\dim_F \overline{D} = d^2$, we conclude that $D' \cong \overline{D}$. Note that the invariants of $\overline{D}$ agree with the invariants of D' given by Theorem 5.1, since in this case $\pi \in F$.

Next, we claim that $\mathrm{End}_L(\phi)$ is a maximal A-order in D'. One can argue as follows: The discriminant of $E_1 \subset \mathrm{End}_L(\phi)$ is $(\mathfrak{r}T^{d-1})^{d(d-1)}$ (cf. Example 2.15), so $\mathrm{End}_L(\phi) \otimes_A A_{\mathfrak{p}}$ is a maximal $A_{\mathfrak{p}}$-order in $\overline{D}_{\mathfrak{p}}$ for all $\mathfrak{p} \neq (T)$. On the other hand, the discriminant of E_{d-1} is $(\mathfrak{r}^{d-1}T)^{d(d-1)}$, so $\mathrm{End}_L(\phi) \otimes_A A_T$ is a maximal A_T-order in $\overline{D}_T$. Since an A-order in $\overline{D}$ is maximal if and only if it is locally maximal at all primes $\mathfrak{p} \lhd A$ (see [27, (11.6)]), we conclude that $\mathrm{End}_L(\phi)$ is a maximal order.

Finally, note that $\mathbb{F}_{q^d}^\times \cong \mathrm{Aut}_L(\phi)$. Indeed, $\mathbb{F}_{q^d}^\times \cong \mathbb{F}_q(h)^\times \subseteq \mathrm{Aut}_L(\phi)$, so the equality holds by part (4) of Theorem 4.1.

This example shows that the bounds on the rank of $\mathrm{End}_L(\phi)$ and the order of $\mathrm{Aut}_L(\phi)$ given by Theorem 4.1 cannot be improved for fields with nonzero A-characteristic.

Proposition 5.3 *Let ϕ be a Drinfeld–Stuhler O_D-module over L. The following are equivalent:*

1. $\dim_F(\mathrm{End}(\phi) \otimes_A F) = d^2$;
2. *some power of π lies in A;*
3. *there is a unique prime $\tilde{\mathfrak{p}}$ in $\widetilde{F}$ lying over $\mathfrak{p}$;*
4. $\phi[\mathfrak{p}]$ *is connected.*

Proof Let L' be a finite extension of L of degree c. The Frobenius of L' is π^c. Applying Theorem 5.1, we see that $\dim_F(\mathrm{End}_{L'}(\phi) \otimes_A F) = d^2$ is equivalent to $F(\pi^c) = F$, and since π is integral over A, this last condition is equivalent to $\pi^c \in A$. This shows that (1) and (2) are equivalent.

Assume (2), i.e., $\pi^c \in A$ for some $c \geq 1$. By Theorem 5.1, $\mathrm{ord}_{\mathfrak{p}}(\pi^c) \neq 0$. This implies $\mathrm{ord}_{\mathfrak{P}}(\pi^c) \neq 0$ for any prime $\mathfrak{P}$ in $\widetilde{F}$ lying over $\mathfrak{p}$, and hence also $\mathrm{ord}_{\mathfrak{P}}(\pi) \neq 0$. Applying Theorem 5.1 again, we conclude that $\mathfrak{P} = \tilde{\mathfrak{p}}$ is unique, which is (3). To prove (3)$\Rightarrow$(2), let $f = \mathrm{Nr}_{\widetilde{F}/F}(\pi)$. We have $\mathrm{ord}_{\mathfrak{p}}(f) > 0$ and $\mathrm{ord}_{\mathfrak{p}'}(f) = 0$ for any prime $\mathfrak{p}' \lhd A$ not equal to $\mathfrak{p}$. Let $\mathrm{ord}_{\tilde{\mathfrak{p}}}(\pi) = u$ and $\mathrm{ord}_{\tilde{\mathfrak{p}}}(f) = w$. The element $\pi^w/f^u \in \widetilde{F}$ has no zeros or poles away from $\widetilde{\infty}$, since $\tilde{\mathfrak{p}}$ is the unique prime over $\mathfrak{p}$ by assumption. This implies that π^w/f^u lies in the algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$ in $\widetilde{F}$. Therefore, $\pi^{w\kappa} = f^{u\kappa} \in A$, where $\kappa = \#\mathbb{F} - 1$.

Assume (2). Then π^c generates $\mathfrak{p}^h$ for some $c, h \geq 1$. This implies that $\phi[\mathfrak{p}]$ is connected, since $\phi[\mathfrak{p}] \subseteq \phi[\mathfrak{p}^h] = \ker(\pi^c)$, and $\ker(\pi^c)$ is obviously connected. Thus, (2)⇒(4). Conversely, assume $\phi[\mathfrak{p}]$ is connected. Then $\phi[\mathfrak{p}^h]$ is connected for all $h \geq 1$. Choose h such that $\mathfrak{p}^h = (a)$ is principal. The assumption that $\phi[a]$ is connected is equivalent to the action of τ on $M(\phi)/aM(\phi)$ being nilpotent, i.e., $\tau^r M(\phi) \subset aM(\phi) \subset \mathfrak{p}M(\phi)$ for all large enough integers r; cf. [15, Thm. 5.9]. By [24, Sect. 6], this last condition implies that $\dim_F(\mathrm{End}(\phi) \otimes_A F) = d^2$. Hence, (4)⇒(1). □

Definition 5.4 A Drinfeld–Stuhler O_D-module ϕ over $\overline{\mathbb{F}}_\mathfrak{p}$ satisfying the equivalent conditions of Proposition 5.3 is called *supersingular*. (In particular, the Drinfeld–Stuhler module ϕ in Example 5.2 is supersingular.)

Theorem 5.5 *Let ϕ be a supersingular Drinfeld–Stuhler O_D-module over $\overline{\mathbb{F}}_\mathfrak{p}$. We have:*

1. $\mathrm{End}(\phi)$ *is a maximal A-order in* $\mathrm{End}(\phi) \otimes F$;
2. *ϕ can be defined over the extension of $\mathbb{F}_\mathfrak{p}$ of degree $d \cdot \#\mathrm{Pic}(A)$;*
3. *the number of isomorphism classes of supersingular Drinfeld–Stuhler O_D-modules over $\overline{\mathbb{F}}_\mathfrak{p}$ is equal to the class number of* $\mathrm{End}(\phi)$;
4. *all supersingular Drinfeld–Stuhler O_D-modules are isogenous over $\overline{\mathbb{F}}_\mathfrak{p}$.*

Proof (1) and (3) are proved in [24, Thm. 6.2], (2) follows from [23, Sect. 5], and (4) follows from [22, (9.13)]. □

Similar to Remark 4.2, when $O_D = M_d(A)$, the statements of Theorem 5.5 are equivalent to some well-known facts about the endomorphism rings of supersingular Drinfeld A-modules of rank d; cf. [12].

5.2 Case 2: $\mathfrak{p} \in \mathrm{Ram}(D)$

We will make a stronger assumption that D is not just ramified at $\mathfrak{p}$, but, in fact, $D_\mathfrak{p} := D \otimes_F F_\mathfrak{p}$ is a division algebra.

Lemma 5.6 *If $D_\mathfrak{p}$ is a division algebra, then a Drinfeld–Stuhler O_D-module over a field L of A-characteristic $\mathfrak{p}$ is necessarily supersingular, i.e., $\phi[\mathfrak{p}]$ is connected.*

Proof The proof is essentially the same as of the analogous fact for quaternionic abelian surfaces; cf. [28, Lem. 4.1].

For each integer $n \geq 0$, let $H_n := \phi[\mathfrak{p}^n](L^{\mathrm{sep}})$. For $n' \geq n$ we have the inclusion $H_n \subset H_{n'}$ compatible with the left O_D-module structures. We define the Tate module of ϕ at $\mathfrak{p}$ as

$$T_\mathfrak{p}(\phi) = \mathrm{Hom}_{A_\mathfrak{p}}\left(F_\mathfrak{p}/A_\mathfrak{p}, \varinjlim_n H_n\right).$$

$T_\mathfrak{p}(\phi)$ is a free $A_\mathfrak{p}$-module of rank $\leq d^2$; cf. [1,15].

It is enough to show that $T_\mathfrak{p}(\phi) = 0$. Let $V_\mathfrak{p}(\phi) = T_\mathfrak{p}(\phi) \otimes_{A_\mathfrak{p}} F_\mathfrak{p}$. The division algebra $D_\mathfrak{p}$ acts on $V_\mathfrak{p}(\phi)$. If $V_\mathfrak{p}(\phi) \neq 0$, then for $0 \neq v \in V_\mathfrak{p}(\phi)$ the $F_\mathfrak{p}$-vector subspace $D_\mathfrak{p} v$ has to have dimension d^2, as $xv = 0$ implies $x^{-1}(xv) = 1v = v = 0$, contrary to the assumption. On the other hand, for any $a \in \mathfrak{p}$, since $\partial\phi_a = 0$, we have $\phi_a \in M_d(L[\tau])\tau$. Hence, $\phi[\mathfrak{p}]$ is not a reduced scheme, which forces $\dim V_\mathfrak{p}(\phi) < d^2$. □

The number of isomorphism classes of supersingular Drinfeld–Stuhler O_D-module over $\overline{\mathbb{F}}_\mathfrak{p}$ is not finite, unlike the case when $\mathfrak{p} \notin \mathrm{Ram}(D)$. To get finiteness results, and also for certain problems involving moduli spaces, one imposes further restrictions on Drinfeld–Stuhler modules in terms of the action of O_D on the tangent space. For the rest of this subsection we assume that $\mathrm{inv}_\mathfrak{p}(D) = 1/d$.

Let ϕ be a Drinfeld–Stuhler O_D-module over $k := \overline{\mathbb{F}}_\mathfrak{p}$. Note that $\partial_\phi : O_D \to M_d(k)$ factors through $O_D/\mathfrak{p}$. Denote by $\mathbb{F}_\mathfrak{p}^{(d)}$ the degree d extension of $\mathbb{F}_\mathfrak{p}$, and $|\mathfrak{p}| := \#\mathbb{F}_\mathfrak{p}^{(d)}$. It is easy to deduce from [27, Sect. 14] or [21, Appendix A.2] that

$$\begin{aligned} O_D/\mathfrak{p} &\cong \mathbb{F}_\mathfrak{p}^{(d)} \oplus \mathbb{F}_\mathfrak{p}^{(d)}\Pi \oplus \cdots \oplus \mathbb{F}_\mathfrak{p}^{(d)}\Pi^{d-1}, \\ &\Pi^d = 0, \quad \Pi\alpha = \alpha^{|\mathfrak{p}|}\Pi. \end{aligned}$$

Note that Π generates a two-sided ideal of $O_D/\mathfrak{p}$, so, in particular, $O_D/\mathfrak{p} \ncong M_d(\mathbb{F}_\mathfrak{p})$ as was mentioned in Remark 2.6. Fix an embedding $\mathbb{F}_\mathfrak{p}^{(d)} \hookrightarrow k$. Via ∂_ϕ, the submodule $\mathbb{F}_\mathfrak{p}^{(d)}$ of $O_D/\mathfrak{p}$ acts on a d-dimensional k-vector space V. It is easy to see that V canonically decomposes into a direct sum $V = V_1 \oplus \cdots \oplus V_d$, where

$$V_i := \left\{ v \in V \mid \partial_\phi(\alpha)v = \alpha^{|\mathfrak{p}|^i} v \text{ for all } \alpha \in \mathbb{F}_\mathfrak{p}^{(d)} \right\}, \quad 0 \le i \le d-1.$$

The following definition is motivated by [9,28] and [16].

Definition 5.7 The *type* of ϕ is the ordered d-tuple $\mathbf{t}(\phi) = (\dim_k(V_0), \dots, \dim_k(V_{d-1}))$ of nonnegative integers. We say that

- ϕ is *special* if $\mathbf{t}(\phi) = (1, 1, \dots, 1)$.
- ϕ is *exceptional* if $\partial_\phi(\Pi) = 0$.

Example 5.8 Let $A = \mathbb{F}_q[T]$, $d = 2$, and $\mathfrak{r}(D) = T(T-1)$. Let D and O_D be as in Example 2.16. Thus, D be the unique quaternion algebra over F ramified at T and $T-1$ only, and

$$O_D = \mathbb{F}_{q^2}[T, z]/(z^2 - T(T-1)),$$

$$zT = Tz, \quad T\alpha = \alpha T, \quad z\alpha = \alpha^q z \text{ for } \alpha \in \mathbb{F}_{q^2}.$$

Let $\mathfrak{p} = T$. Then $O_D/\mathfrak{p} = \mathbb{F}_{q^2}[z]/z^2$, with $z\alpha = \alpha^q z$ for $\alpha \in \mathbb{F}_\mathfrak{p}^{(2)} \cong \mathbb{F}_{q^2}$. Hence, z plays the role of Π.

Define $\phi : O_D \to M_2(k[\tau])$ by

$$\phi_T = \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix}, \qquad \phi_z = \begin{pmatrix} 0 & 1 \\ \tau^2(\tau^2-1) & 0 \end{pmatrix}, \qquad \phi_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^q \end{pmatrix}, \quad \alpha \in \mathbb{F}_{q^2}.$$

It is easy to check that ϕ is a Drinfeld–Stuhler O_D-module which is special but not exceptional.

Now define $\phi : O_D \to M_2(k[\tau])$ by

$$\phi_T = \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix}, \qquad \phi_z = \begin{pmatrix} 0 & \tau \\ \tau(\tau^2-1) & 0 \end{pmatrix}, \qquad \phi_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad \alpha \in \mathbb{F}_{q^2}.$$

This module is exceptional but not special. Its type is $(2, 0)$.

Finally, define $\phi : O_D \to M_2(k[\tau])$ by

$$\phi_T = \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix}, \qquad \phi_z = \begin{pmatrix} 0 & \tau^2 \\ \tau^2 - 1 & 0 \end{pmatrix}, \qquad \phi_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^q \end{pmatrix}, \quad \alpha \in \mathbb{F}_{q^2}.$$

This module is exceptional and special. (Such modules are called *superspecial*.)

We remark that, for $d = 2$, if ϕ is not exceptional then it is special. Indeed, the commutation relation $\Pi\alpha = \alpha^{|\mathfrak{p}|}\Pi$ implies that $\partial_\phi(\Pi)V_0 \subset V_1$ and vice versa. Hence, if $\partial_\phi(\Pi) \neq 0$, then $V_0 \neq V$ and $V_1 \neq V$, so both must be one-dimensional.

Special Drinfeld–Stuhler modules, or rather their $\mathscr{D}$-elliptic sheaf counterparts, play an important role in the construction of moduli schemes of $\mathscr{D}$-elliptic sheaves over $\mathrm{Spec}(A_\mathfrak{p})$ with nice geometric properties; see [16]. On the other hand, endomorphism rings of exceptional Drinfeld–Stuhler modules have properties similar to the properties of supersingular modules in Theorem 5.5:

Theorem 5.9 *Let ϕ be an exceptional Drinfeld–Stuhler O_D-module over k of type* $\mathbf{t}$. *Then* $\mathrm{End}(\phi)$ *is a hereditary A-order in the central division algebra $\overline{D}$ with invariants*

$$\mathrm{inv}_v(\overline{D}) = \begin{cases} -1/d & \text{if } v = \infty, \\ 0 & \text{if } v = \mathfrak{p}, \\ -\mathrm{inv}_v(D) & \text{if } v \neq \mathfrak{p}, \infty. \end{cases}$$

This order is maximal at every finite place $v \neq \mathfrak{p}$, and at $\mathfrak{p}$ it is isomorphic to a hereditary order uniquely determined by $\mathbf{t}$. *Moreover, the number of isomorphism classes of exceptional Drinfeld–Stuhler O_D-module over k of type* $\mathbf{t}$ *is equal to the class number of* $\mathrm{End}(\phi)$.

Proof See [24], where the reader will also find the precise relationship between the type $\mathbf{t}$ and the hereditary order at $\mathfrak{p}$. For example, $\mathrm{End}(\phi)$ is maximal at $\mathfrak{p}$ if and only if, up to a cyclic permutation, $\mathbf{t} = (d, 0, \ldots, 0)$. □

Example 5.10 Let ϕ be the exceptional Drinfeld–Stuhler module over k of type $(2, 0)$ from Example 5.8:

$$\phi_T = \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix}, \qquad \phi_z = \begin{pmatrix} 0 & \tau \\ \tau(\tau^2 - 1) & 0 \end{pmatrix}, \qquad \phi_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad \alpha \in \mathbb{F}_{q^2}.$$

Fix a generator β of $\mathbb{F}_T^{(2)} \cong \mathbb{F}_{q^2}$ over $\mathbb{F}_q$, and let

$$h := \begin{pmatrix} \beta & 0 \\ 0 & \beta^q \end{pmatrix}, \qquad \kappa := \phi_z\tau = \tau\phi_z = \begin{pmatrix} 0 & \tau^2 \\ \tau^2(\tau^2 - 1) & 0 \end{pmatrix}.$$

By a straightforward calculation one checks that $E := \mathbb{F}_q[\phi_T, h, \kappa] \subseteq \mathrm{End}(\phi)$ and

$$\kappa h = h^q\kappa, \qquad \kappa^2 = \phi_{T^2(T-1)}.$$

It is easy to see from this that $\overline{D} := E \otimes_A F$ is the quaternion algebra over F ramified at $(T - 1)$ and ∞. Moreover, as in Example 5.2, one can check that $\mathrm{End}(\phi)$ is a maximal A-order in $\overline{D}$.

6 Fields of moduli

The main results of this section are about the fields of moduli of Drinfeld–Stuhler modules. As an auxiliary tool, we will need a Hilbert's 90th-type theorem for $\mathrm{GL}_d(L^{\mathrm{sep}}[\tau])$, which we prove first. Throughout the section, L is an arbitrary A-field with $\mathrm{char}_A(L) \nmid \mathfrak{r}(D)$.

Lemma 6.1 *We have:*

1. *Every left ideal of $L[\tau]$ is principal.*
2. *Every finitely generated torsion-free left $L[\tau]$-module is free.*

Proof (1) follows from the existence of the right division algorithm for $L[\tau]$ (see [14, Cor. 1.6.3]), and (2) essentially follows from the same fact (see [14, Cor. 5.4.9]). □

Let K be a finite Galois extension of L of degree n. Let $\sigma_1, \sigma_2, \dots, \sigma_n$ be the elements of $G := \mathrm{Gal}(K/L)$. The Galois group G acts on $K[\tau]$ via the obvious action on the coefficients of polynomials, and it acts on the ring $M_d(K[\tau])$ by acting on the entries of matrices. Let $M := K[\tau]^d$ be the free left $K[\tau]$-module of rank d. Then $\mathrm{GL}_d(K[\tau])$ can be identified with the group $\mathrm{Aut}_{K[\tau]}(M)$ of automorphism of M, where $g \in \mathrm{GL}_d(K[\tau])$ acts on M from the right as on row vectors. (Of course, $\mathrm{GL}_d(K[\tau])$ also acts on M from the left as on column vectors, but that action is not $K[\tau]$-linear.) From this identification it is easy to see the validity of the following:

Lemma 6.2 *If $v_1, \dots, v_d \in M$ form a left $K[\tau]$-basis of M, then the matrix S whose rows are $v_1, \dots, v_d$ is in $\mathrm{GL}_d(K[\tau])$. Conversely, the rows of $S \in \mathrm{GL}_d(K[\tau])$ form a left $K[\tau]$-basis of M.*

Remark 6.3 There are matrices in $M_d(K[\tau])$ whose rows are left linearly independent, but whose columns are left linearly dependent over $K[\tau]$, e.g., $\begin{pmatrix} 1 & \tau \\ \alpha+\tau & \tau(\alpha+\tau) \end{pmatrix}$, where $\alpha \in K$ is such that $\alpha^q \neq \alpha$.

Lemma 6.4 *The inclusion $L[\tau] \subset K[\tau]$ makes $K[\tau]$ into a left $L[\tau]$-module. As such, $K[\tau]$ is a free left $L[\tau]$-module of rank n.*

Proof It is obvious that $K[\tau]$ has no torsion elements for the action of $L[\tau]$. Let $\alpha_1, \dots, \alpha_n \in K$ be an L-basis of K. It is enough to show that $K[\tau] = \sum_{i=1}^n L[\tau]\alpha_i$. By the Dedekind's theorem on the independence of characters, $\{\alpha_1, \dots, \alpha_n\}$ form an L-basis of K if and only if $\det(\sigma_i\alpha_j) \neq 0$. On the other hand, $\det(\sigma_i\alpha_j) \neq 0$ if and only if $\det(\sigma_i\alpha_j)^{q^r} = \det(\sigma_i\alpha_j^{q^r}) \neq 0$ for any $r \geq 0$. Hence, $\{\alpha_1^{q^r}, \dots, \alpha_n^{q^r}\}$ is also an L-basis of K. Let $f = a_0 + a_1\tau + \dots + a_k\tau^k \in K[\tau]$. For each a_i we can find $b_{i,1}, \dots, b_{i,n} \in L$ such that $\sum_{j=1}^n b_{i,j}\alpha_j^{q^i} = a_i$. Thus, $f = \sum_{j=1}^n g_j\alpha_j$, where $g_j := \sum_{i=1}^k b_{i,j}\tau^i \in L[\tau]$. □

Definition 6.5 We say that G acts on M by *semi-linear automorphisms* (cf. [3, p. 110]), $G \times M \to M$, $(\sigma, m) \mapsto \sigma * m$, if for all $m, m' \in M$, $\sigma \in G$, and $\lambda \in K[\tau]$ we have

(i) $\sigma * (m + m') = \sigma * m + \sigma * m'$,
(ii) $\sigma * (\lambda m) = \sigma\lambda \cdot \sigma * m$,

where $\sigma\lambda$ denotes the usual action of G on $K[\tau]$ and the dot denotes the action of $K[\tau]$ on M. Let

$$M^G := \{m \in M \mid \sigma * m = m \text{ for all } \sigma \in G\}.$$

It is easy to see that M^G is a left $L[\tau]$-module.

Lemma 6.6 *The left $L[\tau]$-module M^G is free of rank d, i.e., $M^G \cong L[\tau]^d$. Moreover, the map $K \otimes_L M^G \to M$, $\alpha \otimes m \mapsto \alpha m$, is an isomorphism.*

Proof Since every left ideal of $L[\tau]$ is principal (Lemma 6.1), every submodule of a free left $L[\tau]$-module of finite rank is also free of finite rank (cf. [27, (2.44)]). On the other hand, by Lemma 6.4, the left $L[\tau]$-module M is free of finite rank. Hence, the $L[\tau]$-submodule M^G of M is also free of finite rank. To show that the rank of M^G over $L[\tau]$ is d, it is enough to show that the map $K \otimes_L M^G \to M$, $\alpha \otimes m \mapsto \alpha m$, is an isomorphism. This last isomorphism follows from the Galois descent for vector spaces; see [3, Lem. III.8.21]. □

Proposition 6.7 *Let $c : G \to \mathrm{GL}_d(K[\tau])$, $\sigma \mapsto c_\sigma$, be a map which satisfies $c_{\sigma\delta} = \sigma(c_\delta)c_\sigma$ for all $\sigma, \delta \in G$. Then there is a matrix $S \in \mathrm{GL}_d(K[\tau])$ such that $c_\sigma = (\sigma S)^{-1}S$ for all $\sigma \in G$.*

Proof Define a (twisted) action of G on M:

$$\sigma * m = (\sigma m)c_\sigma \quad \text{for all } m \in M, \sigma \in G.$$

One easily checks that $(\sigma\delta) * m = \sigma * (\delta * m)$ for all $\sigma, \delta \in G$ and $m \in M$, so this is indeed an action. Moreover, this action is semi-linear. Using Lemma 6.6, we can choose a basis $v_1, \ldots, v_d$ of the left $L[\tau]$-module $M^G \cong L[\tau]^d$ such that $\sum_{i=1}^d K[\tau]v_i = M$. Since M is a free left $K[\tau]$-module of rank d, the elements $v_1, \ldots, v_d$ form a left $K[\tau]$-basis of M. Let S be the matrix whose rows are $v_1, \ldots, v_d$. By Lemma 6.2, $S \in \mathrm{GL}_d(K[\tau])$. The relations

$$v_i = \sigma * v_i = (\sigma v_i)c_\sigma \text{ for all } i = 1, \ldots, d,$$

are equivalent to the matrix equality $S = (\sigma S)c_\sigma$ for all $\sigma \in G$, and this implies the claim of the lemma. □

Lemma 6.8 *Let ϕ be a Drinfeld–Stuhler O_D-module over K with $\mathrm{Aut}_K(\phi) \cong \mathbb{F}_{q^r}^\times$ (cf. Theorem 4.1). Then*

$$\partial : \mathrm{Aut}_K(\phi) \to \mathrm{GL}_d(K)$$

gives an isomorphism from the group $\mathrm{Aut}_K(\phi)$ to the group $\mathcal{A} := \{\mathrm{diag}(\alpha, \ldots, \alpha) \mid \alpha \in \mathbb{F}_{q^r}^\times\}$.

Proof By Lemma 2.5, $\partial(\mathrm{Aut}_K(\phi))$ lies in the center of $\mathrm{GL}_d(K)$. Since the center of $\mathrm{GL}_d(K)$ consists of scalar matrices, and the $(q^r - 1)$th roots of 1 in K are the elements of $\mathbb{F}_{q^r}^\times$, the restriction of ∂ to $\mathrm{Aut}_K(\phi)$ is indeed a homomorphism into $\mathcal{A}$. Since $\mathrm{Aut}_K(\phi) \cong \mathcal{A}$, to prove that ∂ is an isomorphism it is enough to prove that it is injective. Let $h := q^r - 1$. Assume $\alpha \in \mathrm{Aut}_K(\phi)$ is such that $\partial(\alpha) = 1$. Then we can write $\alpha = 1 + \sum_{i=1}^n B_i\tau^n$ for some $n \geq 1$. Suppose not all B_i are zero, and let m be the smallest index such that $B_m \neq 0$. Then

$$1 = \alpha^h = 1 + hB_m\tau^m + \cdots,$$

which implies $hB_m = 0$. Since h is coprime to the characteristic of K, we must have $B_m = 0$, which is a contradiction. □

Remark 6.9 It is not generally true that the elements of $\mathrm{Aut}_K(\phi)$ are scalar matrices in $\mathrm{GL}_d(K[\tau])$. For example, suppose $d = 2$, $\mathrm{diag}(\alpha, \alpha) \in \mathrm{Aut}_K(\phi)$, and $\alpha \notin \mathbb{F}_q$. Let $S = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(K[\tau])$. Then $\begin{pmatrix} \alpha & (\alpha^q - \alpha)\tau \\ 0 & \alpha \end{pmatrix} \in \mathrm{Aut}_K(\psi)$, where ψ is the Drinfeld–Stuhler module $S\phi S^{-1}$.

Definition 6.10 Let ϕ be a Drinfeld–Stuhler O_D-module over L^{sep}. For $\sigma \in \mathrm{Gal}(L^{\mathrm{sep}}/L)$, let ϕ^σ be the composition

$$\phi^\sigma : O_D \xrightarrow{\phi} M_d(L^{\mathrm{sep}}[\tau]) \xrightarrow{\sigma} M_d(L^{\mathrm{sep}}[\tau]).$$

It is easy to check that ϕ^σ is again a Drinfeld–Stuhler O_D-module. We say that L is a *field of moduli for* ϕ if for all $\sigma \in \mathrm{Gal}(L^{\mathrm{sep}}/L)$ the Drinfeld–Stuhler module ϕ^σ is isomorphic to ϕ.

If L is a field of definition for ϕ, then L is obviously a field of moduli.

Theorem 6.11 *Let ϕ be a Drinfeld–Stuhler O_D-module over L^{sep} with $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^r}^\times$. Assume L is a field of moduli for ϕ. If d and $q^r - 1$ are coprime, then L is a field of definition for ϕ.*

Proof We can find a finite Galois extension K of L such that ϕ is defined over K and all isomorphisms of ϕ to ϕ^σ for every $\sigma \in \mathrm{Gal}(K/L)$ are defined over K. (Take, for example, K such that ϕ and $\phi[a]$ are defined over K, where $a \in A$ is coprime with $\mathrm{char}_A(L)$ and $\mathfrak{r}(D)$.) In particular, $\mathrm{Aut}_K(\phi) = \mathrm{Aut}(\phi)$. Denote $G = \mathrm{Gal}(K/L)$. For each $\sigma \in G$, choose an isomorphism $\lambda_\sigma : \phi \to \phi^\sigma$. Then

$$\lambda_{\sigma\delta}\phi\lambda_{\sigma\delta}^{-1} = \phi^{\sigma\delta} = (\phi^\delta)^\sigma = (\lambda_\delta\phi\lambda_\delta^{-1})^\sigma = \sigma(\lambda_\delta)\phi^\sigma\sigma(\lambda_\delta)^{-1} = \sigma(\lambda_\delta)\lambda_\sigma\phi\lambda_\sigma^{-1}(\sigma\lambda_\delta)^{-1}. \tag{6.1}$$

Hence,

$$\lambda_{\sigma\delta} = \sigma(\lambda_\delta)\lambda_\sigma\alpha_{\sigma,\delta} \tag{6.2}$$

with $\alpha_{\sigma,\delta} \in \mathrm{Aut}(\phi)$.

Let $\underline{\det} : \mathrm{GL}_d(K[\tau]) \to K^\times$ be the composition

$$\underline{\det} : \mathrm{GL}_d(K[\tau]) \xrightarrow{\partial} \mathrm{GL}_d(K) \xrightarrow{\det} K^\times.$$

The assumption that d and $q^r - 1$ are coprime, combined with Lemma 6.8, implies that $\underline{\det} : \mathrm{Aut}(\phi) \xrightarrow{\sim} \mathbb{F}_{q^r}^\times$ is an isomorphism. Denote $\mu_\sigma = \underline{\det}(\lambda_\sigma)$ and $h = q^r - 1$. Then $\mu_{\sigma\delta} = \sigma(\mu_\delta)\mu_\sigma\underline{\det}(\alpha_{\sigma,\delta})$, and $\mu_{\sigma\delta}^h = \sigma(\mu_\delta^h)\mu_\sigma^h$. Hence, $G \to K^\times, \sigma \mapsto \mu_\sigma^h$, is a 1-cocycle. By Hilbert's theorem 90 for $K^\times$, there is $b \in K^\times$ such that $\mu_\sigma^h = b/\sigma(b)$ for all $\sigma \in G$. Let a be an element of L^{sep} such that $a^h = b$. The extension $K' := K(a)$ is Galois over L. Put $G^* = \mathrm{Gal}(K'/L)$, and let $\pi : G^* \to G$ be the natural homomorphism. For every $\sigma \in G^*$, we see that $\mu_{\pi(\sigma)}\sigma(a)/a$ is an hth root of unity; hence, there is a unique $\alpha_\sigma \in \mathrm{Aut}(\phi)$ such that $\mu_{\pi(\sigma)}\underline{\det}(\alpha_\sigma) = a/\sigma(a)$.

Put $c_\sigma = \lambda_{\pi(\sigma)}\alpha_\sigma$. Then $c_\sigma : \phi \to \phi^\sigma$ is an isomorphism and $\underline{\det}(c_\sigma) = a/\sigma(a)$. Repeating the calculation (6.1) for c_σ, we arrive at the relations $c_{\sigma\delta} = \sigma(c_\delta)c_\sigma\beta_{\sigma,\delta}$ for some $\beta_{\sigma,\delta} \in \mathrm{Aut}(\phi)$. But now, taking $\underline{\det}$ of both sides, we have

$$\frac{a}{\sigma\delta(a)} = \frac{\sigma(a)}{\sigma(\delta(a))}\frac{a}{\sigma(a)}\underline{\det}(\beta_{\sigma,\delta}).$$

Thus, $\underline{\det}(\beta_{\sigma,\delta}) = 1$. Since $\underline{\det} : \mathrm{Aut}(\phi) \to K^\times$ is injective, we must have $\beta_{\sigma,\delta} = 1$. Therefore, $c_{\sigma\delta} = \sigma(c_\delta)c_\sigma$ for all $\sigma, \delta \in G^*$. By Proposition 6.7, there is $S \in \mathrm{GL}_d(K'[\tau])$ such that $c_\sigma = (\sigma S)^{-1}S$ for all $\sigma \in G^*$. Put $\psi = S\phi S^{-1}$; this is a Drinfeld–Stuhler module isomorphic to ϕ over K'. For any $\sigma \in G^*$ we have

$$\psi^\sigma = (S\phi S^{-1})^\sigma = (\sigma S)(\phi^\sigma)(\sigma S^{-1}) = (\sigma S)(c_\sigma\phi c_\sigma^{-1})(\sigma S)^{-1} = S\phi S^{-1} = \psi,$$

so ψ is defined over L. □

Remark 6.12 Recall from Theorem 4.1 that $\mathrm{Aut}(\phi) \cong \mathbb{F}_{q^s}^\times$ for some s dividing d. Therefore, the assumption in Theorem 6.11 can be replaced by a universal but stronger assumption that d and $q^d - 1$ are coprime. Note that if $d = p^e$ is a power of the characteristic of F, then the assumption of Theorem 6.11 is always satisfied. On the other hand, if $d = \ell$ is a prime different from p, then the assumption is satisfied if and only if ℓ does not divide $q - 1$.

Theorem 6.13 *Let ϕ be a Drinfeld–Stuhler O_D-module over L^{sep}. Assume L is a field of moduli for ϕ. Then L is a field of definition for ϕ if and only if $O_D \otimes_A L \cong M_d(L)$.*

Proof If L is a field of definition for ϕ, then $O_D \otimes_A L \cong M_d(L)$ by Lemma 2.5, so the condition is necessary.

Now assume $O_D \otimes_A L \cong M_d(L)$. As in the proof of Theorem 6.11, let K/L be a finite Galois extension such that ϕ is defined over K and all isomorphisms of ϕ to ϕ^σ for every $\sigma \in \mathrm{Gal}(K/L) =: G$ are defined over K. We have $\partial_{\phi,K} : O_D \otimes_A K \xrightarrow{\sim} M_d(K)$. The group G acts on $O_D \otimes_A K$ via its action on K. We fix an element $e \in O_D \otimes_A L$, and consider it as an element of $O_D \otimes_A K$ fixed by G.

Denote

$$V(\phi, e) = \ker(\partial_{\phi,K}(e)) \subset K^d.$$

For each $\sigma \in G$ choose an isomorphism $\lambda_\sigma : \phi \xrightarrow{\sim} \phi^\sigma$, and denote $M_\sigma = \partial(\lambda_\sigma) \in \mathrm{GL}_d(K)$. Then

$$M_\sigma V(\phi, e) = \sigma V(\phi, e), \tag{6.3}$$

where $\sigma V(\phi, e)$ denotes the image of $V(\phi, e)$ under the action of σ on K^d; we prove (6.3) in two steps:

(i) Because $\lambda_\sigma\phi = \phi^\sigma\lambda_\sigma$, we have $M_\sigma\partial_{\phi,K}(e) = \partial_{\phi^\sigma,K}(e)M_\sigma$. Hence, the matrices $\partial_{\phi,K}(e)$ and $\partial_{\phi^\sigma,K}(e)$ have the same rank, which implies $\dim_K V(\phi, e) = \dim_K V(\phi^\sigma, e)$. If $v \in V(\phi, e)$, then $\partial_{\phi,K}(e)(v) = 0$, so $M_\sigma(v) \in V(\phi^\sigma, e)$. This implies that $M_\sigma V(\phi, e) \subseteq V(\phi^\sigma, e)$. Since $\dim_K V(\phi, e) = \dim_K V(\phi^\sigma, e)$, we must have $M_\sigma V(\phi, e) = V(\phi^\sigma, e)$.

(ii) If $v \in V(\phi, e)$, then

$$\sigma(\partial_{\phi,K}(e)(v)) = \partial_{\phi^\sigma,K}(\sigma(e))(\sigma(v)) = \partial_{\phi^\sigma,K}(e)(\sigma(v)) = 0,$$

where we have used the assumption that $\sigma(e) = e$. Thus, $\sigma(v) \in V(\phi^\sigma, e)$, which implies $\sigma V(\phi, e) = V(\phi^\sigma, e)$.

We can choose e such that $V(\phi, e) = K\omega$ is one-dimensional, spanned by some $\omega \in K^d$. (Note that the rank of e does not change whether we consider it as an element of $O_D \otimes_A L \cong M_d(L)$ or $O_D \otimes_A K \cong (O_D \otimes_A L) \otimes_L K \cong M_d(K)$.) Then (6.3) becomes

$$M_\sigma \omega = \mu_\sigma \sigma(\omega) \quad \text{for some} \quad \mu_\sigma \in K^\times.$$

Using (6.2), we get

$$M_{\sigma\delta} = (\sigma M_\delta) M_\sigma \partial(\alpha_{\sigma,\delta}), \tag{6.4}$$

where $\alpha_{\sigma,\delta} \in \mathrm{Aut}(\phi)$ is the automorphism appearing in (6.2). By Lemma 6.8, $\partial(\alpha_{\sigma,\delta})$ is a scalar matrix, so applying both sides of (6.4) to ω, we get an equality in $K^\times$

$$\mu_{\sigma\delta} = \sigma(\mu_\delta)\mu_\sigma \partial(\alpha_{\sigma,\delta}).$$

At this point one can repeat the argument in the proof of Theorem 6.11, with $\partial(\alpha_{\sigma,\delta})$ playing the role of $\underline{\det}(\alpha_{\sigma,\delta})$, to obtain a Drinfeld–Stuhler module ψ over L, which is isomorphic to ϕ over L^{sep}. □

Remark 6.14 Theorem 6.11 is the analogue for Drinfeld–Stuhler modules of a theorem of Shimura for abelian varieties [30, Thm. 9.5], and Theorem 6.13 is the analogue of a theorem of Jordan for abelian surfaces with quaternionic multiplication [18, Thm. 1.1].

Corollary 6.15 *Let $\Phi : A \to L^{\mathrm{sep}}[\tau]$ be a Drinfeld A-module of rank $r \geq 1$. If L is a field of moduli for Φ, then L is also a field of definition.*

Proof The claim follows from Theorem 6.13 specialized to $O_D = M_r(A)$ and Theorem 2.20. Alternatively, one can arrive at the same conclusion by repeating for Φ the argument in the proof of Theorem 6.11 with $d = 1$ (but arbitrary rank), which becomes simpler since in that case Proposition 6.7 is just the usual Hilbert's 90. □

Remark 6.16 Corollary 6.15 is the analogue of the well-known fact that the fields of moduli for elliptic curves are fields of definition; cf. [31, Prop. I.4.5]. The proof for elliptic curves uses the j-invariant, an invariant which is not available for Drinfeld modules if A is not the polynomial ring or the rank is greater than 2.

In [22], $\mathscr{D}$-elliptic sheaves are defined over any $\mathbb{F}_q$-scheme S. The functor which associates with S the set of isomorphism classes of $\mathscr{D}$-elliptic sheaves over S modulo the action of $\mathbb{Z}$ possesses a coarse moduli scheme $X^{\mathscr{D}}$ which is a smooth proper scheme over $C' := C - \mathrm{Ram}(D) - \{\infty\}$ of relative dimension $(d-1)$; this follows from Theorems 4.1 and 6.1 in [22], combined with the Keel–Mori theorem. Thanks to Theorems 3.2 and 3.4, the fiber of this moduli scheme over a (not necessarily closed) point x of C' is the coarse moduli space of isomorphism classes of Drinfeld–Stuhler O_D-modules over fields L such that $z(\mathrm{Spec}(L)) = x$ (recall from Definition 3.3 that z is the morphism induced by γ).

Corollary 6.17 *Let $z(\mathrm{Spec}(L)) = x$ and $X^{\mathscr{D}}_x$ be the fiber of $X^{\mathscr{D}}$ over the point x of C'.*

1. *Assume d and $q^d - 1$ are coprime. If $O_D \otimes_A L \ncong M_d(L)$, then $X^{\mathscr{D}}_x(L) = \emptyset$.*

2. *Assume $O_D \otimes_A L \cong M_d(L)$ and $X_x^{\mathscr{D}}(L) \neq \emptyset$. Then there is a Drinfeld–Stuhler O_D-module defined over L.*

Proof Given $a \in A - \mathbb{F}_q$ coprime with $\mathfrak{r}(D)$, one can consider the problem of classifying Drinfeld–Stuhler modules with level-a structures, i.e., classifying pairs (ϕ, ι), where ϕ is a Drinfeld–Stuhler O_D-module and ι is an isomorphism $\iota : \phi[a] \cong O_D/aO_D$. This moduli problem is representable; see [22, (5.1)]. Denote the corresponding moduli scheme by $X^{\mathscr{D},a}$. The forgetful map $(\phi, \iota) \mapsto \phi$ gives a Galois covering $X_x^{\mathscr{D},a} \to X_x^{\mathscr{D}}$. Suppose there is an L-rational point P on $X_x^{\mathscr{D}}$. Then a preimage P' of P in $X_x^{\mathscr{D},a}$ is defined over a Galois extension L' of L. Since $X_x^{\mathscr{D},a}$ is a fine moduli scheme, there is a Drinfeld–Stuhler O_D-module ϕ defined over L' which corresponds to P'. For any $\sigma \in \mathrm{Gal}(L'/L)$, the Drinfeld–Stuhler modules ϕ and ϕ^σ are isomorphic over L', since ϕ arises from an L-rational point on $X_x^{\mathscr{D}}$. Hence, L is a field of moduli of ϕ.

If d and $q^d - 1$ are coprime, then Theorem 6.11 and Remark 6.12 imply that ϕ can be defined over L. Now Lemma 2.5 implies that $O_D \otimes_A L \cong M_d(L)$. This proves part (1). Part (2) follows from Theorem 6.13. ◻

Remark 6.18 It is known that in general the fields of moduli for abelian varieties are not necessarily fields of definition.

For example, let B be an indefinite quaternion division algebra over $\mathbb{Q}$, and let X^B be the associated Shimura curve over $\mathbb{Q}$, which is the coarse moduli scheme of abelian surfaces equipped with an action of B. The main result in [19] provides examples of non-Archimedean local fields L failing to split B with $X^B(L) \neq \emptyset$ (see also [18, Sect. 1]); a necessary condition for this phenomenon is that 2 ramifies in B. If we let $A = \mathbb{F}_q[T]$, $F = \mathbb{F}_q(T)$, and $d = 2$, then $X_F^{\mathscr{D}}$ is the function field analogue of X^B; cf. [22,26]. However, examples similar to those constructed by Jordan and Livné do not exist in this setting since for any finite extension L of F_v, $v \in \mathrm{Ram}(D)$, which does not split D we have $X_F^{\mathscr{D}}(L) = \emptyset$ by Theorem 4.1 in [26]. This leaves open the interesting question whether in general the fields of moduli of Drinfeld–Stuhler modules are fields of definition.

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References

1. Anderson, G.: *t*-Motives. Duke Math. J. **53**(2), 457–502 (1986)
2. Becker, M.F., MacLane, S.: The minimum number of generators for inseparable algebraic extensions. Bull. Am. Math. Soc. **46**, 182–186 (1940)
3. Berhuy, G.: An Introduction to Galois Cohomology and Its Applications, London Mathematical Society Lecture Note Series, vol. 377
4. Blum, A., Stuhler, U.: Drinfeld modules and elliptic sheaves. In: Vector Bundles on Curves—new Directions (Cetraro, 1995), Lecture Notes in Math., vol. 1649, pp. 110–193. Springer, Berlin (1997)
5. Böckle, G., Gvirtz, D.: Division algebras and maximal orders for given invariants. LMS J. Comput. Math. **19**, no. suppl. A, 178–195 (2016)
6. Bornhofen, M., Hartl, U.: Pure Anderson motives and abelian τ-sheaves. Math. Z. **268**(1–2), 67–100 (2011)
7. Deligne, P., Husemöller, D.: Survey of Drinfeld modules. In: Current Trends in Arithmetical Algebraic Geometry (Arcata, Calif., 1985), Contemp. Math., vol. 67, Am. Math. Soc., Providence, RI, pp. 25–91 (1987)
8. Drinfeld, V.G.: Elliptic modules. Mat. Sb. (N.S.) **94**, 594–627 (1974)
9. Drinfeld, V.G.: Coverings of *p*-adic symmetric domains. Funkcional. Anal. i Priložen. **10**(2), 29–40 (1976)

10. Drinfeld, V.G.: Commutative subrings of certain noncommutative rings. Funkcional. Anal. i Priložen. **11**(1), 11–14, 96 (1977)
11. Gekeler, E.-U.: Zur Arithmetik von Drinfeld–Moduln. Math. Ann. **262**(2), 167–182 (1983)
12. Gekeler, E.-U.: On finite Drinfeld modules. J. Algebra **141**(1), 187–203 (1991)
13. Gekeler, E.-U., van der Put, M., Reversat, M., Van Geel, J. (eds.): Drinfeld Modules, Modular Schemes and Applications. World Scientific Publishing Co., Inc, River Edge (1997)
14. Goss, D.: Basic structures of function field arithmetic. In: Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 35. Springer, Berlin (1996)
15. Hartl, U.: Isogenies of Abelian Anderson *A*-modules and *A*-motives (preprint)
16. Hausberger, T.: Uniformisation des variétés de Laumon-Rapoport-Stuhler et conjecture de Drinfeld–Carayol. Ann. Inst. Fourier (Grenoble) **55**, 1285–1371 (2005)
17. Hayes, D.: Explicit class field theory in global function fields. In: Studies in Algebra and Number Theory. Adv. Math. Suppl. Stud., vol. 6, Academic Press, New York, pp. 173–217 (1979)
18. Jordan, B.: Points on Shimura curves rational over number fields. J. Reine Angew. Math. **371**, 92–114 (1986)
19. Jordan, B., Livné, R.: Local Diophantine properties of Shimura curves. Math. Ann. **270**(2), 235–248 (1985)
20. Lafforgue, L.: Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson, Astérisque, no. 243, ii+329 (1997)
21. Laumon, G.: Cohomology of Drinfeld modular varieties. Part I, Cambridge Studies in Advanced Mathematics, vol. 41
22. Laumon, G., Rapoport, M., Stuhler, U.: $\mathscr{D}$-elliptic sheaves and the Langlands correspondence. Invent. Math. **113**(2), 217–338 (1993)
23. Papikian, M.: Modular varieties of $\mathscr{D}$-elliptic sheaves and the Weil–Deligne bound. J. Reine Angew. Math. **626**, 115–134 (2009)
24. Papikian, M.: Endomorphisms of exceptional $\mathscr{D}$-elliptic sheaves. Math. Z. **266**(2), 407–423 (2010)
25. Papikian, M.: On finite arithmetic simplicial complexes. Proc. Am. Math. Soc. **139**(1), 111–124 (2011)
26. Papikian, M.: Local diophantine properties of modular curves of $\mathscr{D}$-elliptic sheaves. J. Reine Angew. Math. **664**, 115–140 (2012)
27. I. Reiner, Maximal orders, London Mathematical Society Monographs. New Series, vol. 28, The Clarendon Press, Oxford University Press, Oxford: Corrected reprint of the 1975 original. With a foreword by M.J. Taylor (2003)
28. Ribet, K.: Bimodules and abelian surfaces, Algebraic number theory, Adv. Stud. Pure Math., vol. 17. Academic Press, Boston, MA, pp. 359–407 (1989)
29. Serre, J.-P.: Local fields. In: Graduate Texts in Mathematics, vol. 67
30. Shimura, G.: On the real points of an arithmetic quotient of a bounded symmetric domain. Math. Ann. **215**, 135–164 (1975)
31. Silverman, J.: Advanced topics in the arithmetic of elliptic curves. In: Graduate Texts in Mathematics, vol. 151. Springer, New York (1994)
32. Taelman, L.: On *t*-motifs, 2007, Thesis (Ph.D.)—The University of Groningen
33. van der Heiden, G.-J.: Weil pairing for Drinfeld modules. Monatsh. Math. **143**(2), 115–143 (2004)
34. Vignéras, M.-F.: Arithmétique des algèbres de quaternions. Lecture Notes in Mathematics, vol. 800. Springer, Berlin (1980)