

KUIPER'S THEOREM FOR HILBERT MODULES

Dedicated to

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in Honor of their Sixtieth Birthdays

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Let A be a C^* -algebra with a countable approximate identity, let K denote the C^* -algebra of compact operators on a separable Hilbert space, and denote by $M(K \otimes A)$ the multiplier algebra of $K \otimes A$. The purpose of this note is to prove that the unitary group of $M(K \otimes A)$ is a contractible topological space. The result is due to Mingo [7] in the case that A is unital, although the proof given here is different and simpler. The first theorem of this type is due to Kuiper [5], who showed that the unitary group of a separable, infinite dimensional Hilbert space is contractible: the relationship between this result and ours is that $M(K \otimes A)$ can be viewed as the algebra of adjointable operators on the "standard" Hilbert A -module $\ell^2 A$ (see [4]). As with Kuiper's theorem, the result on $M(K \otimes A)$ has implications for the representability of K -theory functors in terms of Fredholm families -- see [1] and [7].

The proof rests on one or two facts concerning projections in $M(K \otimes A)$. Let us call a projection P in a unital C^* -algebra *proper* if both P and $1 - P$ are equivalent to the identity in the sense of Murray and von Neumann. It is useful to introduce another notion

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of equivalence: let us say that two projections P and Q in a unital C^* -algebra are *strongly equivalent*, denoted $P \approx Q$, if there exists a unitary W , connected to the identity, such that $WPW^* = Q$. This is the same as requiring that P and Q be connected by a path of projections.

LEMMA 1. *Let P and Q be equivalent projections in a unital C^* -algebra. If $\|PQ\| < 1$ then P and Q are strongly equivalent.*

PROOF. Suppose first that $PQ = 0$. Let V be a partial isometry such that $V^*V = P$ and $VV^* = Q$. If R denotes the projection $1 - P - Q$ then the element $W = V + V^* + R$ is a self-adjoint unitary such that $WPW^* = Q$. In the general case, write Q as a matrix

$$\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

with respect to the decomposition of the identity $1 = P + (1 - P)$. Since Q is a projection, a and c are positive elements with norm less than or equal to one, and the relations

$$(1) \quad a = a^2 + bb^*$$

$$(2) \quad b = ab + bc$$

$$(3) \quad c = c^2 + b^*b$$

are satisfied. (These conditions are in fact necessary and sufficient for

$$\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

to be a projection.) The second relation may be written as $(1 - a)b = bc$, and by approximating with polynomials we get that $f(1 - a)b = bf(c)$ for any continuous function $f: [0,1] \rightarrow \mathbb{C}$. The hypothesis $\|PQ\| < 1$ implies that $\|a\| < 1$, say $\|a\| < 1 - \epsilon$, where $\epsilon > 0$. Let f_t ($t \in [0,1]$), be a norm continuous family of continuous functions on $[0,1]$, with values in $[0,1]$ such that:

$$(4) \quad f_0(x) = x \quad \text{for all } x \in [0,1],$$

$$(5) \quad \begin{array}{ll} f_t(x) \leq x & \text{for all } t \in [0,1] \text{ and all } x \text{ near } 0, \text{ and} \\ f_t(x) \geq x & \text{for all } t \in [0,1] \text{ and all } x \text{ near } 1, \end{array}$$

$$(6) \quad f_1(x) = 1 \quad \text{for all } x \in [\epsilon,1].$$

Now, let:

$$a_t = 1 - f_t(1 - a)$$

$$b_t = b \left[\frac{f_t(c) - f_t(c)^2}{c - c^2} \right]^{1/2}$$

$$c_t = f_t(c)$$

(condition (5) ensures that b_t is well defined). Then

$$\begin{bmatrix} a_t & b_t \\ b_t^* & c_t \end{bmatrix}$$

is a projection because the relations (1)-(3) are satisfied. Since $\|a\| < 1 - \epsilon$, from (6) we have that $f_1(1 - a) = 1$. In particular, $f_1(1 - a)^2 = f_1(1 - a)$, so that

$$0 = (f_1(1 - a) - f_1(1 - a)^2)b = b(f_1(c) - f_1(c)^2),$$

and thus b_1 is equal to zero. Therefore

$$Q = \begin{bmatrix} a_0 & b_0 \\ b_0^* & c_0 \end{bmatrix} \approx \begin{bmatrix} a_1 & b_1 \\ b_1^* & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & f_1(c) \end{bmatrix},$$

and since

$$P \begin{bmatrix} 0 & 0 \\ 0 & f_1(c) \end{bmatrix} = 0,$$

we conclude that

$$P \approx \begin{bmatrix} 0 & 0 \\ 0 & f_1(c) \end{bmatrix} \approx Q. \quad \square$$

LEMMA 2. (i) *If P is a proper projection in $M(K \otimes A) \otimes C(X)$ (where X is a compact space) and $U \in M(K \otimes A) \otimes C(X)$ is a unitary such that $PUP = P$, then U is connected to the identity by a path of unitaries.*

(ii) *If $Q \in M(K \otimes A) \otimes C(X)$ is a projection such that Q and $1-Q$ contain subprojections equivalent to the identity, then Q is proper.*

PROOF. (i) This follows from the fact that $K_1(M(K \otimes A) \otimes C(X)) = 0$ (cf. [3], Lemma 3.2). Indeed, U is path connected to the identity in $M_n(M(K \otimes A) \otimes C(X))$ for some n , and since U is of the form

$$\begin{bmatrix} U' & 0 \\ 0 & 1 \end{bmatrix}$$

in $M(K \otimes A) \otimes C(X)$ in the matrix decomposition with respect to P , we may transfer this path to one in $M(K \otimes A) \otimes C(X)$ by conjugating with an appropriate isometry.

(ii) This follows from the fact that $K_0(M(K \otimes A) \otimes C(X)) = 0$ (cf. [3], Lemma 3.2 once more) in a similar manner. \square

For the next two lemmas, we fix an infinite sequence $\{e_n\}_{n=1}^\infty$ of pairwise orthogonal, rank one projections in K . It is convenient to assume that $1 - \sum_{n=1}^\infty e_n$ is an infinite dimensional projection in $B(H)$.

LEMMA 3. *Let $\{e_n\}_{n \in K}$ be a subsequence of $\{e_n\}_{n=1}^\infty$ and let $\{u_n\}_{n=1}^\infty$ be an approximate unit for A such that $u_{n+1}u_n = u_n$ for all n . There exists a proper projection $P \in M(K \otimes A)$ such that $P \preceq \sum_{n \in K} e_n \otimes u_n$, and in fact $(\sum_{n \in K} e_n \otimes u_n)P = P$.*

We remark that $\sum_{n \in K} e_n \otimes u_n$ is easily seen to converge in the strict topology of $M(K \otimes A)$.

PROOF. Let $d_n = (u_{n-1} - u_{n-2})^{1/2}$ (where we set $u_0 = u_{-1} = 0$). Choose partial isometries $v_{mn} \in K$ such that $v_{mn}^*v_{mn} = e_m$ and the $v_{mn}v_{mn}^*$ are pairwise orthogonal projections, each equal to some e_k for which $k \in K$ and $k \geq n$, and let $V_n = \sum_{m=1}^\infty v_{mn}$. The series $\sum_{n=1}^\infty V_n \otimes d_n$ converges in the strict topology (cf. [2], Lemma 2.4) to some element V , which is an isometry because $V^*V = \sum 1 \otimes d_n^2 = 1$. Since $u_k d_n = d_n$ if $k \geq n$, it follows easily that $(\sum_{k \in K} e_k \otimes u_k)V = V$. Thus if $P = VV^*$ then $(\sum_{k \in K} e_k \otimes u_k)P = P$. That P is proper follows from part (ii) of Lemma 2, together with the fact that $1 - P$ majorizes the proper projection $1 - \sum_{k \in K} e_k \otimes 1$. \square

If A is unital and we take u_n to be identically equal to 1 then the above lemma is of course trivial.

LEMMA 4. *Let A be a C^* -algebra with a countable approximate unit*

and let $P, Q \in M(K \otimes A) \otimes C(X)$ be proper projections. There exist proper projections P' and Q' such that $P \geq P', Q \geq Q', \|P'Q'\| < 1$, and $P'Q' \in K \otimes A \otimes C(X)$.

PROOF. Let $E = \sum_{n=1}^{\infty} e_n \otimes 1 \otimes 1$; then since any two proper projections are unitarily equivalent, there are unitaries V and W such that $P = VEV^*$ and $Q = WEW^*$. Without loss of generality we may assume that $V = 1$ and so $P = E$. Setting $g_n = e_n \otimes u_n \otimes 1$ and $h_n = Wg_nW^*$ (where u_n is as in the previous lemma) we have that $\lim_{m \rightarrow \infty} g_m h_n = 0$ for fixed n , and conjugating with W^* , also $\lim_{n \rightarrow \infty} g_m h_n = 0$ for fixed m . Choose subsequences g_{m_i} and h_{n_j} such that $\|g_{m_i} h_{n_j}\| < 2^{-i-j}$. By Lemma 3, there are proper projections

$$P' = P'' \otimes 1 \leq \left[\sum e_{m_i} \otimes u_{m_i} \right] \otimes 1 \leq P$$

and

$$Q' = W(Q'' \otimes 1)W^* \leq W \left[\left(\sum e_{n_j} \otimes u_{n_j} \right) \otimes 1 \right] W^* \leq Q.$$

Since $P'Q' = P'(\sum_i g_{m_i})(\sum_j h_{n_j})Q'$, the lemma follows from the facts that $\sum_{ij} g_{m_i} h_{n_j} \in K \otimes A \otimes C(X)$ and $\|\sum_{ij} g_{m_i} h_{n_j}\| < 1$. \square

COROLLARY. Let A be a C^* -algebra with a countable approximate identity and let P and Q be proper projections in $M(K \otimes A) \otimes C(X)$. There exists a unitary W , connected to the identity, such that $WPW^* = Q$.

PROOF. Choose projections P' and Q' as in the preceding lemma. By Lemma 1,

$$P \approx 1 - P \approx P' \approx Q' \approx 1 - Q \approx Q. \quad \square$$

THEOREM. If A has a countable approximate identity then the unitary group of $M(K \otimes A)$ is contractible.

PROOF. Since the unitary group is homotopy equivalent to an open subset of a normed linear space (namely the group of invertible elements, for example), it is homotopy equivalent to a CW-complex (see [6], Lemma 5.2). Thus it suffices to show that any map from a compact space X into the unitary group is homotopic

to the identity, or in other words, that the unitary group of $M(K \otimes A) \otimes C(X)$ is connected. Given any element U of this group, and any proper projection P , by the corollary above, there is a unitary W , connected to the identity, such that $WUW^* = P$. Thus we may assume that

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$$

in the matrix decomposition with respect to P . Since P is equivalent to $1 - P$, we have that

$$U \sim \begin{bmatrix} U_1 U_2 & 0 \\ 0 & 1 \end{bmatrix},$$

and so the theorem follows from Lemma 2. \square

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